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ROLLING OF A BODY ON A PLANE OR A SPHERE: A GEOMETRIC POINT OF VIEW

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A pair of bodies rolling on each other is an interesting example of nonholonomic systems in control theory. Here the controllability of rolling bodies is investigated with a global approach. By using simple geometric facts, this problem has been completely solved in the special case where one of them is a plane or a sphere.

1. INTRODUCTION

In this paper we study the controllability of two rigid bodies rolling on each other. The main question is: "Beginning from an initial contact configuration of two rigid bodies, which configurations could be achieved only by rolling them on each other?"

This problem has its origin in dexterous manipulations, and has been studied in several papers: Montana [8] derived a differential-geometric model of the rolling constraint between general bodies and discussed its application to robotic manipulation. Li and Canny [6] showed that the plate ball system as well as two unequal spheres is controllable. Levi [5] gave explicit formulas for evaluating the final configuration of the ball after a circular motion of the plate. Marigo and Bicchi [7] showed that the generated involutive distribution of this system at each point of the phase space is either two dimensional or the entire five dimensional space. Moreover, in the first case, the two bodies should be specular images of each other, that is, around the contact point they are locally mirror symmetries of each other. Then they concluded that the reachable manifolds of such a system are some disjoint two dimensional and five dimensional manifolds. Agrachev and Sachkov [1] showed that the generated involutive distribution is two dimensional if and only if the Gaussian curvatures of the surfaces at the contact point are equal. Also they showed that two-dimensional orbits are in one to one correspondence with the isometries of the surfaces.

The approach of this paper differs from the usual techniques of control theory. The main idea is that if a homogeneous space S, like a plane or a sphere, rolls along a closed curve on a given rigid body, it returns to its previous location, so this rolling gives an isometry of S. We show that the set of isometries obtained in this manner is

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an arcwise connected subgroup of the isometry group of S which is known to be a Lie group. Therefore it should be a Lie subgroup of the isometry group by Yamabe theorem. This is exactly the same process as the one used for showing that the holonomy group of a connection on a vector bundle is a Lie subgroup of general linear group of the fibre. Because of simplicity of the isometry group of the plane and the sphere, we are able to show that when S is the plane this subgroup must be the entire isometry group, and when it is the sphere this subgroup may be the entire isometry group or the trivial subgroup. The latter case occurs exactly when the other rigid body is also a sphere with the same radius as the S. The controllability results will be obtained from these facts.

In Section 1 we first review some terminology that we shall use later and then study the rolling constraint and its equivalent conditions. In Section 2 we show that all contact configurations are accessible when one of the bodies is a plane. In Section 3 this proposition will be proved when one of the bodies is a sphere, except in the case where the other is also a sphere with the same radius. In all cases the main tools are some elementary facts in differential geometry.

2. Rolling of Two Bodies

Rigid transformations in Euclidean spaces. The group of orientation preserving rigid transformations in Euclidean space $V = \mathbb{R}^n$ is denoted by $SE(V)$ which is a Lie group. For each $g \in SE(V)$ there is a unique rotation $R \in SO(V)$ and a unique translation by $T \in V$ such that $g(p) = Rp+T$. The map $r : SE(V) \to SO(V)$; $r(g) = R$, is a Lie group homomorphism whose kernel is all translations in V. In fact $r(g)$ gives the rotation part of the rigid transformation g.

Rigid motions. Suppose $A \subset \mathbb{R}^n$ is a subset that is not contained in any hyperplane of $\mathbb{R}^n$. Each rigid motion of A is uniquely represented by $g^s(A)$ where $a \leqslant s \leqslant b$ and $g^s = \big(R(s), T(s)\big)$ is a continuous path in $SE(\mathbb{R}^n)$. We may interpret $g^s(A)$ as the location of A at instant s.

Phase space of contact configurations. Suppose A and B are two smooth strictly convex subsets of $\mathbb{R}^3$. Since the Gauss map $n_B : B \to S^2$ is one to one in this case, for each $x \in A$ and $R \in SO(3)$, there is a unique translation that makes $R.B := \{Rx \mid x \in B\}$ tangent to A at $x \in A$. Therefore the phase space for all contact configurations of two bodies can be represented by $A \times SO(3)$.

Using this terminology the controllability problem of rolling bodies can be stated in the following way: "Beginning from a given initial configuration, which points of $A \times\ SO(3)$ are accessible by rolling the bodies on each other?"

Assume A and B are smooth strictly convex as before. First note that if $g_1^s(A)$ and $g_2^s(B)$ represent a rolling of A and B on each other in $\mathbb{R}^3$, then for another rigid motion h^s, $h^s \circ g_1^s(A)$ and $h^s \circ g_2^s(B)$ are also a rolling of A and B, because in both cases the

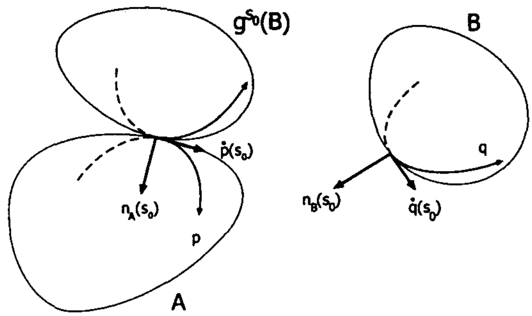

Figure 1: At instant s_0 the location of B is given by $g^{s_0}(B)$, $n_B(s_0)$ coincides with $n_A(s_0)$ and $\dot{q}(s_0)$ coincides with $\dot{p}(s_0)$.

motion of B relative to A is given by $(g_1^s)^{-1} \circ g_2^s(B)$. So we may assume A is fixed in $\mathbb{R}^3$ and $g^s(B) = R(s).B + T(s)$ represents a rolling of B and A on each other.

Now suppose $p(s)$ and $q(s)$ are two curves on A and B traced out by the rolling. These curves are smooth if the motion g^s is smooth. In the following theorem we consider the conditions that are forced on these curves by the rolling constraint, and show that these curves uniquely determine the rolling.

THEOREM 1.

(a) *If p and q are two curves on A and B corresponding to a rolling then they have the same speed and geodesic curvature. That is, for every s: $|\dot{p}(s)| = |\dot{q}(s)|$ and $k_1(s) = k_2(s)$.*

(b) *Any two curves p on A and q on B with the same speed and geodesic curvature, uniquely identify a rolling of A and B.*

(b) *Furthermore, if B is tangent to A at $x \in A$, a curve from x on A uniquely identifies a rolling of A and B on each other.*

In the case of part (c) we usually say B (or A) rolls along that curve.

PROOF: Since at each instant s, the two bodies are tangent to each other at $p(s)$ and $q(s)$ we should have

$$p(s) = g^s\big(q(s)\big), \tag{2.1}$$

$$n_A(s) = R(s)n_B(s), \tag{2.2}$$

where $n_A(s)$ is the inward normal to A at $p(s)$ and $n_B(s)$ is the outward normal to B at $q(s)$. Without loss of generality we may assume

$$\big|\dot{p}(s)\big| = 1. \tag{2.3}$$

During the rolling, the motion of $q(s_0)$ is given by $g^s\big(q(s_0)\big)$. The non-slipping assumption at the instant s_0 means

$$\frac{d}{ds}_{|s=s_0} g^s(q(s_0)) = 0. \tag{2.4}$$

By (2.1)

$$\begin{aligned}\dot{p}(s_0) &= \dot{R}(s_0)q(s_0) + R(s_0)\dot{q}(s_0) + \dot{T}(s_0)\\ &= \Big(\frac{d}{ds}_{|s=s_0} g^s(q(s_0))\Big) + R(s_0)\dot{q}(s_0).\end{aligned}$$

Thus (2.4) is equivalent to

$$\dot{p}(s) = R(s)\dot{q}(s). \tag{2.5}$$

Now suppose p and q are two curves on A and B parameterised by arc length. For each instant s_0 there is only one $g^{s_0} = \big(R(s_0), T(s_0)\big) \in SE(\mathbb{R}^3)$ such that relations (2.1), (2.2) and (2.5) hold. So for each two curves p on A and q on B with the same length, there is a unique non-slipping motion corresponding to them. But at each instant s, B may spin around $n_A(s)$. The non-spinning constraint at instant s_0 means that the angular velocity $\omega(s_0)$ is perpendicular to $n_A(s_0)$. Let us recall that $\dot{R}(s)v = \omega(s) \times R(s)v$ for each vector v. By replacing $\dot{q}(s_0)$ with v we find that the non-spinning constraint is equivalent to

$$\dot{R}(s_0)\dot{q}(s_0) \parallel n_A(s_0). \tag{2.6}$$

By (2.5)

$$\ddot{p}(s) = \dot{R}(s)\dot{q}(s) + R(s)\ddot{q}(s).$$

So (2.6) means $\ddot{p}(s)$ and $R(s)\ddot{q}(s)$ have the same projection on $\mathrm{T}_{p(s)}A$. On the other hand

$$\begin{aligned}k_1(s) &= \big(\ddot{p}(s)\,,\, n_A(s) \times \dot{p}(s)\big) && \text{(by definition)}\\ &= \Big(R(s)\ddot{q}(s)\,,\, R(s)\big(n_B(s) \times \dot{q}(s)\big)\Big) && \text{(by (2.2),(2.5) and (2.6))}\\ &= \Big(\ddot{q}(s)\,,\, \big(n_B(s) \times \dot{q}(s)\,\big)\Big) = k_2(s)\end{aligned}$$

where $k_1(s)$ is geodesic curvature of $p(s)$ and $k_2(s)$ is geodesic curvature of $q(s)$. Therefore by assuming (2.5), $k_1(s) = k_2(s)$ is equivalent to (2.6).

For the third part of the theorem it is sufficient to note that every curve on a regular oriented surface is uniquely identified by its geodesic curvature and its initial conditions. □

REMARK 1. Note that the above choice of orientation (that is, directions of n_A and n_B) was for simplicity. By changing orientation, only the sign of the geodesic curvature will change.

REMARK 2. The regularity assumption on A and B was necessary for the above calculations. But strict convexity was too strong for this theorem. In fact it can be replaced by the following weaker assumption

1. For each contact configuration, A and B meet each other only at one point.

Having such a condition one of the bodies may have concave parts or rolls inside of the other. Only the orientations should be chosen so that normal vectors at contact points coincide. Furthermore, in light of Theorem 1, one can generalise rolling motion for any two regular surfaces by considering two curves on them having the same speed and geodesic curvature. However, in some contact configurations they intersect each other or coincide through a region.

In this paper we assume strict convexity for simplicity, although our arguments go through or can easily be modified for more general cases.

COROLLARY 1. *If B rolls along a geodesic on A, the corresponding curve on B is also a geodesic with the same length.*

As a consequence of this corollary, to obtain geodesics of A one can roll it along straight lines of a plane.

COROLLARY 2. *If two bodies B_1 and B_2 roll simultaneously along a curve on A, they also roll on each other.*

PROOF: Suppose p, q_1 and q_2 are curves respectively on A, B_1 and B_2 corresponding to this rolling. By Theorem 1, these curves have the same speed and geodesic curvature. But by another use of Theorem 1, the motion of B_1 and B_2 corresponding to curves q_1 and q_2, should be a rolling. $\square$

COROLLARY 3. *If initially B_1 and B_2 are mirror images of each other relative to the common tangent plane, at each time during the rolling they will also be mirror images of each other with respect to the new common tangent plane. It means only a section of the fibre bundle $A \times SO(3) \xrightarrow{\pi_1} A$ is accessible from the first configuration, which is a connected two dimensional submanifold of the phase space.*

PROOF: Suppose P is a plane and q is an arbitrary smooth curve on B_1. When B_1 rolls on P along q, its mirror image with respect to P also rolls on P simultaneously. So by the previous corollary, B_1 and its image should roll on each other along q. This is nothing except rolling of B_1 and B_2 on each other along q and at each instant they are mirror images with respect to P which is their common tangent plane. $\square$

2.1. A GEOMETRIC INTERPRETATION We know from elementary courses in differential geometry that if there is a vector field $Y(s)$ along a regular curve $q(s)$ on a surface B such that it is linearly independent of $\dot{q}(s)$ and satisfyies $\dot{n}_B(s).Y(s) = 0$, then the ruled surface

$$u(s,t) = q(s) + tY(s) \qquad t \in I,\ s \in (-\varepsilon, \varepsilon) \tag{2.7}$$

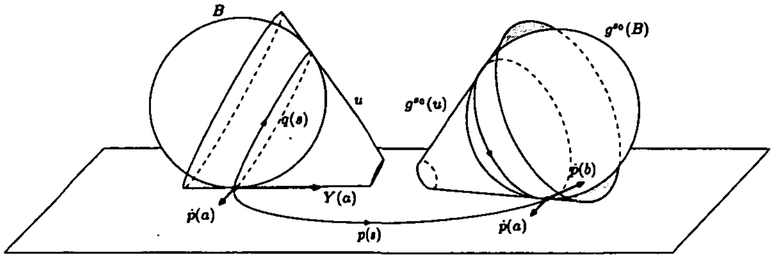

Figure 2: During the rolling B on the plane P along the curve q, the corresponding developable surface rolls simultaneously on P and this rolling gives an isometry between it and P.

is a regular developable surface (that is, n, the normal map of u, is constant along the lines $p(s_0) + tY(s_0)$). One of the important properties of such an osculating developable surface is that its Gaussian curvature vanishes identically. So it is locally isometric to an open set of the plane. It means that it can be locally flattened on the plane (see for example [**4**]).

Suppose $g^s \in SE(\mathbb{R}^3)$ represents a rolling of B on a plane P and q and p are the corresponding curves on B and P. We claim that by rolling of B, u rolls as well on P and this rolling induces an isometry between u and an open subset of P. Thus the geodesic curvature of q on u is equal to the geodesic curvature of p on P. But the geodesic curvatures of q on u and B are the same. This might be considered as an intuitive argument for Theorem 1. Moreover it implies that a parallel vector field along q on B induces a vector field along p on P which is parallel in the usual sense.

Let us prove our intuitive claim:

First note that at instant s_0, line $q(s_0)+tY(s_0)$ lies entirely on P, so $g^{s_0}(u)$ is tangent to P along $g^{s_0}\big(q(s_0) + tY(s_0)\big) = g^{s_0}\big(q(s_0)\big) + t\widetilde{Y}(s_0)$, where $\widetilde{Y}(s_0) = R(s_0)Y(s_0)$. Our claim is that the map $u(s,t) \to \widetilde{u}(s,t) = p(s) + t\widetilde{Y}(s)$ is a local isometry. But we have

$$\frac{\partial u}{\partial s}(s,t) = \dot p(s) + t\dot Y(s) \qquad \frac{\partial u}{\partial t}(s,t) = Y(s)$$
$$\frac{\partial \widetilde{u}}{\partial s}(s,t) = \dot p(s) + t\dot{\widetilde{Y}}(s) \qquad \frac{\partial \widetilde{u}}{\partial t}(s,t) = \widetilde{Y}(s).$$

Since $R(s)\dot q(s) = \dot p(s)$ and $R(s)Y(s) = \widetilde{Y}(s)$, our claim will be proved if we show $R(s)\dot Y(s) = \dot{\widetilde{Y}}(s)$.

$$\dot{\widetilde{Y}}(s) = \frac{d}{ds}R(s)Y(s) = \dot R(s)Y(s) + R(s)\dot Y(s) = \omega(s) \times R(s)Y(s) + R(s)\dot Y(s).$$

By the choice of Y, $R(s)Y(s)$ is perpendicular to $R(s)n_B(s) = n_P$ and $R(s)\dot n_B(s)$. also

$\omega(s) \perp n_P$ and

$$R(s)n_B(s) = n_P = \text{const} \;\Rightarrow\; \dot{R}(s)n_B(s) = -R(s)\dot{n}_B(s)$$
$$\Rightarrow \omega(s) \times R(s)n_B(s) = -R(s)\dot{n}_B(s).$$

This implies $\omega(s) \perp R(s)\dot{n}_B(s)$, so we should have $\omega(s) \| R(s)Y(s)$.

3. Rolling on a plane

Suppose P is a plane and A is tangent to P at a point $a \in A$.

Theorem 2. *By rolling alone, all contact configurations could be achieved.*

Proof: At first we pose this question: "By rolling, which of the points of P can the point a coincide with?" Note that such a rolling corresponds to a closed curve on A with base a. Let $\mathcal{P}_a A$ be the set of all closed piecewise smooth curves on A with base at a. By rolling along $\gamma \in \mathcal{P}_a A$, the tangent space $\mathrm{T}_a A$ moves rigidly and coincides with P again with the same orientation, so it induces a rigid transformation $j(\gamma) \in SE(P)$ where $j : \mathcal{P}_a A \to SE(P)$.

For $\gamma_1, \gamma_2 \in \mathcal{P}_a A$, a rolling along γ_1 and then along γ_2, is a rolling along $\gamma_2 * \gamma_1 \in \mathcal{P}_a A$. Also by rolling along γ and rolling back along it we obtain the identity map of P, that is

$$j(\gamma_2 * \gamma_1) = j(\gamma_2)j(\gamma_1) \qquad j(\gamma^-) = j(\gamma)^{-1} \tag{3.8}$$

where γ^- is γ with the reverse orientation. Thus $G := j(\mathcal{P}_a A)$ is a subgroup of $SE(P)$. For every $\gamma \in \mathcal{P}_a A$ there is a contraction of γ to a by curves $\gamma^s \in \mathcal{P}_a A$ whose length decrease to zero. Clearly $j(\gamma^s)$ is a path in $SE(P)$ joining $j(\gamma)$ to $1_{SE(P)}$. Now by a theorem of Yamabe [**3**] every path connected subgroup of a Lie group is a Lie subgroup. Connected Lie subgroups of a Lie group are in one to one correspondence with Lie subalgebras of that group so in simple cases they could be characterised easily. Path connected Lie subgroups of $SE(P)$ are

1. The trivial subgroup $\{1\}$.
2. All rotations around a fixed point.
3. All translations along a line.
4. All translations in P.
5. $SE(P)$.

Now it is enough to prove the following.

Lemma 1. *G is equal to $SE(P)$.*

This lemma says that only by rolling, all configurations in which A is tangent to P at a, are accessible. Assume this lemma for a moment and consider any arbitrary contact configuration of A and P. By rolling along a curve γ on A joining the point of contact to a, A will be tangent to P at a. By the lemma there is a rolling which transfers A to this last configuration. So by rolling back along γ we get the desired configuration. $\square$

PROOF OF LEMMA 1: If G is of types (1), (2), or (3), a might only coincide with the points of a circle or a line. Let S be a circle and L be a line in P and $p \in P$ be a point whose distance from L and S is more than $M := 2(\max_{a,b\in A} d_A(a,b) + 1)$. If we roll A along a curve on P with the end at p and then along a curve on A that joins the point of contact to a with length less than $M/2$, then a coincides with a point $q \in P$ with distance less than $M/2$ from p. Thus it couldn't lie on S or L. This argument shows that G couldn't be of types (1),(2) or(3).

We claim that the case (4) is also impossible for G: If γ is a closed smooth Jordan curve on A with base at a, by the Gauss-Bonnet theorem

$$\int_S K + \int_\gamma k_g = 2\pi,$$

where K is the curvature of A, S is the region inside γ (S is on the left side of γ), and k_g is the geodesic curvature of γ. Suppose $\widetilde{\gamma}$ is a curve on P corresponding to the rolling of A along γ. We know that for any regular curve $\widetilde{\gamma} : [a,b] \to P$:

$$\int_{\widetilde{\gamma}} \widetilde{k}_g = \measuredangle\big(\dot{\widetilde{\gamma}}(a), \dot{\widetilde{\gamma}}(b)\big).$$

If G is of type (4) we should have $\displaystyle\int_{\widetilde{\gamma}} \widetilde{k}_g = 2l\pi$ for some $l \in \mathbb{Z}$ (see Figure 2). Since $|\dot{\gamma}| = |\dot{\widetilde{\gamma}}|$ and $\widetilde{k}_g = k_g$, we have $\displaystyle\int_{\widetilde{\gamma}} \widetilde{k}_g = \int_\gamma k_g$. Thus

$$\int_S K = 2l_0\pi \qquad l_0 = 1 - l.$$

By continuous deformation of γ, $\displaystyle\int_S K$ varies continuously and for small regions S, it is also small. Therefore

$$\int_S K = 0$$

for every such S. But it implies that $K = 0$ every where in A. This contradicts the compactness of A. So the only possible case for G is (5). $\square$

4. ROLLING ON A SPHERE

Suppose B is a sphere with radius r_B.

THEOREM 3. *Starting from an initial state, all contact configurations of A and B could be achieved, except in the case where A is a sphere equal to B. In this case, the reachable manifold is a two dimensional submanifold of the phase space.*

PROOF: Let $\mathcal{P}A$ be the set of all piecewise smooth curves on A. Also suppose that B is initially tangent to A at $x \in A$ and, after rolling along $\gamma \in \mathcal{P}A$, its position is given by $g(B)$, where $g \in SE(\mathbb{R}^3)$. Define a map $j : \mathcal{P}A \to SO(3)$ by $j(\gamma) := r(g)$, that is, the amount of rotation of B after this rolling, and let $G_x = j(\mathcal{P}_x A)$.

Similarly to the previous section it can easily be proved that

$$j(\gamma_2 * \gamma_1) = j(\gamma_2)j(\gamma_1)j(\gamma^-) = j(\gamma)^{-1} \tag{4.9}$$

and the subgroup $G_x \subseteq SO(3)$ is pathwise continuous, so by the Yamabe theorem [**3**] it is a Lie subgroup of SO(3). In addition, if $\gamma \in \mathcal{P}A$ joins x to y, then

$$\left.\begin{array}{l} j(\gamma^-)G_y j(\gamma) \subseteq G_x \\ j(\gamma)G_x j(\gamma^-) \subseteq G_y \end{array}\right\} \Rightarrow j(\gamma)^{-1}G_y j(\gamma) = G_x. \tag{4.10}$$

Connected Lie subgroups of $SO(3)$ are

1. Trivial subgroup $\{1\}$.
2. All rotation around a fixed direction.
3. $SO(3)$.

CASE 3. If $G_x = SO(3)$, then by (4.10) $G_y = SO(3)$ for all $y \in A$, so from the initial state all points of the phase space $A \times SO(3)$ are accessible.

CASE 1. If $G_x = \{1\}$ then $G_y = \{1\}$ for all $y \in A$, so there is exactly one point of B that can be in contact with y. We denote it by $f(y)$. According to corollary 1 if γ is a geodesic on A with length l, $f(\gamma)$ is also a geodesic with length l. Therefore $f : A \to B$ is a local isometry. Since B has constant curvature $1/r_B > 0$, A has also constant curvature $1/r_B$. Therefore by the sphere rigidity theorem (see [**2**]), A should be a sphere with radius r_B.

By corollary 3, when A and B are equal spheres, the reachable manifold is two dimensional submanifold of the phase space.

CASE 2 . We shall show that G_x could not be of type (2) and this completes the proof of Theorem 3.

Assume, on the contrary, that G_x is equal to the group of rotations around an axis I_x. By (4.10), G_y is also the group of rotations around $I_y = j(\gamma)I_x$. Therefore these axes can be identified by a fixed direction on B, described by $\overline{qq'}$, which rotates as B rotates when it rolls on A (see Figure 3).

Also the set of points on B that could be in contact with $y \in A$ is exactly a circle lying on a plane orthogonal to that fixed direction on B. In other words, they are the points whose distances from q are constant. We denote this circle by C_y (see Figure 3). By a rolling, we may assume that $q \in B$ coincides with a point $p \in A$. Let

$$\begin{aligned} &V = \mathrm{T}_p A = \mathrm{T}_q B, \\ &f_1 = \exp_A : V \to A, \qquad f_2 = \exp_B : V \to B, \\ &B_r = \{x \in V : |x| < r\}, \qquad S_r = \{x \in V : |x| = r\}. \end{aligned}$$

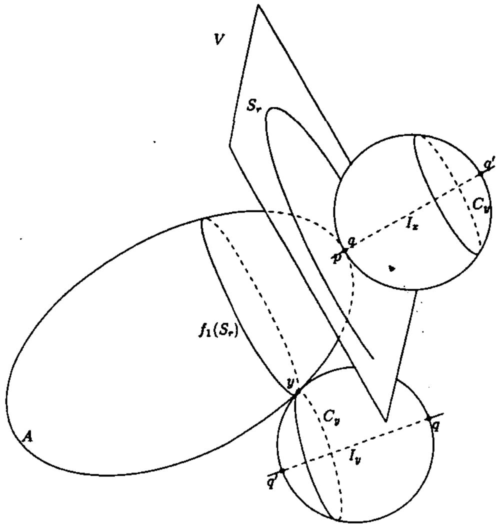

Figure 3: We consider C_y as a fixed subset of B, moving as B moves.

and suppose for $R > 0$, $f_1 : B_R \to A$ and $f_2 : B_R \to B$ are regular maps (that is, full rank maps).

According to Corollary 1, for each $u \in S^1 \subset V$, the geodesics $\gamma_1(s) = f_1(us)$ and $\gamma_2(s)$
$= f_2(us)$ correspond to a single rolling. It means

$$\forall r > 0,\ \forall y \in f_1(S_r) \ : \ C_y = f_2(S_r). \tag{4.11}$$

Thus rolling along $f_1(S_r)$ will also be along $f_2(S_r)$. By the Theorem 1, the geodesic curvature of $f_1(S_r)$ is equal to the geodesic curvature of $f_2(S_r)$. But since B is a sphere $f_2(S_r)$, has constant geodesic curvature. We shall show that this implies that $f_1(B_R)$ and $f_2(B_R)$ are local isometric:

Lemma 2. *The metrics (g_{ij}) and $(\widetilde{g}_{ij})$ on B_R induced by $f_1 : B_R \to A$ and $f_2 : B_R \to B$ respectively, are the same. It means $f_1 \circ f_2^{-1}$ is a local isometry between $f_1(B_R)$ and $f_2(B_R)$.*

Proof: First we recall that if (g_{ij}) is a metric on an open set $U \subseteq \mathbb{R}^n$, the Christoffel symbols may be obtained by

$$\forall i, j \qquad \sum_l g_{kl}\Gamma^l_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial i}(g_{jk}) + \frac{\partial}{\partial j}(g_{ki}) - \frac{\partial}{\partial k}(g_{ij})\right) \tag{4.12}$$

and with these symbols we can compute the covariant derivative as

$$\nabla_{e_i} e_j = \sum_k \Gamma^k_{ij} e_k \tag{4.13}$$

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where e_i are the standard vector fields on U. Also in a geodesic polar coordinate of a surface we have

$$(4.14) \qquad g_{11} = 1 \qquad g_{12} = g_{21} = 0 \qquad \lim_{r\to 0} g_{22} = 0 \qquad \lim_{r\to 0} \frac{\partial}{\partial r}\sqrt{g_{22}} = 1$$

(see [**2**, p. 287]). Since e_r and e_θ are orthogonal and $|e_r| = 1$, geodesic curvature of the curves $r = const$ could be obtained as

$$\begin{aligned}
\frac{e_r \,.\, \nabla_{e_\theta} e_\theta/|e_\theta|}{|e_\theta|} &= \frac{e_r \,.\, \nabla_{e_\theta} e_\theta}{|e_\theta|^2} + \frac{e_r \,.\, ((\partial/\partial\theta)(1/|\theta|))e_\theta}{|e_\theta|} = \frac{e_r.\nabla_{e_\theta} e_\theta}{|e_\theta|^2} \\
&= \frac{\Gamma^1_{22}}{g_{22}} && \text{by (4.13)} \\
&= -\frac{1}{2g_{22}}\frac{\partial}{\partial r} g_{22} && \text{by (4.12)}.
\end{aligned}$$

Now by putting $g = g_{22}$, $\widetilde{g} = \widetilde{g}_{22}$ in the case of the lemma we should have

$$\begin{aligned}
&-\frac{1}{2g}\frac{\partial g}{\partial r} = -\frac{1}{2\widetilde{g}}\frac{\partial \widetilde{g}}{\partial r} \;\Rightarrow\; \frac{\partial}{\partial r}\ln g = \frac{\partial}{\partial r}\ln \widetilde{g} \;\Rightarrow\; \ln g = \ln \widetilde{g} + C_1(\theta) \\
&\quad\Rightarrow\; g = C_2(\theta)\widetilde{g} \;\Rightarrow\; \sqrt{g} = \sqrt{C_2(\theta)}\sqrt{\widetilde{g}} \;\Rightarrow\; \frac{\partial}{\partial r}\sqrt{g} = \sqrt{C_2(\theta)}\frac{\partial}{\partial r}\sqrt{\widetilde{g}} \\
&\qquad\qquad \Rightarrow\; 1 = \sqrt{C_2(\theta)} \text{ (by last equation when } r \to 0\text{)} \;\Rightarrow\; g = \widetilde{g}.
\end{aligned}$$

This completes the proof of the lemma. $\square$

Assume $R_1 = \pi r_B$ and R is the largest number for which $f_1 : B_R \to A$ is regular. By the lemma, if $R < R_1$, in B_R we have

$$\left|\frac{\partial f_1}{\partial r}\right| = \left|\frac{\partial f_2}{\partial r}\right| = 1 \qquad \left(\frac{\partial f_1}{\partial r}, \frac{\partial f_1}{\partial \theta}\right) = \left(\frac{\partial f_2}{\partial r}, \frac{\partial f_2}{\partial \theta}\right) = 0 \qquad \left|\frac{\partial f_1}{\partial \theta}\right| = \left|\frac{\partial f_2}{\partial \theta}\right| > C_R > 0.$$

Noting that $f_1 : V \to A$ is smooth, the above relations imply that f_1 is also regular on an open neighbourhood of B_R, a contradiction to the maximality of R. So $R \geqslant R_1$ and therefore $f := f_1 \circ f_2\big|^{-1}_{B_{R_1}} : B/\{q'\} \to A$ is a regular map. As a consequence of the lemma, the set $\mathcal{F}$ of points on A which may coincide with q or q' is discrete. On the other hand, by a rolling along geodesics with length R_1, B will be tangent to A at q'. Therefore $f_1(S_{R_1}) \subseteq \mathcal{F}$. Since $f_1(S_{R_1})$ is a connected set, it should be a point. So we can consider f as a well defined continuous map from B to A which is smooth except perhaps at $q' \in B$. For sufficiently small r, $f : f_2(B_r) \to f_1(B_r)$ is a diffeomorphism and by (4.11):

$$\forall 0 \leqslant r_1 \leqslant r_2 \leqslant R_1 \qquad f_1(S_{r_1}) \cap f_1(S_{r_2}) = \emptyset,$$

which implies

$$f(f_2(B_r)^c) \subseteq f_1(B_r)^c.$$

But $f_1(B_r)^c$ and $f_2(B_r)^c$ are homeomorphic to the unite disk $\mathcal{D}^2 \subset \mathbb{R}^2$ and f homeomorphically maps the boundary of the former to the boundary of the latter, so

$f : f_2(B_r)^c \to f_1(B_r)^c$ is surjection. Since $f : A \to B$ is surjective and by Lemma 2 it is a local isometry, A has constant curvature $1/r_B$. Thus by the sphere rigidity theorem, A should be a sphere with radius r_B. But as mentioned before, in this case $G_x = \{1\}$. This proves our claim that G_x cannot be of type (2). $\square$

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