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WEIGHTED COMPOSITION OPERATORS ON FUNCTIONAL HILBERT SPACES

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Let X be a non-empty set and let $H(X)$ denote a Hilbert space of complex-valued functions on X. Let T be a mapping from X to X and θ a mapping from X to C such that for all f in $H(X)$, $f \circ T$ is in $H(x)$ and the mappings C_T taking f to $f \circ T$ and M taking f to $\theta.f$ are bounded linear operators on $H(X)$. Then the operator $C_T M_\theta$ is called a weighted composition operator on $H(X)$. This note is a report on the characterization of weighted composition operators on functional Hilbert spaces and the computation of the adjoint of such operators on L^2 of an atomic measure space. Also the Fredholm criteria are discussed for such classes of operators.

1. Introduction

Let X be a non-empty set and let $V(X)$ denote a Banach space of complex valued functions on X. Let T from X to X be a mapping such that for every f in $V(X)$, the composite function $f \circ T$ is also in $V(X)$ and the mapping C_T taking f to $f \circ T$ is a bounded linear operator on $V(X)$. Then C_T is called the composition operator on $V(X)$ induced by T. If $\theta : X \to \mathbf{C}$, the field of complex numbers, is a

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function such that the mapping taking f to $\theta.f$ is a bounded linear operator on $V(X)$, then this operator is called the multiplication operator induced by θ and we denote it by M_θ. The product $C_T M_\theta$ of C_T and M_θ is called a weighted composition operator on $V(X)$. In the last twelve years or so an extensive study of composition operators and multiplication operators has been made on different Banach spaces of functions (see [1] and [3]).

In this note a characterization of weighted composition operators on a functional Hilbert space is obtained. The adjoint of weighted composition operators on L^2 of an atomic measure space is computed. A necessary and sufficient condition for the adjoint of a weighted composition operator to be a composition operator is obtained. Also the dimensions of the kernels of weighted composition operators are determined.

By $B(H)$ we denote the Banach algebra of all bounded linear operators on H. The symbol $\ker A$ stands for the kernel of $A \in B(H)$. If θ is a complex valued function on X, then Z_θ and Z'_θ denote the zero set of θ and the complement of the zero set of θ respectively.

2. Characterization of weighted composition operators

Let $H(X)$ denote a Hilbert space of complex valued functions on a non-empty set X with algebraic operations of pointwise addition and pointwise scalar multiplication. $H(X)$ is said to be a functional Hilbert space if the evaluation functional taking f to $f(x)$ is continuous for every x in X. The Hardy space H^2 and the L^2 space of atomic measure space are well-known examples of functional Hilbert spaces. Let $x \in X$. Then by the Riesz representation theorem, there exists a vector K_x in $H(X)$ such that $f(x) = \langle f, K_x \rangle$ for every f in $H(X)$. Let $K = \{K_x : x \in X\}$. Then an operator A on $H(X)$ is a composition operator if and only if the set K is invariant under A^*. This has been established by Caughram and Schwartz [2]. A result of Shields and Wallen [8] shows that an operator A on $H(X)$ is a multiplication operator if and only if the elements of K are the eigen-vectors of A^*. The characterization of weighted composition operators on a functional Hilbert space is given in the following theorem.

THEOREM 2.1. *An operator* A *on a functional Hilbert space* $H(X)$ *is a weighted composition operator if and only if* $A^*(K) \subset M_\phi(K)$ *for some function* ϕ *on* X *. In this case* θ *and* T *are determined such that* $A^*K_x = \overline{\theta}.K_{T(x)}$.

Proof. Suppose A is a weighted composition operator on $H(X)$. Then $A = C_T M_\theta$ for some $T : X \to X$ and $\theta : X \to \mathbb{C}$. Then, for every f in $H(X)$,

$$\begin{aligned} \langle f, A^*K_x \rangle &= \langle Af, K_x \rangle \\ &= \langle C_T M_\theta f, K_x \rangle \\ &= \langle M_\theta f, C_T^* K_x \rangle \\ &= \langle \theta f, K_{T(x)} \rangle \\ &= \langle f, \overline{\theta} K_{T(x)} \rangle . \end{aligned}$$

Consequently $A^*K_x = \overline{\theta} K_{T(x)}$. Taking $\phi(x) = \overline{\theta}(x)$, we see that $A^*(K) \subset M_\phi(K)$.

Conversely, suppose $A^*(K) \subset M_\phi(K)$ for some function ϕ . Let $x \in X$. Then there exists $z \in X$ such that $A^*K_x = \phi.K_z$. Define $T(x) = z$ and $\theta(x) = \overline{\phi}(x)$. Then

$$\begin{aligned} (Af)(x) &= \langle Af, K_x \rangle \\ &= \langle f, A^*K_x \rangle \\ &= \langle f, \overline{\theta} K_{T(x)} \rangle \\ &= \langle \theta f, C_T^* K_x \rangle \\ &= \langle C_T M_\theta f, K_x \rangle \\ &= (C_T M_\theta f)(x) \end{aligned}$$

for every f in $H(X)$ and $x \in X$. Hence $A = C_T M_\theta$. This completes the proof of the theorem.

3. Adjoint of weighted composition operators

A measure space (X, S, λ) is said to be a standard Borel space if X

is a subset of a complete metric space Y with the sigma algebra S of Borel sets restricted to X. It has been proved in [4] that the adjoint of a composition operator C_T on L^2 of a standard Borel space is a composition operator if and only if it is invertible and $f_0 = 1$ (almost everywhere), where f_0 is the Radon-Nikodym derivative of the measure λT^{-1} with respect to λ. In the following theorem we obtain almost an analogous result for the adjoint of a weighted composition operator.

THEOREM 3.1. *The adjoint of a weighted composition operator $C_T M_\theta$ on L^2 of a standard Borel space is a composition operatir if and only if C_T is invertible and $\theta f_0 = 1$ (almost everywhere).*

Proof. Suppose $(C_T M_\theta)^*$ is a composition operator. Then there exists a non-singular measurable transformation S on X such that $(C_T M_\theta)^* = C_S$. This implies that $M_{\overline{\theta}} C_T^* = C_S$ and hence $M_{\overline{\theta}} C_T^* C_T = C_S C_T = C_{T\circ S}$. By the proof of Theorem 1 of Singh [4, p. 348], $C_T^* C_T = M_{f_0}$ and hence $M_{\overline{\theta} f_0} = C_{T\circ S}$. Again by Theorem 2 of Singh [4, p. 349], $\overline{\theta} f_0 = 1$ (almost everywhere) and C_T is left invertible. Since f_0 is a real valued essentially bounded function $\theta f_0 = 1$ (almost everywhere) and θ and f_0 are invertible.

Let f be in $L^2(\lambda)$. Since $C_T M_\theta = M_{\theta\circ T} C_T$, $C_S^* = M_{\theta\circ T} C_T$ and hence $C_S^* C_S = M_{\theta\circ T} C_T C_S$. This shows that $M_{g_0(\theta\circ T)^{-1}} = C_{S\circ T}$, where g_0 is the Radon-Nikodym derivative of λS^{-1} with respect to λ. Hence by Theorem 1 of [4, p. 348], C_T is right invertible. This completes the necessary part of this theorem.

Conversely, suppose C_T is invertible and $\theta f_0 = 1$ (almost where). Then

$$\begin{aligned}(C_TM_\theta)^* &= M_{\bar\theta}C_T^* \\ &= M_{\bar\theta}C_T^*C_TC_{T^{-1}} \\ &= M_{\bar\theta f_0}C_{T^{-1}} \\ &= C_{T^{-1}} .\end{aligned}$$

Hence $(C_TM_\theta)^*$ is a composition operator.

Note. The necessary part of this theorem is true when (X, S, λ) is any sigma-finite measure space. The following corollaries are an immediate consequence of this theorem.

COROLLARY 3.2. *The adjoint of a composition operator* C_T *on* $L^2(\lambda)$ *is a composition operator if and only if* C_T *is invertible and* $f_0 = 1$ *(almost everywhere).*

COROLLARY 3.3. *Let* $T : X \to X$ *be injective. Then* $(M_\theta C_T)^*$ *is a composition operator if and only if* C_T *is invertible and* $(\theta f_0) \circ T = 1$ *(almost everywhere).*

Proof. Since T is injective, by the proof of Theorem 2 of Singh [4, p. 349], C_T has dense range and hence $C_TC_T^* = M_{f_0\circ T}$. Thus the result follows from Theorem 3.1.

COROLLARY 3.4. *Let* $C_TM_\theta \in B(l^2)$ *, where* l^2 *is the Hilbert space of square summable sequences of complex numbers. Then* $(C_TM_\theta)^*$ *is a composition operator if and only if* $\theta = 1$ *and* $f_0 = 1$.

Proof. Since C_T is invertible implies C_T is an isometry, $f_0 = 1$. Hence the necessary part follows from this.

Conversely, suppose $\theta = 1$ and $f_0 = 1$. Then C_T is unitary if and only if C_T is invertible and hence C_T^* is a composition operator.

THEOREM 3.5. *Let* $C_TM_\theta \in B(l^2)$. *Then the following are equivalent:*

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(i) $C_T M_\theta$ *is unitary and* $|\theta| = 1$ *;*

(ii) $C_T M_\theta$ *is invertible and* $|\theta| = 1$ *;*

(iii) $(C_T M_\theta)^*$ *is a composition operator.*

4. Atomic measure spaces and weighted composition operators

Let (X, S, λ) be a sigma-finite measure space. A non-null $E \in S$ is said to be an atom if every non-null measurable subset F of E is such that either $\lambda(F) = 0$ or $\lambda(F) = \lambda(E)$. A measure space (X, S, λ) is said to be atomic if every measurable set contains an atom. A non-singular measurable transformation $T : X \to X$ is injective (almost everywhere) if the inverse image of every atom under T contains at most one atom. A measurable transformation T is surjective (almost everywhere) if the inverse image of every atom under T contains at least one atom. If (X, S, λ) is a sigma-finite atomic measure space, we write X as a countable union of atoms $\{x_i\}$, where the ith atom is denoted by x_i. We compute the adjoint of a weighted composition operator on L^2 of an atomic measure space.

THEOREM 4.1. *Let* $C_T M_\theta \in B(L^2(\lambda))$ *and let* $A \in B(L^2(\lambda))$ *be defined as*

$$(Af)(x_i) = \frac{\theta(x_i)}{\lambda\{x_i\}} \int_{T^{-1}\{x_i\}} f d\lambda \quad \textit{almost everywhere}$$

for every atom x_i *in* X. *Then* $A = (C_T M_\theta)^*$.

Proof. Let $f, g \in L^2(\lambda)$. Then

$$\langle C_T M_\theta f, g \rangle = \int_X (C_T M_\theta f)\bar{g} d\lambda$$

$$= \sum_{i=1}^{\infty} \int_{T^{-1}\{x_i\}} (\theta \circ T)(f \circ T)\bar{g} d\lambda$$

$$= \sum_{i=1}^{\infty} \theta(x_i) f(x_i) \int_{T^{-1}\{x_i\}} \bar{g} d\lambda$$

$$= \sum_{i=1}^{\infty} f(x_i) \lambda(x_i) A\bar{g}(x_i)$$

$$= \sum_{i=1}^{\infty} \int_{\{x_i\}} f(A\bar{g}) d\lambda$$

$$= \int_X f(A\bar{g}) d\lambda = \langle f, Ag \rangle .$$

Hence $A = (C_T M_\theta)^*$.

We compute the dimension of the kernel of weighted composition operators on L^2 of an atomic measure space in the following theorem.

THEOREM 4.2. *Let* $C_T M_\theta \in B(L^2(\lambda))$. *Then* dim ker $C_T M$ *equals the number of atoms in* $(X \backslash T(X)) \cup Z_\theta$.

Proof. Suppose x_i is an atom such that $x_i \in (X - T(X)) \cup Z_\theta$. If $x_i \in X - T(X)$, then $x_i \notin T(X)$ and therefore $\lambda T^{-1}\{x_i\} = 0$. This implies that $C_T X_{\{x_i\}} = 0$, where $X_{\{x_i\}}$ is the characteristic function of $\{x_i\}$. Hence $M_{\theta \circ T} C_T X_{\{x_i\}} = C_T M_\theta X_{\{x_i\}} = 0$. If x_i is an atom in Z_θ , then $\theta(x_i) = 0$ and this shows that $M_\theta X_{\{x_i\}} = 0$. Hence $C_T M_\theta X_{\{x_i\}} = 0$.

Conversely, suppose x_i is an atom such that $x_i \notin (X - T(X)) \cup Z_\theta$. Then $\lambda T^{-1}(x_i) \neq 0$ and $\theta(x_i) \neq 0$. Hence $C_T X_{\{x_i\}} \neq 0$ and $M_\theta X_{\{x_i\}} \neq 0$. This implies that

$$\theta(x_i) . C_T X_{\{x_i\}} = C_T \theta(x_i) X_{\{x_i\}} = C_T M_\theta X_{\{x_i\}} \neq 0 .$$

This completes the proof of the theorem.

COROLLARY 4.3. *The weighted composition operator* $C_T M_\theta$ *on* $L^2(\lambda)$

is an injection if and only if T *is surjective (almost everywhere) and* $\theta \neq 0$ *(almost everywhere).*

Proof. Since C_T is an injection if and only if T is a surjection (almost everywhere) the result is obvious.

COROLLARY 4.4. *If* $\theta \neq 0$ *(almost everywhere), then* $\dim \ker C_T M_\theta$ *equals the number of atoms in* Z_θ .

COROLLARY 4.5. *If* $\theta \neq 0$ *(almost everywhere), then* $\ker(C_T M_\theta)^* = \ker C_T^*$.

COROLLARY 4.6. *Let* $C_T M_\theta \in B(L^2(\lambda))$. *Then* $C_T M_\theta$ *has dense range if and only if* T *is an injection (almost everywhere) and* $\theta \neq 0$ *(almost everywhere).*

It has been proved in [7] that if $C_T \in B(L^2(\lambda))$, then $\dim \ker C_T^* = \sum_{n=1}^{\infty} \beta_n$, where β_n denotes the number of atoms minus one if $T^{-1}(x_n)$ has more than one atom, otherwise zero.

THEOREM 4.7. *Let* $C_T M_\theta \in B(L^2(\lambda))$. *Then* $C_T M_\theta$ *has closed range if and only if* θ *and* f_0 *are bounded away from zero in* $Z'_\theta \cap Z'_{f_0}$.

Proof. Suppose $C_T M_\theta$ has closed range. Then $(C_T M_\theta)^* C_T M_\theta$ and hence $M_{|\theta|^2 f_0}$ has closed range. This implies that $|\theta|^2 f_0$ is bounded away from zero in $Z'_{|\theta|^2 f_0}$ and hence θf_0 is bounded away from zero in $Z'_{\theta f_0}$. Since $Z_{\theta f_0} = Z_\theta \cup Z_{f_0}$, θ and f_0 are bounded away from zero in $Z'_\theta \cap Z'_{f_0}$.

DEFINITION. An operator $A \in B(H)$ is said to be Fredholm if the kernel and co-kernel of A are finite dimensional and the range of A is closed.

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The following theorem characterises Fredholm weighted composition operators on $L^2(\lambda)$.

THEOREM 4.8. *The weighted composition operator* $C_T M_\theta$ *on* $L^2(\lambda)$ *is Fredholm if and only if*

(i) $X - T(X)$ *and* Z_θ *contain finite numbers of atoms,*

(ii) *on the complement of a set containing a finite number of atoms,* T *is an injection (almost everywhere),*

(iii) θ *and* f_0 *are bounded away from zero in* $Z'_\theta \cap Z'_{f_0}$.

Proof. The proof follows from Theorem 4.2, Theorem 4.7 and Theorem 5 of [7, p. 263].

COROLLARY 4.9. *Let* $C_T M_\theta \in B(l^2)$. *Then* $C_T M_\theta$ *is Fredholm if and only if*

(i) $X - T(X)$ *and* Z_θ *contain finite numbers of elements,*

(ii) *on the complement of a set containing a finite number of elements,* T *is an injection,*

(iii) θ *is bounded away from zero in* $Z'_\theta \cap Z'_{f_0}$.

Proof. Since the range of C_T on l^2 is closed [6], the result follows from Theorem 4.8.

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