Can. J. Math., Vol. XLII, No. 2, 1990, pp. 304–314

THE PEXIDER FUNCTIONAL EQUATIONS IN DISTRIBUTIONS

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1. Introduction. The Cauchy functional equations have been studied recently for Schwartz distributions by Koh in [**3**]. When the solutions are locally integrable functions, the equations reduce to the classical Cauchy equations (see [**1**]):

(1) $$f(x+y) = f(x) + f(y)$$

(2) $$f(x+y) = f(x)f(y)$$

(3) $$f(xy) = f(x) + f(y)$$

(4) $$f(xy) = f(x)f(y).$$

Earlier efforts to study functional equations in distributions were given by Fenyö [**2**] for the Hosszu' equations

$$f(x+y-xy) + f(xy) = f(x) + f(y),$$

by Neagu [**4**] for the Pompeiu equation

$$f(x+y+xy) = f(x) + f(y) + f(x)f(y)$$

and by Swiatak [**6**].

In this paper, we interpret and study the Pexider functional equations in distributions. These are direct generalizations of Cauchy's equations and involve several unknown functions or distributions. The classical Pexider equations are given by

(5) $$f(x+y) = g(x) + h(y)$$

(6) $$f(x+y) = g(x)h(y)$$

(7) $$f(xy) = g(x) + h(y)$$

(8) $$f(xy) = g(x)h(y)$$

where f, g and h are the unknown functions. A discussion of these equations and their solutions may be found in [**1**]. We now proceed to define the Pexider

Received May 2, 1989.

equations in distributions. In the next section we summarize the background material needed for our work.

2. Notation and preliminary results. Let $I = (0, \infty) \subset \mathbf{R}$ and $I^2 = I \times I \subset \mathbf{R}^2$. We denote by $\mathcal{D}(I)$ and $\mathcal{D}(I^2)$ the spaces of infinitely differentiable complex-valued functions with compact support on I and I^2 respectively. Likewise, we denote by $\mathcal{E}(I)$ and $\mathcal{E}(I^2)$ the spaces of infinitely differentiable complex-valued functions on I and I^2 respectively. The dual of these spaces will be denoted by a prime and we note that $\mathcal{D}(I) \subset \mathcal{E}(I) \subset \mathcal{E}'(I) \subset \mathcal{D}'(I)$ (see [**5**]). The second inclusion will be interpreted by identifying the smooth function in $\mathcal{E}(I)$ with the regular distribution it generates in $\mathcal{E}'(I)$. The topologies for these spaces will be the usual convergence concepts for $\mathcal{D}(I)$ and $\mathcal{E}(I)$ as given by Schwartz [**5**] and weak topologies for their duals. $L_{\text{LOC}}(I)$ and $L_{\text{LOC}}(I^2)$ denote the spaces of equivalence classes of locally integrable functions on I and I^2 respectively. Two functions are equivalent if they are equal almost everywhere. In the sequel we shall denote by λ_f the distribution corresponding to a locally integrable function f. For example, if $f \in L_{\text{LOC}}(I)$, then

$$\langle \lambda_f, \phi \rangle = \int_I f(x)\phi(x)dx$$

for $\phi \in \mathcal{D}(I)$.

Throughout, D denotes the differentiation operator in $\mathcal{D}'(I)$ whereas D_1 and D_2 are the partial differentiation operators in $\mathcal{D}'(I^2)$ with respect to the first and second variable from I^2 respectively. Let E_1 and E_2 be integration operators from $\mathcal{D}(I^2)$ onto $\mathcal{D}(I)$ given by

$$E_1[\phi](x) = \int_I \phi(x, y)dy \tag{9}$$

and

$$E_2[\phi](y) = \int_I \phi(x, y)dx \tag{10}$$

for any $\phi \in \mathcal{D}(I^2)$. These are continuous linear operators and we shall denote this by membership in $L[\mathcal{D}(I^2); \mathcal{D}(I)]$. Their adjoints are E_1^* and E_2^* from $\mathcal{D}'(I)$ onto $\mathcal{D}'(I^2)$ and are defined by

$$\langle E_1^*[T], \phi \rangle = \langle T, E_1[\phi] \rangle = \left\langle T(x), \int_I \phi(x, y)dy \right\rangle \tag{11}$$

and

$$\langle E_2^*[T], \phi \rangle = \langle T, E_2[\phi] \rangle = \left\langle T(y), \int_I \phi(x, y)dx \right\rangle, \tag{12}$$

https://doi.org/10.4153/CJM-1990-017-6 Published online by Cambridge University Press

for any $T \in \mathcal{D}'(I)$ and $\phi \in \mathcal{D}(I^2)$. Again, we note that E_1^* and E_2^* both belong to the linear space $L[\mathcal{D}'(I); \mathcal{D}'(I^2)]$.

We shall now list several propositions concerning the properties of the above operators. We shall omit their proofs since they are analogous to proofs given in [**3**].

PROPOSITION 2.1. a. *If $f \in L_{\mathrm{LOC}}(I)$, and $\phi(x,y) = f(x)$ for $(x,y) \in I^2$, then $\phi \in L_{\mathrm{LOC}}(I^2)$ and $E_1^*(\lambda_f) = \phi$.*

Likewise, if $\psi(x,y) = f(y)$ for $(x,y) \in I^2$, then $\psi \in L_{\mathrm{LOC}}(I^2)$ and $E_2^*(\lambda_f) = \psi$.

b. *If $\alpha \in \mathcal{E}(I)$, then $E_1^*[\alpha]$ and $E_2^*[\alpha]$ both belong to $\mathcal{E}(I^2)$.*

c. *If $\alpha \in \mathcal{E}(I)$ and $T \in \mathcal{D}'(I)$, then, for $i = 1,2$,*

$$E_i^*[\alpha T] = E_i^*[\alpha]E_i^*[T]. \tag{13}$$

d. *If $T \in \mathcal{D}'(I)$, then*

$$D_1E_1^*[T] = E_1^*[DT] \quad \text{and} \quad D_1E_2^*[T] = 0 \tag{14}$$

$$D_2E_2^*[T] = E_2^*[DT] \quad \text{and} \quad D_2E_1^*[T] = 0. \tag{15}$$

We now define the operator $P : \mathcal{D}'(I) \times \mathcal{D}'(I) \longrightarrow \mathcal{D}'(I^2)$ for any $S, T \in \mathcal{D}'(I)$ and any $\phi \in \mathcal{D}(I^2)$ by

$$\begin{aligned} \langle P[S;T], \phi\rangle &= \langle S(x), \langle T(y), \phi(x,y)\rangle\rangle \\ &= \langle T(y), \langle S(x), \phi(x,y)\rangle\rangle. \end{aligned}$$

It is known [**5**] that P is a continuous linear operator in both variables and that

$$D_1P[S;T] = P[DS;T] \quad \text{and} \quad D_2P[S;T] = P[S;DT]. \tag{16}$$

The following proposition will be heavily utilized. Since the proof of parts (a) and (b) is immediate, we shall only give the proof for part (c).

PROPOSITION 2.2. a. *If $T \in \mathcal{D}'(I)$, then*

$$E_1^*[T] = P[T;1] \quad \text{and} \quad E_2^*[T] = P[1;T]. \tag{17}$$

b. *If $\alpha, \beta \in \mathcal{E}(I)$ and $S, T, U \in \mathcal{D}'(I)$, then*

$$E_1^*[\alpha]P[S;U] + E_1^*[\beta]P[T;U] = P[\alpha S + \beta T; U] \tag{18}$$

$$E_2^*[\alpha]P[S;T] + E_2^*[\beta]P[S;U] = P[S; \alpha T + \beta U]. \tag{19}$$

c. *Suppose S, T, U and V are non-zero distributions in $\mathcal{D}'(I)$. Then $P[S;T] = P[U;V]$ if and only if there exist non-zero real numbers c_1 and c_2 such that $S = c_1U$ and $T = c_2V$.*

Proof. We shall include a proof of (c) above. For any $\phi_1, \phi_2 \in \mathcal{D}(I)$ define

$$(\phi_1 \oplus \phi_2)(x,y) = \phi_1(x)\phi_2(y) \quad \text{for all } (x,y) \in I^2.$$

Clearly

$$\phi_1 \oplus \phi_2 \in \mathcal{D}(I^2) \quad \text{and} \quad \langle P[S;T], \phi_1 \oplus \phi_2\rangle = \langle S, \phi_1\rangle\langle T, \phi_2\rangle$$

whenever $S, T \in \mathcal{D}'(I)$ and $\phi_1, \phi_2 \in \mathcal{D}(I)$. Since $T \neq 0$), there exists $\phi_0 \in \mathcal{D}(I)$ such that $T(\phi) \neq 0$. Then, for every $\phi \in \mathcal{D}(I)$,

$$\begin{aligned}\langle S, \phi\rangle\langle T, \phi_0\rangle &= \langle P[S;T], \phi \oplus \phi_0\rangle \\ &= \langle P[U;V], \phi \oplus \phi_0\rangle \\ &= \langle U, \phi\rangle\langle V, \phi_0\rangle.\end{aligned}$$

Thus $\langle S, \phi\rangle = c_1 U(\phi)$ for all $\phi \in \mathcal{D}(I)$ with $c_1 = \langle V, \phi_0\rangle / \langle T, \phi_0\rangle$. That is $S = c_1 U$. By the same token there is a $c_2 = \langle U, \phi_0\rangle / \langle S, \phi_0\rangle$ such that $T = c_2 V$.

To come up with the generalization of Pexider equations to distributions, we need to define the following operators Q and R from $\mathcal{D}(I^2)$ into $\mathcal{D}(I)$ by

$$Q[\phi](x) = \int_{-\infty}^{\infty} \phi(x-y, y)dy = \int_{-\infty}^{\infty} \phi(y, x-y)dy \tag{20}$$

and

$$R[\phi](x) = \int_I \frac{\phi(x/y, y)dy}{y} = \int_I \frac{\phi(y, x/y)}{y} dy \tag{21}$$

and their respective adjoints Q^* and R^* from $\mathcal{D}'(I)$ into $\mathcal{D}'(I^2)$ by

$$\langle Q^*[T], \phi\rangle = \langle T, Q[\phi]\rangle = \langle T(x), Q[\phi](x)\rangle \tag{22}$$

and

$$\langle R^*[T], \phi\rangle = \langle T, R[\phi]\rangle = \langle T(x), R[\phi](x)\rangle \tag{23}$$

for any $\phi \in \mathcal{D}(I^2)$ and $T \in \mathcal{D}'(I)$.

Some of the properties of these operators are given in the following propositions:

PROPOSITION 2.3. *Suppose* $f \in L_{\mathrm{LOC}}(I)$.
a. *If we let* $g(x,y) = f(x+y)$ *for* $(x,y) \in I^2$, *then*

$$g \in L_{\mathrm{LOC}}(I^2) \quad \textit{and} \quad Q^*[\lambda_f] = \lambda_g.$$

b. *If we let* $h(x,y)=f(xy)$ *for* $(x,y)\in I^2$, *then*

$$h\in L_{\mathrm{LOC}}(I^2) \quad \textit{and} \quad R^*[\lambda_f]=\lambda_h.$$

PROPOSITION 2.4. *If* $\alpha\in\mathcal{E}(I)$, *then* $Q^*[\alpha]$ *and* $R^*[\alpha]\in\mathcal{E}(I^2)$.

PROPOSITION 2.5. *If* $\alpha\in\mathcal{E}(I)$ *and* $T\in\mathcal{D}'(I)$ *then*

$$Q^*[\alpha T]=Q^*[\alpha]Q^*[T] \tag{24}$$

$$R^*[\alpha T]=R^*[\alpha]R^*[T]. \tag{25}$$

PROPOSITION 2.6. *If* $T\in\mathcal{D}'(I)$, *then*

$$D_1(Q^*[T])=D_2(Q^*[T])=Q^*[DT] \tag{26}$$

$$D_1(R^*[T])=E_2^*(\Omega)R^*(DT) \tag{27}$$

$$D_2(R^*[T])=E_1^*(\Omega)R^*(DT) \tag{28}$$

where $\Omega=x\in\mathcal{E}(I)$.

3. The Pexider equations in distributions. In this section we shall make use of the operators Q^* and R^* to write the Pexider equations in distributions and utilize the propositions stated in Section 2 to exhibit the solutions of these equations. We shall indicate that for regular distributions, that is, locally integrable functions, the results reduce to the "classical" solutions of the Pexider equations (5)–(8).

Let T,U and V be in $\mathcal{D}'(I)$. The equations

$$Q^*[T]=E_1^*[U]+E_2^*[V] \tag{29}$$

$$Q^*[T]=P[U;V] \tag{30}$$

$$R^*[T]=E_1^*[U]+E_2^*[V] \tag{31}$$

$$R^*[T]=P[U;V] \tag{32}$$

will be referred to as the Pexider equations in distributions.

Although the proof of the next proposition can be deduced from the Proposition 2.1 – Proposition 2.6, we give the proof for the sake of clarity and completeness.

PROPOSITION 3.1. *If* T,U *and* V *are regular distributions, that is, locally integrable function* f,g, *and* h *respectively, then equations* (29)–(32) *reduce to the Pexider functional equations* (5)–(8) *respectively.*

Proof. (i) We shall first show that equation (29) reduces to equation (5) for regular distributions. We perform the following computations for $f \in L_{\mathrm{LOC}}(I)$ and $\phi \in \mathcal{D}(I^2)$:

$$\begin{aligned}\langle Q^*[\lambda_f], \phi\rangle &= \langle \lambda_f, Q[\phi](x)\rangle \\ &= \left\langle \lambda_f, \int_{-\infty}^{\infty} \phi(x-y, y)dy \right\rangle \\ &= \int_I f(x) \int_{-\infty}^{\infty} \phi(x-y, y)dydx \\ &= \int_{I^2} f(x+y)\phi(x,y)dydx = \langle f(x+y), \phi(x,y)\rangle\end{aligned}$$

and for g and $h \in L_{\mathrm{LOC}}(I)$ and any $\phi \in \mathcal{D}(I^2)$

$$\begin{aligned}\langle E_1^*[\lambda_g], \phi\rangle &= \langle \lambda_g, E_1[\phi](x)\rangle = \int_{I^2} g(x)\phi(x,y)dydx \\ &= \langle g(x), \phi(x,y)\rangle \\ \langle E_2^*[\lambda_h], \phi\rangle &= \langle \lambda_h, E_2[\phi](y)\rangle = \int_{I^2} h(y)\phi(x,y)dxdy \\ &= \langle h(y), \phi(x,y)\rangle.\end{aligned}$$

The above calculation shows that if $f, g, h \in L_{\mathrm{LOC}}(I)$ and if $Q^*[\lambda_f] = E_1^*[\lambda_g] + E_2^*[\lambda_h]$ then

$$\int_{I^2} \{f(x+y) - g(x) - h(y)\}\phi(x,y)dxdy = 0$$

for all $\phi \in \mathcal{D}(I^2)$. Hence we conclude that $f(x+y) = g(x) + h(y)$ for almost every $(x,y) \in I^2$.

(ii) We shall show that equation (30) reduces to equation (6) for regular distributions.

$$\begin{aligned}\langle P[\lambda_g; \lambda_h], \phi\rangle &= \langle \lambda_g, \langle \lambda_h, \phi(x,y)\rangle\rangle \\ &= \int_{I^2} g(x)h(y)\phi(x,y)dydx \\ &= \langle g(x)h(y), \phi(x,y)\rangle.\end{aligned}$$

Again, this shows that if $f, g, h \in L_{\mathrm{LOC}}(I)$ and if $Q^*[\lambda_f] = P[\lambda_g; \lambda_h]$ then

$$\int_{I^2} \{f(x+y) - g(x)h(y)\}\phi(x,y)dydx = 0$$

for all $\phi \in \mathcal{D}(I^2)$. Hence we conclude that $f(x+y) = g(x)h(y)$ for almost every $(x,y) \in I^2$.

(iii) We now show that equation (31) reduces to equation (2) for regular distributions.

$$\begin{aligned}\langle R^*[\lambda_f], \phi\rangle = \langle \lambda_f, R[\phi](x)\rangle &= \left\langle \lambda_f, \int_I \frac{\phi(x/y, y)}{y}\,dy\right\rangle \\ &= \int_{I^2} \frac{f(x)\phi(x/y, y)}{y}\,dydx = \int_{I^2} f(xy)\phi(x, y)dydx \\ &= \langle f(xy), \phi(x, y)\rangle.\end{aligned}$$

Thus if $f, g, h \in L_{\mathrm{LOC}}(I)$ and if $R^*[\lambda_f] = E_1^*[\lambda_g] + E_2^*[\lambda_h]$ then

$$\int_{I^2} \{f(xy) - g(x) - h(y)\}\phi(x, y)dydx = 0$$

for all $\phi \in \mathcal{D}(I^2)$. Hence $f(xy) = g(x) + h(y)$ for almost every $(x, y) \in I^2$.

(iv) From (ii) and (iii) we conclude that if $f, g, h \in L_{\mathrm{LOC}}(I)$ and if $R^*[\lambda_f] = P[\lambda_g; \lambda_h]$, then

$$\int_{I^2} \{f(xy) - g(x)h(y)\}\phi(x, y)dydx = 0$$

for every $\phi \in \mathcal{D}(I^2)$. Thus $f(xy) = g(x)h(y)$ for almost every $(x, y) \in I^2$.

We have thus shown that for regular distributions the Pexider equations in distributions as defined in (29)–(32) reduce to the ‘classical’ Pexider equations given in (5)–(8). We shall now present the solutions of equations (29)–(32).

THEOREM 3.1. *If* $T, U, V \in \mathcal{D}'(I)$ *satisfy equation* (29) $Q^*[T] = E_1^*[U] + E_2^*[V]$, *then there exists* $a, b, c \in \mathbf{R}$ *such that* $T = \lambda_f, U = \lambda_g$ *and* $V = \lambda_h$ *where* $f(x) = cx + a + b$, $g(x) = cx + a$, *and* $h(x) = cx + b$ *for all* $x \in I$.

Proof. Proposition 2.6, Proposition 2.1(d), Proposition 2.2(a) and (c) imply that

$$Q^*[DT] = D_1Q^*[T] = D_1(E_1^*[U] + E_2^*[V]) = D_1E_1^*[U] = P[DU; 1]$$

while

$$Q^*[DT] = D_2Q^*[T] = D_2(E_1^*[U] + E_2^*[V]) = D_2E_2^*[V] = P[1; DV].$$

Hence $P[DU; 1] = P[1; DV]$ which yields the system $DU = c$ and $DV = c$ for some $c \in R$. Thus $U = cx + a$ and $V = cx + b$ with $a, b \in \mathbf{R}$. Further, for any $\phi \in \mathcal{D}(I^2)$

$$\begin{aligned}\langle T, Q[\phi]\rangle &= \langle Q^*[T], \phi\rangle \\ &= \langle E_1^*[U] + E_2^*[V], \phi\rangle \\ &= \langle U, E_1[\phi]\rangle + \langle V, E_2[\phi]\rangle \\ &= \langle U(x), E_1[\phi](x)\rangle + \langle V(y), E_2[\phi](y)\rangle \\ &= \left\langle cx + a, \int_I \phi(x, y)dy \right\rangle + \left\langle cy + b, \int_I \phi(x, y)dx \right\rangle \\ &= \int_{I^2} (cx + cy + a + b)\phi(x,\ y)dydx \\ &= \int_I (cx + a + b) \int_{-\infty}^{\infty} \phi(x - y,\ y)dydx \\ &= \langle cx + a + b, Q[\phi]\rangle.\end{aligned}$$

It follows that $T = \lambda_f$ where $f(x) = cx + a + b \in \mathcal{E}(I)$.

COROLLARY 3.1. *If f, g and $h \in L_{\mathrm{LOC}}(I)$ satisfy equation* (29), *then $f(x), g(x)$ and $h(x) \in \mathcal{E}(I)$ with $f(x) = cx + a + b, g(x) = cx + a$ and $h(x) = cx + b$ for some $a, b, c \in \mathbf{R}$.*

Proof. Theorem 3.1 implies that there exists $a, b, c \in \mathbf{R}$ such that $f(x) = cx + a + b, g(x) = cx + a$ and $h(x) = cx + b$ hold for almost all $x \in I$. Let $\tilde{f}(x) = f(x) - cx - a - b, \tilde{g}(x) = g(x) - cx - a$ and $\tilde{h}(x) = h(x) - cx - b$ for all $x \in I$. Clearly $\tilde{f}(x + y) = \tilde{g}(x) + \tilde{h}(y)$ for all $x, y \in I$ and $\tilde{f}, \tilde{g}$ and $\tilde{h}$ vanish almost everywhere on I. Let

$$A = \{x \in I \mid \tilde{g}(x) = 0\} \quad \text{and} \quad B = \{x \in I \mid \tilde{h}(x) = 0\}.$$

Since A and B have full measure, it follows that $A + B = I$; that is,

$$I = \{x + y \mid x \in A, y \in B\}.$$

Thus $\tilde{f}(z) = 0$ for all $z \in I$. Given $x \in I$. Choose $y \in I$ such that $\tilde{h}(y) = 0$. This will then imply that $\tilde{g}(x) = 0$. Thus $\tilde{g} = 0$. Similarly one can show that $\tilde{h} = 0$. Therefore the conclusion of Corollary 3.1 holds for all $x \in I$.

This is the classical solution of the Pexider equation (5).

THEOREM 3.2. *If $T, U, V \in \mathcal{D}'(I)$ satisfy equation* (30) $Q^*[T] = P[U; V]$, *then there exist real numbers $a, b, c \in \mathbf{R}$ such that $T = \lambda_f, U = \lambda_g$ and $V = \lambda_h$ where $f(x) = abe^{cx}, g(x) = ae^{cx}$ and $h(x) = be^{cx}$ for all $x \in I$.*

Proof. From Proposition (2.6) and equations (16) we have

$$Q^*[DT] = D_1Q^*[T] = D_1P[U; V] = P[DU; V]$$
$$Q^*[DT] = D_2Q^*[T] = D_2P[U; V] = P[U; DV]$$

and thus $P[DU;V] = P[U;DV]$. This implies that $DU = cU$ and $DV = cV$ for some $c \in \mathbf{R}$. The solution of this system is $U = ae^{cx}$ and $V = be^{cx}$ with $a, b \in R$. Now, for any $\phi \in \mathcal{D}(I^2)$,

$$\langle Q^*[T], \phi \rangle = \langle T(x), Q[\phi](x) \rangle$$

while

$$\begin{aligned}\langle P[U;V], \phi \rangle &= \langle ae^{cx}, \langle be^{cy}, \phi(x,y) \rangle \rangle \\ &= \int_{I^2} abe^{c(x+y)} \phi(x,y)dydx \\ &= \int_{I^2} abe^{cx} \phi(x-y,y)dydx \\ &= \langle abe^{cx}, Q[\phi](x) \rangle.\end{aligned}$$

Hence $T = \lambda_f$ where $f(x) = abe^{cx}$.

COROLLARY 3.2. *If f, g and h belonging to $L_{\mathrm{LOC}}(I)$ satisfy equation* (30), *then $f(x), g(x)$, and $h(x)$ belong to $\mathcal{E}(I)$ with $f(x) = abe^{cx}, g(x) = ae^{cx}$ and $h(x) = be^{cx}, a, b, c \in \mathbf{R}$.*

This is the classical solution of the Pexider equation (6). The proof of this corollary is similar to that of Corollary 3.1.

THEOREM 3.3. *If $T, U, V \in \mathcal{D}'(I)$ satisfy equation* (31) *$R^*[T] = E_1^*[U] + E_2^*[V]$, then there exist $\alpha, \beta, c \in \mathbf{R}$ such that $T = \lambda_f, U = \lambda_g$ and $V = \lambda_h$ where $f(x) = cln(\alpha\beta x), g(x) = cln(\alpha x)$ and $h(x) = cln(\beta x)$ for all $x \in I$.*

Proof. Proposition (2.6) and equations (14) and (15) imply that

$$E_2^*(\Omega)R^*[DT] = D_1(R^*[T]) = D_1E_1^*[U] = E_1^*[DU] = P[DU;1]$$
$$E_1^*(\Omega)R^*[DT] = D_2(R^*[T]) = D_2E_2^*[V] = E_2^*[DV] = P[1;DV].$$

This will in turn imply that

$$E_1^*(\Omega)P[DU;1] = E_2^*(\Omega)P[1;DV].$$

Hence, $P[\Omega DU;1] = P[1;\Omega DV]$. From this we obtain the system $\Omega DU = c$ and $\Omega DV = c$, for come $c \in \mathbf{R}$. The solution of this system is

$$U = cln(\alpha\Omega) \quad \text{and} \quad V = cln(\beta\Omega), \quad \alpha, \beta \in R.$$

Now, for any $\phi \in \mathcal{D}(I^2)$, we have

$$\begin{aligned}\langle R^*[T], \phi \rangle &= \langle E_1^*[cln(\alpha\Omega)] + E_2^*[cln(\beta\Omega)], \phi \rangle \\ &= \langle cln(\alpha x), E_1[\phi](x) \rangle + \langle cln(\beta y), E_2[\phi](y) \rangle \\ &= \int_{I^2} cln(\alpha x)\phi(x,y)dydx + \int_{I^2} cln(\beta y)\phi(x,y)dxdy \\ &= \int_{I^2} cln(\alpha\beta xy)\phi(x,y)dxdy = \langle cln(\alpha\beta xy), \phi(x,y) \rangle\end{aligned}$$

while

$$\begin{aligned}\langle R^*[T],\phi\rangle &= \langle T(x), R[\phi](x)\rangle\\ &= \left\langle T(x), \int \frac{\phi(x/y,y)}{y}\,dy\right\rangle\\ &= \int_{I^2} T(x)\frac{\phi(x/y,y)}{y}\,dydx = \int_{I^2} T(xy)\phi(x,y)dydx\\ &= \langle T(xy),\phi(x,y)\rangle.\end{aligned}$$

Hence $T(xy) = cln(\alpha\beta xy)$. Equivalently, $T = cln(\alpha\beta\Omega)$.

COROLLARY 3.3. *If f, g, and h belonging to $L_{LOC}(I)$ satisfy equation* (31), *then $f(x), g(x)$ and $h(x)$ belong to $\mathcal{E}(I)$ with*

$$f(x) = cln(\alpha\beta x) \quad g(x) = cln\alpha x \quad \text{and} \quad h(x) = cln\beta x.$$

These are the solutions of the Pexider equation (7).

Finally, we have

THEOREM 3.4. *If $T, U, V \in \mathcal{D}'(I)$ satisfy equation* (32) $R^*[T] = P[U;V]$, *then there exist $a, b, c \in R$ such that $T = \lambda_f, U = \lambda_g, V = \lambda_h$ where $f(x) = abx^c, g(x) = ax^c$ and $h(x) = bx^c$ for all $x \in I$.*

Proof. From Proposition (2.6) and equations (16) we obtain

$$\begin{aligned}E_2^*(\Omega)R^*[DT] &= D_1(R^*[T]) = D_1(P[U;V]) = P[DU;V]\\ E_1^*(\Omega)R^*[DT] &= D_2(R^*[T]) = D_2P[U;V] = P[U;DV]\end{aligned}$$

and this implies that

$$E_1^*(\Omega)P[DU;V] = E_2^*(\Omega)P[U;DV].$$

Thus $P[\Omega DU;V] = P[U;\Omega DV]$ which yields the system $\Omega DU = cU$ and $\Omega DV = cV, c \in \mathbf{R}$. The solution of this system is $U = a\Omega^c$ and $V = b\Omega^c, a, b \in \mathbf{R}$. Now, for any $\phi \in \mathcal{D}(I^2)$,

$$\begin{aligned}\langle R^*[T],\phi\rangle &= \left\langle T(x), \int_I \frac{\phi(x/y,y)}{y}\,dy\right\rangle\\ &= \int_{I^2} T(x)\frac{\phi(x/y,y)}{y}\,dydx\\ &= \int_{I^2} T(xy)\phi(x,y)dydx\\ &= \langle T(xy),\phi(x,y)\rangle,\end{aligned}$$

and on the other hand,

$$\begin{aligned}\langle P[U;V],\phi\rangle &= \langle U(x),\langle V(y),\phi(x,y)\rangle\rangle\\ &= \langle ax^c,\langle by^c,\phi(x,y)\rangle\rangle\\ &= \int_{I^2} abx^c y^c \phi(x,y)dydx\\ &= \langle ab(xy)^c,\phi(x,y)\rangle.\end{aligned}$$

Hence $T(xy) = ab(xy)^c$. Equivalently $T = ab\Omega^c$.

COROLLARY 3.4. *If f, g and h are locally integrable functions satisfying equation* (32), *then $f(x), g(x)$ and $h(x)$ belong to $\mathcal{E}(I)$ with*

$$f(x) = abx^c, g(x) = ax^c \quad \text{and} \quad h(x) = bx^c, \quad a, b, c \in \mathbf{R}.$$

This is the classical solution of the Pexider equation (8).

Acknowledgement. The authors thank a referee for his suggestions to improve the presentation and wording of several results in this paper as well as supply the proof to Corollary 3.1.

REFERENCES

1. J. Aczel, *Lectures on functional equations and their applications* (Academic Press, New York, 1966).
2. I. Fenyö, *On the general solution of a functional equation in the domain of distributions*, Aequationes Math. *3* (1969), 236–246.
3. E. L. Koh, *The Cauchy functional equations in distributions*, Proc. Amer. Math. Soc. *106* (1989), 641–647.
4. M. Neagu, *About the Pompeiu equation in distributions*, Inst. Politehn. "Traran Vaia" Timis. Sem. Mat. Fiz. (1984), 62–66.
5. L. Schwartz, *Theorie des distributions* (Hermann, Paris, 1966).
6. H. Swiatak, *On the regularity of the distributional and continuous solutions of the functional equations* $\sum_{i=1}^{k} a_i(x,t)f(x+\phi_i(t)) = b(x,t)$, Aequationes Math. *1* (1968), 6–19.

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