Can. J. Math., Vol. XXXIII, No. 6, 1981, pp. 1338–1355

VALUATION RINGS AND RIGID ELEMENTS IN FIELDS

ROGER WARE

1. Introduction. In [**20**], T. A. Springer proved that if A is a complete discrete valuation ring with field of fractions F, residue class field of characteristic not 2, and uniformizing parameter π then any anisotropic quadratic form q over F has a unique decomposition as $q = q_1 \perp \langle \pi \rangle q_2$, where q_1 and q_2 represent only units of A, modulo squares in F (compare [**14**, Satz 12.2.2], [**19**, §4], [**18**, Theorem 8.9]). Consequently the binary quadratic form $x^2 + \pi y^2$ represents only elements in $\dot{F}^2 \cup \pi \dot{F}^2$, where $\dot{F}^2$ denotes the set of nonzero squares in F. Szymiczek [**21**] has called a nonzero element a in a field F *rigid* if the binary quadratic form $x^2 + ay^2$ represents only elements in $\dot{F}^2 \cup a\dot{F}^2$. It is fairly easy to show (see Example 2.2 (iii)) that if F is any field and A is a valuation ring of F with maximal ideal M such that $1 + M \subset F^2$ then any element in F whose value is not divisible by 2 must be rigid. One of the objects of this paper is to show that in "most" cases rigid elements in fields arise in this way; that is, if a and $-a$ are both rigid (and there exists a non rigid element in $F \backslash \pm F^2$) then there is a "2-henselian" valuation v on F (see Section 4) such that $v(a)$ is not divisible by 2.

More generally, we will consider T-rigid elements where T is a multiplicative subgroup of $\dot{F} = F \backslash \{0\}$ containing $\dot{F}^2$ (see Definition 2.1) and investigate the connection between such elements and the existence of valuation rings of F with $1 + M \subset T$. The method used to construct such valuation rings is based on an idea of Bill Jacob, who was dealing with formally real fields. We have been able to extend and modify his construction to include non real fields, as well. As applications of these investigations we obtain new proofs of [**6**, Theorem 2.7] and of [**4**, Theorem 5] and [**22**, Theorem 2] (compare [**3**, Corollary 3.2]) as well as a valuation theoretic characterization (Theorem 4.4 (4)) of the "fields of class C" studied in [**24**], [**25**].

For the definition of the Witt ring, $W(F)$, of symmetric bilinear forms over a field F (possibly of characteristic 2) the reader is referred to [**14**] or [**16**]. This will only be needed for the statements and proofs of Theorems 4.4 and 4.8 in Section 4. The term *preordering* will be used to describe a subgroup of $\dot{F}$ which contains $\dot{F}^2$, is closed under addition, and

Received May 27, 1980 and in revised form July 28, 1980. Support from the National Science Foundation is gratefully acknowledged.

does not contain -1. In [**18**], preorderings are called pre-positive cones of F and in [**1**] they are called pre-orderings of level one.

I would like to thank Joe Yucas, Alex Rosenberg, and Teruo Kanzaki for patiently reading this paper and making numerous helpful suggestions.

2. Rigid elements and valuation rings. Throughout this paper, F will be an arbitrary (commutative) field, $\dot{F}$ its multiplicative group, and T a subgroup of $\dot{F}$ containing $\dot{F}^2$. If A is a valuation ring of F then M, U, k, Γ will denote, respectively, its maximal ideal, group of units, residue class field, and value group.

Definition 2.1. An element x in $\dot{F}$ is called *T-rigid* if $T + xT \subset T \cup xT$.

Note that, because T is a group, x is T-rigid if and only if $1 + xT \subset T \cup xT$. The term "rigid" was introduced by K. Szymiczek in [**21**] and was subsequently used in [**3**], [**4**] (with $T = \dot{F}^2$).

Examples 2.2. (i) ([**2**, Satz 20]). A preordering T is a *fan* if and only if every element $x \in \dot{F}\backslash(-T)$ is T-rigid.

(ii) Let A be a valuation ring of F (with maximal ideal M and group of units U). If T is any subgroup of $\dot{F}$ containing $1 + M$ then any element in $\dot{F}\backslash UT$ is T-rigid.

Note. If T is a preordering then $1 + M \subset T$ means that the preordering T is "compatible" with the valuation ring A (see, for example, [**1**], [**6**], [**18**]). If $T = \dot{F}^2$ and A is nondyadic then $1 + M \subset T$ means that A is 2-henselian in the sense of [**1**] or [**6**]. (See Section 4.)

(iii) Let A be a valuation ring and let $T = (1 + M)\dot{F}^2$. Then T is the smallest subgroup of $\dot{F}$ containing $\dot{F}^2$ with which A is "compatible". Moreover, $UT = U\dot{F}^2$ whence any element in F whose value in Γ is not divisible by 2 is T-rigid.

(iv) Let F be a formally real field, let $T = \Sigma\dot{F}^2$ the set of nonzero sums of squares in F, let X_F be the Boolean space of orderings on F (see [**16**, pp. 63–65], [**18**, § 6], or [**15**, § 3]), and for a in $\dot{F}$ let $H(a) = \{P \in X_F | a \in P\}$ be the associated Harrison (subbasic) open set. Then an element $x \notin T$ is T-rigid if and only if $H(x)$ is a maximal element in the set $\{H(a) | a \notin T\}$. Thus, if $x \notin \pm T$ then x and $-x$ are both T-rigid if and only if $H(x)$ is both maximal and minimal in the set $\{H(a) | a \notin \pm T\}$.

Proof of (ii). Let $x \in \dot{F}\backslash UT$ and let $t \in T$. Then xt is not a unit in A so either xt or $(xt)^{-1}$ lies in M. If $xt \in M$ then $1 + xt \in 1 + M \subset T$ while if $(xt)^{-1} \in M$ then $1 + xt \in xT$. Example 2.2 (iii) follows from this.

Proof of (iv). It follows from Artin's characterization of totally positive elements in a field (see [**16**, Exercise 2.3, p. 61] or [**18**, Corollary

1.9, p. 6]) that $H(x) = H(y)$ if and only if $xT = yT$. Moreover, from Pfister's generalization of Artin's theorem, [**17**, Satz 21] (compare [**15**, Theorem 4.8]), it follows that $H(x) \subset H(y)$ if and only if $y \in T + xT$. Consequently, $H(x)$ is a maximal element in the set $\{H(a)|a \notin T\}$ if and only if $x \notin T$ and x is T-rigid.

Definition 2.3. (cf. [**3**], [**4**]). An element b of $\dot{F}$ is called *T-basic* if either $b \in \pm T$ or b and $-b$ are not both T-rigid. The set of T-basic elements will be denoted by $B(T)$.

PROPOSITION 2.4. (cf. [**4**, Theorem 1]). *Let $\hat{T} = T \cup \{0\}$. If for every x in $\dot{F}$ the set $D_T(x) = \dot{F} \cap (\hat{T} + x\hat{T})$ is a group, then $B(T)$ is a group.*

Proof (following [**4**]). First observe that if a is T-rigid then so is ta for any t in T, whence $\pm TB(T) \subset B(T)$. Suppose there exist x, y in $B(T)$ with xy not in $B(T)$. Then xy is T-rigid and we may assume that x and y are not T-rigid and do not lie in $\pm T$. If $a \in D_T(x) \cap D_T(y)$ and $a \notin T$ then $-x, -y \in D_T(-a)$ whence (by assumption) $xy \in D_T(-a)$. Then $a \in D_T(-xy)$ (because $xy \notin T$) and, since $T \subset D_T(-xy)$, it follows that $D_T(x) \cap D_T(y) \subset D_T(-xy)$. Moreover, $-y \notin T \cup xyT = D_T(xy)$ so $-xy \notin D_T(y)$. Thus $-xy \notin D_T(x) \cap D_T(y)$ and so $T = D_T(x) \cap D_T(y)$. Now

$$\begin{aligned} D_T(x)D_T(y) &\subset [(\hat{T} + xy\hat{T}) + x(\hat{T} + xy\hat{T})] \cap \dot{F} \\ &= [(\hat{T} \cup xy\hat{T}) + x(\hat{T} \cup xy\hat{T})] \cap \dot{F} \\ &= D_T(x)(T \cup yT) \cup D_T(y)(T \cup xT). \end{aligned}$$

Since $D_T(x)D_T(y)$ is a group and no group can be a union of 2 proper subgroups we may assume, by symmetry, that

$$D_T(x)D_T(y) \subset D_T(x)(T \cup yT).$$

Then, for any a in $D_T(y)$ we have $a \in D_T(x) \cup yD_T(x)$. If $a \in D_T(x)$ then

$$a \in D_T(x) \cap D_T(y) = T$$

and if $a \in yD_T(x)$ then

$$ay^{-1} \in D_T(x) \cap D_T(y) = T.$$

Hence y is T-rigid, contrary to assumption.

Examples 2.5. (i) (Cordes, Berman). If $T = \dot{F}^2$ then it is well known that the set $D_T(x)$ is a group for all x in F. Hence $B(\dot{F}^2)$ is a group. In [**3**] and [**4**], $B(F^2)$ is denoted $A(F)$.

(ii) Let T be a preordering of F. Then $D_T(x)$ is clearly a group for any x in F and hence $B(T)$ is a group.

LEMMA 2.6 (cf. [**4**, Proposition 1]). *Let H be a subgroup of $\dot{F}$ containing $B(T)$. If $-y \notin T$ then $T + yT \subset H \cup yT$.*

Proof. If there exist $t_1, t_2 \in T$ with $t_1 + yt_2 \notin H$ then

$$T + (-t_1 - yt_2)T \subset T \cup (-t_1 - yt_2)T$$

and so

$$-yt_2 = t_1 - t_1 - yt_2 \in T \cup (-t_1 - yt_2)T.$$

Since $-y \notin T$, it follows that $-yt_2 \in (-t_1 - yt_2)T$, i.e., $t_1 + yt_2 \in yT$.

Definition 2.7. (compare [**12**, § 1], [**13**, § 1]). For a subgroup H of F we define

$$O_1 = O_1(H, T) = \{x \in F | x \notin H \text{ and } 1 + x \in T\},$$
$$O_2 = O_2(H, T) = \{x \in H | xO_1 \subset O_1\}, \text{ and}$$
$$O(H, T) = O_1 \cup O_2.$$

PROPOSITION 2.8. *Let H be a subgroup of $\dot{F}$ which contains $B(T)$. If $A = O(H, T)$ is a valuation of F then*

(1) $UT \subset H$

(2) $O_1 \subset M$

(3) *The following are equivalent:*

 (a) $A = F$

 (b) $H = \dot{F}$

 (c) *The value group Γ is 2-divisible (i.e., $\Gamma = \Gamma^2$).*

(4) $1 + M \subset T$

(5) *If $-1 \notin T$ then A is nondyadic.*

(6) *If for each t in T there exists $x \in F$ with $1 - tx^2 \notin H$ then $T = (1 + M)\dot{F}^2$.*

Proof. (1). It suffices to show $U \subset H$. If $x \notin H$ then x is T-rigid so either $1 + x \in T$ or $1 + x \in xT$. If $1 + x \in xT$ then $x \notin A$ so $x \notin U$. If $1 + x \in T$ then $1 + x^{-1} \in x^{-1}T = xT$. But then $x^{-1} \notin A$ so $x^{-1} \notin U$ and $x \notin U$.

(2). Assume $x \notin M$. If $x \in U$ then $x \in H$ so $x \notin O_1$ and if $x \notin U$ then $x \notin A$ so $x \notin O_1$.

(3). The implication $(a) \Rightarrow (c)$ is obvious.

$(c) \Rightarrow (b)$: if Γ is 2-divisible, then $U\dot{F}^2 = \dot{F}$. But $U\dot{F}^2 \subset UT \subset H$ so this forces $\dot{F} = H$.

$(b) \Rightarrow (a)$. If $H = \dot{F}$ then $O_1 = \{0\}$, $O_2 = \dot{F}$, and $A = F$.

(4). Let $x \in M$. If $x \in O_1$ then $1 + x \in T$ so we may assume that $x \notin O_1$. Then $x \in O_2 \subset H$. Moreover, $x^{-1} \notin A$ so there exists y in O_1 such that $1 + x^{-1}y \notin T$. Since $x^{-1} \in H$ and y is not in H, we have $x^{-1}y \notin H$. Then $x^{-1}y$ is T-rigid and hence $1 + x^{-1}y \in x^{-1}yT$. Also, $T \subset H$ implies that $x^{-1}yT \cap H$ is empty. Hence $1 + x^{-1}y \notin H$. Since $-1 \in H$, it follows that $-y \notin H$ and so $-y \in O_1$ (because A is a ring). Thus $1 - y \in T$ and

$$1 + x^{-1}y = 1 + x^{-1} - x^{-1} + x^{-1}y \in (1 + x^{-1})T + (-x^{-1})T.$$

Since $x \in M$, $1 + x \in U \subset H$ and

$$1 + x^{-1} = x^{-1}(1 + x) \in H.$$

By Lemma 2.6, we must have

$$-(1 + x^{-1})(-x^{-1}) \in T$$

whence $1 + x \in x^2T \subset T$.

(5). If $2 \in M$ then $-1 = 1 - 2 \in 1 + M \subset T$.

(6). Let $t \in T$. Then there exists an x in F with $tx^2 - 1 \notin H$. Then $tx^2 - 1 \in O_1 \subset M$ whence $tx^2 \in 1 + M$ and $t \in (1 + M)\dot{F}^2$. The inclusion $(1 + M)\dot{F}^2 \subset T$ follows from (4).

The following two examples show that it can happen that $T \neq (1 + M)\dot{F}^2$ and $H \neq U\dot{F}^2$ in this construction.

Examples 2.9. (i) Let $F = \mathbf{Q}((x))$, the field of formal Laurent series in one variable over the field of rational numbers, and let $T = \Sigma\dot{F}^2$ be the group of all non zero sums of squares in F. The ring $\mathbf{Q}[[x]]$ is easily seen to be the smallest valuation ring in F which is compatible with T; indeed in the notation of [**1**], $\mathbf{Q}[[x]] = A(P_1) = A(P_2) = A_T$ where P_1 and P_2 are the two orderings on F. Since $\mathbf{Q}[[x]]$ is a rank one valuation ring, it is the only proper valuation ring of F which is compatible with T. Moreover, T is a fan and it is easy to check that for $a \notin \pm T$, $1 + a \in T$ implies $1 - a \in T$ (compare Proposition 3.7, Section 3). Hence by Theorem 2.12 (to be proved), $A = O(\pm T, T)$ is a proper valuation ring of F which is, by Proposition 2.8 (4), compatible with T. Hence $O(\pm T, T) = \mathbf{Q}[[x]]$. But if M is the maximal ideal of $\mathbf{Q}[[x]]$ then $1 + M \subset \dot{F}^2$ whence $(1 + M)\dot{F}^2 = \dot{F}^2 \neq T$.

Note that for this field we have $B(\dot{F}^2) = \pm T (= \pm\Sigma\dot{F}^2)$ and by Proposition 2.13 (i), $O(\pm T, \dot{F}^2)$ is a valuation ring so that we also have $\mathbf{Q}[[x]] = O(\pm T, \dot{F}^2)$.

(ii) Let $F = \mathbf{Q}((x))((y))$ be the field of iterated formal Laurent series in two independent variables over the field of rational numbers and let $T = \dot{F}^2$. Then $B(T) = \dot{\mathbf{Q}}\dot{F}^2$. Let H be any subgroup of $\dot{F}$ with $B(T) \subset H$, $y \in H$, $x \notin H$, and $(\dot{F} : H) = 2$. Since $B(T) \neq \pm T$ (i.e., $\dot{\mathbf{Q}}\dot{F}^2 \neq \pm\dot{F}^2$) it follows from Proposition 2.13 that $A = O(H, \dot{F}^2)$ is a valuation ring of F. Moreover, $x \in O_1$, $1 + x + y \in \dot{F}^2$, and $1 + x^{-1}y \in \dot{F}^2$, i.e., $x + y \in x\dot{F}^2$. Hence $x + y \in O_1$, as well. Since $O_1 \subset M$ it follows that $y \in M$. We assert that $y \notin U\dot{F}^2$ (and so $H \neq U\dot{F}^2$).

If $a^2y \in U$ for some a in F then

$$x + a^2y = a^2y(1 + (a^2y)^{-1}x) \in a^2y(1 + M) \subset a^2y\dot{F}^2 = y\dot{F}^2.$$

Now x is a unit in the complete local ring $Q((x))[[y]]$, so if $a \in \mathbf{Q}((x))[[y]]$, then $x + a^2y \in x\dot{F}^2$. Hence we can write

$$a = \sum_{i=-n}^{\infty} r_i y^i \quad \text{with } r_i \in \mathbf{Q}((x)) \quad \text{and} \quad n > 0.$$

Then $a^2 = r_{-n}{}^2 y^{-2n}$ + terms of higher degree, whence

$$1 + a^2 y(x + y) = 1 + \sum_{i=1}^{\infty} s_i y^{-2n+i},$$

where $s_1 = xr_{-n}^2$ and $s_i \in \mathbf{Q}((x))$, $i > 1$. Thus

$$y^{2n}(1 + a^2 y(x + y)) = y^{2n} + \sum_{i=1}^{\infty} s_i y^i = y\left(s_1 + y^{2n-1} + \sum_{i=2}^{\infty} s_i y^{i-1}\right).$$

Since $b = y^{2n-1} + \sum_{i=2}^{\infty} s_i y^{i-1}$ lies in the maximal ideal of $\mathbf{Q}((x))[[y]]$ and s_1 is a unit, $s_1 + b \in s_1\dot{F}^2$, whence

$$1 + a^2y(x + y) \in ys_1\dot{F}^2 \neq \dot{F}^2.$$

Because $x + y \in O_1$, this means that a^2y is not in O_2. By Proposition 2.8 (2), $U \subset O_2$ so $a^2y \notin U$. Thus $y \in H\backslash UF^2$ and $H \neq U\dot{F}^2$.

LEMMA 2.10. *Let T be a subgroup of $\dot{F}$ and let a, x be elements of $\dot{F}$ such that $a \notin T$ and x, ax are T-rigid. If $1 + a$, $1 - x \in T$ then $1 + ax \in T$.*

Proof. $1 + ax = 1 - x + x + ax \in (1 - x)T + x(1 + a)T \subset T + xT \subset T \cup xT$. But $1 + ax \in T + axT \subset T \cup axT$. Since $a \notin T$, $xT \cap axT$ is empty, whence $1 + ax \in T$.

LEMMA 2.11. *Let H be a subgroup of $\dot{F}$ containing $B(T)$. Assume that for all $x \notin H$, $1 + x \in T$ implies $1 - x \in T$ (i.e., $-O_1(H, T) \subset O_1(H, T)$). Then, for any $x \notin H$, $2 + x \in H$ implies $1 + x \in T$.*

Proof. Let $x \notin H$. If $1 + x \in T$ then $1 + x \in xT$. Hence $1 + x \notin H$ and $1 - x \in (-x)T$. Then $1 - (1 + x) \in (-x)T$ and so $2 + x = 1 + (1 + x) \notin T$. Since $1 + x$ is T-rigid, $2 + x$ lies in $(1 + x)T$ and because $(1 + x)T \cap H$ is empty, $2 + x \notin H$.

THEOREM 2.12 (cf. [**12**, Proposition 1]). *Let H be a subgroup of $\dot{F}$ containing $B(T)$. Suppose that*

(1) $-O_1(H, T) \subset O_1(H, T)$.

(2) *For each x in O_1 there exists $a = a(x)$ in $F\backslash T$ such that $1 - a \in H$, $a^{-1}x \in O_1$, and if $y \in O_1$, with $y \notin aT$ then $ay \in O_1$.*

Then $A = O(H, T)$ is a valuation ring of F.

Note that if $-1 \notin T$ and (1) holds then we can take $a = -1$ in (2).

Proof. We first show that A is closed under multiplication. To do this it suffices to show

M1. If $x, y \in O_1$ and $xy \notin H$ then $xy \in O_1$;

and

M2. If $x, y \in O_1$ and $xy \in H$ then $xy \in O_2$.

M1 follows from assumption (1) and Lemma 2.10.

M2: Let $z \in O_1$. Now

$$1 + xy = 1 - x + x(1 + y) \in (1 - x)T + x(1 + y)T \subset T + xT \subset T \cup xT$$

and similarly, $1 + xy \in T \cup yT$. Thus, if $xT \neq yT$ it follows that $1 + xy \in T$ and, in addition,

$$1 + xyz = 1 + xy - xy(1 - z) \in (1 + xy)T + (-xy)(1 - z)T \subset T + (-xy)T \subset H,$$

by Lemma 2.6. Now $xyz \notin H$ so, if $xT \neq yT$ then $1 + xyz \in T$, by Lemma 2.10, and so $xy \in O_2$. $yT \neq zT$ then $1 + xyz \in T$. Thus we may assume that $xT = yT = zT$. In this case, use assumption (2) and multiply x by $a^{-1} = a(x)^{-1}$ to obtain x' in O_1 with $x'T \neq yT = zT$. Thus $x'yzT = x'T$ so $x'yz \notin H$. Also, if $x'y \in H$ then the above argument shows that $1 + x'yz \in T$ while if $x'y \notin H$ we get

$$1 + x'yz \in (T \cup x'T) \cap (T \cup (-x'y)T).$$

Since $x'T \cap -x'yT$ is empty we have $1 + x'yz \in T$ in either case. Hence $x'yz \in O_1$. Since $xyz \notin T$, $x'yz \notin aT$ and $xyz = a(x'yz) \in O_1$. Thus $xy \in O_2$.

To see that A is closed under addition we must prove

A1. If $x, y \in O_1$ and $x + y \notin H$ then $x + y \in O_1$.

A2. If $x, y \in O_2$ and $x + y \in H$ then $x + y \in O_2$.

A3. If $x, y \in O_2$ and $x + y \notin H$ then $x + y \in O_1$.

A4. If $x, y \in O_1$ and $x + y \in H$ then $x + y \in O_2$.

A5. If $x \in O_1$, $y \in O_2$, and $x + y \in H$ then $x + y \in O_2$.

A6. If $x \in O_1$, $y \in O_2$, and $x + y \notin H$ then $x + y \in O_1$.

A1: First assume $-1 \notin T$. In this case,

$$2 + x + y \in (1 + x)T + (1 + y)T \subset T + T \subset H.$$

Since $x + y \notin H$ and $-O_1 \subset O_1$ it follows from Lemma 2.11 that $x + y \in O_1$.

Now suppose $-1 \in T$. Since

$$1 + x + y \in (1 + x)T + yT = T + yT \subset T \cup yT$$

and (similarly) $1 + x + y \in T \cup xT$ it follows that, if $xT \neq yT$ then $1 + x + y \in T$. Thus we assume that $xT = yT$. Choose $a = a(x) \in F$ as in assumption (2). Then $a^{-1}x$, $ax \in O_1$ and $axT = a^{-1}xT \neq yT$ so

$1 + a^{-1}x + y$ and $1 + ax + y$ lie in T. Thus

$$1 + x + y = 1 + a^{-1}x + y + (1 - a^{-1})x = 1 + ax + y + (1 - a)x$$

lies in $(T \cup xT) \cap (T + (1 - a^{-1})xT) \cap (T + (1 - a)xT)$. Since $1 - a \in H$, $(1 - a)x$ is T-rigid. Also, $(1 - a^{-1})x = (1 - a)(-a^{-1}x)$ and $-a^{-1}x \notin H$ so $(1 - a^{-1})x$ is T-rigid. Hence

$$T + (1 - b)xT = T \cup (1 - b)xT \text{ for } b = a \text{ or } a^{-1}.$$

If $xT = (1 - a)xT = (1 - a^{-1})xT$ then $T = (1 - a)T$ and $aT = (a - 1)T$. But $-1 \in T$, forcing $T = aT$, i.e., $a \in T$, a contradiction. Hence

$$(T \cup xT) \cap (T \cup (1 - a^{-1})xT) \cap (T \cup (1 - a)xT) = T$$

and $1 + x + y$ lies in T.

A2: Let $z \in O_1$. Then $xz, yz \in O_1$. Also, $xz + yz = (x + y)z \notin H$. By A1, $(x + y)z \in O_1$, i.e., $x + y \in O_2$.

A3: Since $x, y \in H$ and $x + y \notin H$, Lemma 2.6 implies that $-xy \in T$. Moreover, $x^{-1} \in H$ so $1 + x^{-1}y = x^{-1}(x + y) \notin H$. Since $1 - (1 + x^{-1}y) = -x^{-1}y \in -xyT = T$, it follows from assumption (1) that $1 + x^{-1}y \in O_1$. Hence $x + y = x(1 + x^{-1}y) \in O_1$.

A4: If $xy \notin H$ then xy is T-rigid which implies that

$$x + y \in x(T + xyT) \subset xT \cup yT.$$

Since $x \notin H$ and $y \notin H$, $H \cap (xT \cup yT)$ is empty. Hence $x + y \in H$ implies $xy \in H$. Moreover,

$$x = (x + y)\left(\frac{x}{x + y}\right)^2 + xy(x + y)\left(\frac{1}{x + y}\right)^2 \in (x + y)T + (xy)(x + y)T,$$

so since $x \notin H$, Lemma 2.6 forces $-xy \in T$. Now let $z \in O_1$. If $xz \in H$ then $yz \in H$ in which case, by M2, xz and yz lie in O_2. Then, because $(x + y)z \notin H$, A3 implies that $(x + y)z \in O_1$. If $xz \notin H$ then $yz \notin H$ and by M1, xz and yz lie in O_1. By A1, $(x + y)z \in O_1$ and $x + y \in O_2$.

A5: Let $z \in O_1$. Then $(x + y)z \notin H$ and $yz \in O_1$. If $xz \notin -T$ then

$$1 + xz + yz \in xzT + T \subset H \cup xzT.$$

But $1 + xz + yz \in T \cup (xz + yz)T$ and, because $x \notin H$ and $x + y \in H$, $(xz + yz)T \cap (H \cup xzT)$ is empty. Hence, in this case, $1 + xz + yz \in T$.

Now suppose $xz \in -T$. Multiply z by $a^{-1} = a(z)^{-1}$, as in assumption (2), to obtain z' in O_1 with $xz' \notin -T$. Then $(x + y)z' \notin H$ and $1 + (x + y)z' \in T$, i.e., $(x + y)z' \in O_1$. Since $(x + y)z \notin T$, $(x + y)z' \notin aT$ and $(x + y)z = a(x + y)z' \in O_1$. Thus $x + y \in O_1$.

A6: If $y \notin -T$ then

$$1 + x + y \in (1 + x)T + yT \subset H \cup yT.$$

Since $x + y \notin H$ this implies $1 + x + y \in T$. If $y \in -T$, choose $a = a(x)$ (as in (2)) and let $x' = a^{-1}x \in O_1$ and $y' = a^{-1}y$. Then $y' \notin -T$ and

$$1 + x' + y' \in (1 + x')T + y'T = T + y'T \subset H \cup y'T.$$

Now $(x' + y')T \neq y'T$ so if $x' + y' \notin H$ then $1 + x' + y' \in T$. If $x' + y' \in H$ then $a^{-1} \notin H$, whence $y' \notin H$. But $x'y' = a^{-2}xy \notin H$, whence

$$x' + y' \in x'(T + x'y'T) \subset x'(T \cup x'y'T) = x'T \cup y'T.$$

Thus $x' + y' \notin H$ and therefore $1 + x' + y' \in T$, i.e., $x' + y' \in O_1$. Now $x + y \notin T$ implies that $x' + y' \notin aT$. Hence $x + y = a(x' + y') \in O_1$, as desired.

Thus A is a subring of F. To show A is a valuation ring of F we must prove

V1. If $x \notin H$ then $x \notin O_1$ if and only if $x^{-1} \in O_1$.

V2. If $x \in H$ and $x \notin O_2$ then $x^{-1} \in O_2$.

Since $x \notin H$ implies $1 + x \in T \cup xT$ and $1 + x^{-1} = x^{-1}(1 + x)$, V1 is clear.

To prove V2, let $x \in H \backslash O_2$. Then there exists y in O_1 with $1 + xy \in xyT$. Then $1 + x^{-1}y^{-1} \in T$ and by (1), $1 - x^{-1}y^{-1} \in T$. Let $z \in O_1$. We must show $x^{-1}z \in O_1$.

First assume that $-1 \notin T$. Since $1 + x^{-1}y^{-1}z$ lies in

$$\begin{aligned}((1 - x^{-1}y^{-1})T + x^{-1}y^{-1}(1 + z)T) \cap ((1 + x^{-1}y^{-1})T \\ + (-x^{-1}y^{-1}(1 - z))T) \subset (T + x^{-1}y^{-1}T) \\ \cap (T + (-x^{-1}y^{-1}T))\end{aligned}$$

and $x^{-1}y^{-1} \notin H$ we have $1 + x^{-1}y^{-1}z \in T$. If $x^{-1}y^{-1}z \notin T$ then by Lemma 2.10, $1 + x^{-1}z \in T$. If $x^{-1}y^{-1}z \in T$, let $z' = -z$. Then $z' \in O_1$, $1 + x^{-1}y^{-1}z' \in T$, and $x^{-1}y^{-1}z' \notin T$. Hence $1 + x^{-1}z' \in T$, i.e., $-x^{-1}z \in O_1$. But then $x^{-1}z \in O_1$, as well.

Now suppose $-1 \in T$. If $x^{-1}y^{-1}z \notin H$ then

$$1 + x^{-1}y^{-1}z \in T \cup x^{-1}y^{-1}zT.$$

On the other hand, we saw above that

$$1 + x^{-1}y^{-1}z \in T \cup x^{-1}y^{-1}T.$$

Since $z \notin T$ this means that if $x^{-1}y^{-1}z \notin H$ then $1 + x^{-1}y^{-1}z \in T$. If $x^{-1}y^{-1}z \in H$ and $x^{-1}y^{-1}z \notin -T = T$ then $1 + x^{-1}y^{-1}z \in H$. Since $x^{-1}y^{-1} \notin H$ this implies $1 + x^{-1}y^{-1}z \in T$. Thus if $x^{-1}y^{-1}z \notin T$ then $1 + x^{-1}y^{-1}z \in T$ and by Lemma 2.10, $x^{-1}z \in O_1$. Thus we assume that

$x^{-1}y^{-1}z \in T$. Choose $a = a(z)$ as in assumption (2) and let $z' = a^{-1}z$. Then $z' \in O_1$ and $x^{-1}y^{-1}z' \notin T$ so by the above argument, $x^{-1}z' \in O_1$. Since $y \notin T$, $x^{-1}z \notin T$ whence $x^{-1}z' \notin aT$. Hence $x^{-1}z = a(x^{-1}z') \in O_1$.

PROPOSITION 2.13. *Let H be a subgroup of $\dot{F}$ containing $B(T)$. Suppose H contains an element a with a, $-a$, $1 + a$, and $1 - a^{-1}$ lying in $\dot{F}\backslash T$. Then the pair (H, T) satisfies assumptions* (1) *and* (2) (*of Theorem* 2.12) *and hence $A = O(H, T)$ is a valuation ring of F.*

If either (i) *H contains a non T-rigid element x with $\pm x \notin T$ or* (ii) *H contains an element x with $\pm x$, $1 - x \notin T$ and $1 + x \in T$, then H contains such an element a.*

Proof. First suppose H contains an element a with $\pm a$, $1 + a$, $1 - a^{-1} \notin T$. By Lemma 2.6, $1 - a \in H$. Thus it suffices to show that for any x in O_1, ax, $-x$, and $a^{-1}x$ are all in O_1. Note that $a \pm 1$, $a^{-1} \pm 1$ are also in H so that ax, $a^{-1}x$, $(a \pm 1)x$, and $(a^{-1} \pm 1)x$ do not lie in H and therefore are T-rigid.

$$ax \in O_1 : 1 + ax = 1 + x + (a - 1)x \in T + (a - 1)xT \subset T \cup (a - 1)xT$$

and

$$1 + ax \in T \cup axT.$$

Since $1 - a^{-1} \notin T$, $(a - 1)xT \cap axT$ is empty and hence $1 + ax \in T$.

$$-x \in O_1 : 1 - x = 1 + ax - ax - x \in T + -x(1 + a)T \subset T \cup -x(1 + a)T$$

and

$$1 - x \in T \cup -xT.$$

Since $1 + a \notin T$, $-x(1 + a)T \cap -xT$ is empty, whence $1 - x \in T$.

$$a^{-1}x \in O_1 : 1 + a^{-1}x = 1 - x + x(1 + a^{-1}) \in T + x(1 + a^{-1})T \subset T \cup x(1 + a^{-1})T$$

and

$$1 + a^{-1}x \in T \cup a^{-1}xT.$$

Since $a + 1 \notin T$, $x(1 + a^{-1})T \cap a^{-1}xT$ is empty and $1 + a^{-1}x \in T$.

To prove the second part of the proposition, suppose H contains a non T-rigid element x with $\pm x \notin T$. Then there must exist y in F with $y \notin T \cup xT$ and $y = 1 + tx$, $t \in T$. Let $a_1 = tx$. Then $\pm a_1 \notin T$ and $1 + a_1 \notin T$. Thus, if $1 - a_1^{-1} \notin T$ then we can take $a = a_1$. If $1 - a_1^{-1} \in T$ take $a = a_1^{-1}$. Then

$$1 - a^{-1} = 1 - a_1 \in -a_1T \neq T.$$

Also, $1 + a = 1 + t^{-1}x^{-1}$ lies in $(t + x^{-1})T = (tx + 1)x^{-1}T = yx^{-1}T \neq T$.

Finally, suppose H contains an element x with $x \notin \pm T$, $1 + x \in T$, $1 - x \notin T$. Let $a = -x$. Then $\pm a \notin T$, $1 + a \notin T$, and $1 - a^{-1} = x^{-1}(x + 1)$ lies in $x^{-1}T \neq T$.

Definition 2.14. We say that a valuation ring A of F is *T-compatible* if $1 + M \subset T$.

LEMMA 2.15. *Let A be a T-compatible valuation ring of F, let $\pi : A \to k$ be the natural map, and let $\bar{T} = \pi(U \cap T)$. Then*

(1) *$\pi^{-1}(\pi(z)\bar{T}) = z(T \cap U)$ for any unit z in A.*

(2) *A unit z is T-rigid if and only if $\pi(z)$ is $\bar{T}$-rigid.*

(3) *$\pi(U \cap B(T)) = B(\pi(U \cap T))$.*

(4) *$k/\bar{T} \cong UT/T$.*

Proof. (1). Let $x \in \pi^{-1}(\pi(z)\bar{T})$. Then there exists t in $U \cap T$ and b in the maximal ideal M of A such that $x = zt + b$. But

$$zt + b = zt(1 + (zt)^{-1}b) \in z(T \cap U)(1 + M) = z(T \cap U).$$

(2). If z is T-rigid then $\pi(z)$ is certainly $\bar{T}$-rigid. Now assume $\pi(z)$ is $\bar{T}$-rigid and let $y = 1 + tz$ with t in T. If t is not a unit in A then either $1 + tz$ or $1 + (tz)^{-1}$ lies in $1 + M \subset T$ whence $y \in T \cup zT$. Thus we can assume t is a unit. Then $y \in A$ and

$$\pi(y) \in \bar{T} + \pi(z)\bar{T} \subset \bar{T} \cup \pi(z)\bar{T},$$

and by (1), $y \in T \cup zT$.

(3) follows from (1) (with $z = \pm 1$) and (2).

(4). The natural map π induces a surjection $U/U \cap T \to k/\bar{T}$ which has trivial kernel by (1). Hence

$$UT/T \cong U/U \cap T \cong k/\bar{T}.$$

THEOREM 2.16. *Assume that $B(T)$ is a group (e.g. $D_T(x)$ is a group for all x in F: see Proposition 2.4). If $B(T) \neq \pm T$ then*

(1) *There is a T-compatible valuation ring A on F with $B(T) = UT$.*

(2) *For $x \in F$, x and $-x$ are both T-rigid if and only if there is a T-compatible valuation ring A on F with $x \notin UT$.*

(3) *If $B(T) \neq \dot{F}$ there is a T-compatible valuation ring on F whose value group is not 2-divisible.*

(4) *If $x \in \dot{F} \backslash B(T)$ then $1 + x \in T$ if and only if $1 - x \in T$.*

Proof. (1). Since $B(T) \neq \pm T$, $A = O(B(T), T)$ is a valuation ring by Proposition 2.13 (i). By Proposition 2.8 (4) and (1), A is T-compatible with $UT \subset B(T)$ and by Example 2.2 (ii), $B(T) \subset UT$.

(2). Assume x, $-x$ are both T-rigid and let $A = O(B(T), T)$. Then $x \notin B(T) = UT$. The converse is Example 2.2 (ii).

(3). Assume $B(T) \neq \dot{F}$ and let $A = O(B(T), T)$. By Proposition 2.8 (4), (3), A is T-compatible with non 2-divisible value group.

(4). Since $O(B(T), T)$ is a ring we must have $O_1 = -O_1$.

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COROLLARY 2.17. *If $B(T) \neq \pm T$ then $B(T)$ is a group if and only if there is a T-compatible valuation on F with group of units U such that $B(T) = UT$.*

Remark 2.18. If $F = \mathbf{Q}_5$, the field of 5-adic numbers and $T = \dot{F}^2$ then $2 \in \dot{F} \backslash B(T)$, $1 - 2 \in T$, and $1 + 2 \notin T$. There is no T-compatible valuation ring A on F with $2 \notin UT$. Of course, $B(T) = T$.

Definition 2.19. The field F is called *T-basic* if $B(T) = \dot{F}$.

PROPOSITION 2.20. *Assume that $B(T) \neq \pm T$ is a group. Let $A = O(B(T), T)$, let $\pi : A \rightarrow k$ be the natural map, and let $\bar{T} = \pi(U \cap T)$. Then k is $\bar{T}$-basic. Moreover, if F is not T-basic (i.e., $A \neq F$) then the value group A is not 2-divisible.*

Proof. Apply Proposition 2.8 (1), (3) and Lemma 2.15 (3).

3. T-rigid fields.

Definition 3.1. We say that F is *T-rigid* (where T is a subgroup of $\dot{F}$ containing $\dot{F}^2$) if $B(T) = \pm T$; that is, if all elements of F not in $\pm T$ are T-rigid. Following [**21**], we call *F rigid* if F is $\dot{F}^2$-rigid.

Remarks 3.2. (i) Rigid fields of characteristic not 2 are the "fields of class C" studied in [**24**], [**25**]. They have also been studied in [**21**]. They were characterized in [**24**] as those fields whose Witt rings are group rings over $\mathbf{Z}$, $\mathbf{Z}/4\mathbf{Z}$, or $\mathbf{F}_2$; in [**25**] it was shown that a field (of characteristic $\neq 2$) is rigid if and only if the 2-part of its absolute Galois group is metabelian (as a topological group). Non formally real rigid fields having a finite number of square classes were first studied by C. Cordes [**8**] who called them $\bar{C}$-fields. These include nondyadic local fields. Formally real rigid fields were first studied in [**9**] and since then in [**1**], [**5**], [**6**], [**7**], [**11**], [**12**], [**21**], [**23**], [**24**]. Bröcker [**5**] called such fields strictly pythagorean while Elman and Lam [**11**] named them superpythagorean.

(ii) Let T be a preordering of F. Then F is T-rigid if and only if T is a fan. Fans were introduced in [**2**] and have since been studied in several papers, including [**1**], [**6**], [**12**], [**13**]. If F is formally real and $T = \Sigma \dot{F}^2$, the set of all nonzero sums of squares in F, then F is T-rigid (i.e., T is a fan) if and only if F is superordered in the sense of [**7**].

THEOREM 3.3. (cf. [**1**, Theorem 14], [**6**, Theorem 2.7], [**7**, Theorem 1], [**12**, Proposition 1]). *Let F be a T-rigid field. Then there is a T-compatible valuation ring A on F with residue field k such that k is $\bar{T}$-rigid and $(\dot{k} : \bar{T}) \leqq 4$. Here $\bar{T}$ is the subgroup of $\dot{k}$ induced by T. Moreover,*

(1) *If for all $x \notin \pm T$, $1 + x \in T$ implies $1 - x \in T$ then $(\dot{k} : \bar{T}) \leqq 2$.*

(2) *Suppose F contains an element $x \notin \pm T$ with $1 + x \in T$ and $1 - x \notin T$.*

https://doi.org/10.4153/CJM-1981-103-0 Published online by Cambridge University Press

(a) *If* $-1 \in T$ *then* $(\dot{k} : \bar{T}) = 2$.

(b) *If* $-1 \notin T$ *then* $(\dot{k} : \bar{T}) = 4$ *and for any* T*-compatible valuation on* F *with residue class* L *we have* $(\dot{L} : \bar{T}) \geqq 4$, *where* $\bar{T}$ *is the subgroup induced by* T *in* L.

Proof. If F carries a T-compatible valuation with residue field k then by Lemma 2.15, F is T-rigid if and only if k is T-rigid. Thus it remains to show that F has a T-compatible valuation whose residue field k satisfies statements (1) and (2).

(1). Assume that for all $x \notin \pm T$, $1 + x \in T$ implies $1 - x \in T$. If $-1 \notin T$ then for any subgroup H containing $B(T) = \pm T$, the hypotheses of Theorem 2.12 are satisfied and so $A = O(\pm T, T)$ is a T-compatible valuation ring with $UT = \pm T$. Then $(\dot{k} : \bar{T}) = (UT : T) = (\pm T : T) = 2$.

Now suppose $-1 \in T$. If assumption (2) of Theorem 2.12 is satisfied for the pair (T, T), then as above we obtain a T-compatible valuation with residue field k such that $(\dot{k} : \bar{T}) = 1$. If assumption (2) (Theorem 2.12) is not satisfied then there exists an element $x_0 \notin T$ with $1 + x_0 \in T$ such that for all $a \notin T \cup x_0T$ with $1 - a \in T$ we have $1 + a^{-1}x_0 \in a^{-1}x_0T$. Note that if $y \notin T \cup aT$ and $1 + y \in T$ then ay is T-rigid and $1 + ay \in T$ by Lemma 2.10, so the last part of assumption (2) is automatically satisfied. Let $H = T \cup x_0T$ and let $O_1 = O_1(H, T)$. Then $1 - x_0 \in T \subset H$. If $a \in O_1$ then $a \notin T \cup x_0T$ and $1 - a \in T$ so $1 + a^{-1}x_0 \in a^{-1}x_0T$. Hence $1 + x_0^{-1}a \in T$ and $x_0^{-1}a \in O_1$. If $y \in O_1$ then $x_0y \notin H$, whence x_0y is T-rigid and by Lemma 2.10 (together with the assumption $1 + x \in T \Rightarrow 1 - x \in T$, for $x \notin T$) it follows that $x_0y \in O_1$. Hence assumptions (1) and (2) of Theorem 2.12 are fulfilled by the pair (H, T) and $A = O(H, T)$ is a T-compatible valuation ring of F whose residue field k satisfies $(\dot{k} : \bar{T}) = (H : T) = 2$.

(2). Assume F contains an element $x \notin \pm T$ with $1 + x \in T$ and $1 - x \notin T$. Let $H = (\pm T) \cup (\pm xT)$. By Proposition 2.13, $A = O(H, T)$ is a (T-compatible) valuation ring of F (with $UT \subset H$). Moreover neither x (because $1 - x \notin T$) nor x^{-1} (because $1 + x^{-1} \notin T$) can be in the maximal ideal of A so x is a unit. Thus $UT = (\pm T) \cup (\pm xT)$ and $(\dot{k} : \bar{T}) = (UT : T) \leqq 4$. If $-1 \in T$ then $UT = T \cup xT$ and $(\dot{k} : \bar{T}) = 2$. Finally, if $-1 \notin T$, let A' be any T-compatible valuation ring on F with residue class field L and group of units U'. As above, $x \in U'$ whence

$$(\dot{L} : \bar{T}) = (U'T : T) \geqq (\pm T \cup \pm xT : T) = 4.$$

Remark 3.4. If T is a preordering then F is T-rigid if and only if T is a fan and a fan T is trivial if and only if $(\dot{F} : T) \leqq 4$. Thus we obtain the theorem of L. Bröcker [**6**, 2.7] which states that if T is a fan then there exists a compatible valuation ring such that the induced fan $\bar{T}$ is trivial.

COROLLARY 3.5. *Suppose* $-1 \notin T$. *Then the following statements are equivalent:*

(1) *F is T-rigid and for all* $x \notin \pm T$, $1 + x \in T$ *implies* $1 - x \in T$.

(2) *There is a T-compatible valuation ring A on F such that* $UT = \pm T$ (*and so* $(k : \bar{T}) = 2$).

Proof. The implication (1) $\Rightarrow$ (2) follows from Theorem 3.3.

(2) $\Rightarrow$ (1). By Example 2.2 (ii), F is T-rigid and by Theorem 3.3 (2) (b), $1 + x \in T$ implies $1 - x \in T$, whenever $x \notin \pm T$.

PROPOSITION 3.6. (compare [**13**, Theorem 1]). *For a subgroup T of $\dot{F}$ with $\dot{F}^2 \subset T$ the following statements are equivalent:*

(1) *T is a fan and for all* $x \notin \pm T$, $1 + x \in T$ *implies* $1 - x \in T$.

(2) *There is a T-compatible valuation ring A of F with residue class k such that $\bar{T}$ is an ordering on k.*

(3) *T is a preordering and every element of $\dot{F}$, not in $\pm T$, is either infinitely large over* **Q** *with respect to all orderings containing T or infinitely small over* **Q** *with respect to all orderings containing T.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Corollary 3.5.

(2) $\Rightarrow$ (3). In order to show that T is a preordering, it suffices to prove that $1 + t \in T$ for all $t \in T$. If $t \in U$ then, because $\bar{T}$ is an order on k, $1 + \bar{t} \in \bar{T}$. Hence $1 + t = t_1 + b$ for some $t_1 \in T \cap U$ and $b \in M$. Then $1 + t = t_1(1 + bt_1^{-1}) \in T$, since A is T-compatible. If $t \in M$ then $1 + t \in T$. Finally, if $t \notin A$ then $t^{-1} \in M$ whence $t^{-1}(1 + t) = 1 + t^{-1} \in T$, forcing $1 + t \in T$.

Now let x be an element of $\dot{F}$ not contained in $\pm T$. Since $\bar{T}$ is an order on k and A is T-compatible, x cannot be a unit in A. Hence either x or x^{-1} lies in M. If $x \in M$ then for any integer $n > 0$, $nx \in M$, whence $1 \pm nx \in T$. Since T is a preordering, $n \in T$, and because T is a group it follows that $n^{-1} \pm x \in T$. Hence x is infinitely small over **Q** with respect to all orderings containing T. If x^{-1} lies in M then x^{-1} is infinitely small and therefore x is infinitely large over **Q** with respect to all orderings containing T.

(3) $\Rightarrow$ (1). Let $x \in \dot{F} \backslash \pm T$. If x is infinitely small over **Q** with respect to all orderings P containing T then $1 \pm x \in \bigcap_{P \supset T} P = T$ (by [**18**, Corollary 1.6]), while if x is infinitely large over **Q** with respect to all such P then x^{-1} is infinitely small and $1 + x = x(1 + x^{-1}) \in xT$.

COROLLARY 3.7. *Assume T satisfies the equivalent conditions of Proposition* 3.6. *Then every x in $\dot{F} \backslash \pm T$ is transcendental over* **Q**.

THEOREM 3.8. *If* $(\dot{F} : T) \geqq 8$ *and* $B(T)$ *is a group then the following statements are equivalent:*

(1) *There exists x in $\dot{F}$ with x and $-x$ both T-rigid.*

(2) *There exists a T-compatible valuation ring A on F with* $UT \neq \dot{F}$.

Proof. (1) $\Rightarrow$ (2): We take $A = O(B(T), T)$ if $B(T) \neq \pm T$, while if $B(T) = \pm T$ we let A be the appropriate valuation ring constructed in the proof of Theorem 3.3. Then A is T-compatible and if $B(T) \neq \pm T$ then $B(T) = UT$ so by (1), $UT \neq \dot{F}$. If $B(T) = \pm T$ then, by Theorem 3.3, $(UT : T) \leqq 4$ and the assumption $(\dot{F} : T) \geqq 8$ forces $UT \neq \dot{F}$.

The implication (2) $\Rightarrow$ (1) is example 2.2 (ii).

4. 2-henselian valuations.

Definition 4.1. A valuation ring of F will be called *square compatible* if it is $\dot{F}^2$-compatible. Following [**1**] or [**6**], a valuation ring of F will be called *2-henselian* if it has a unique extension to the quadratic closure of F.

Remark 4.2. By [**10**, § 1], a valuation ring is 2-henselian if and only if Hensel's lemma holds for quadratic polynomials. If A is a nondyadic valuation ring then the concepts of 2-henselian and square compatible are equivalent for A. The ring of 2-adic integers provides an example of a 2-henselian valuation ring which is not square compatible. However, when char $F \neq 2$, square compatible valuations are always 2-henselian:

LEMMA 4.3. *Let A be a valuation ring whose field of fractions F is not of characteristic 2. If A is square compatible then A is 2-henselian.*

Proof. Let $K = F(\sqrt{c})$ be a quadratic extension of F and let $y = a + b\sqrt{c}$, $a, b \in F$, be an element in $K \backslash F$. Then the minimal polynomial of y is $x^2 - 2ax + a^2 - b^2c$. By [**10**, § 1], it suffices to show that if $a^2 - b^2c$ lies in A then a lies in A. If $a \notin A$ then a^{-2} lies in the maximal ideal M of A, whence $1 - a^{-2}b^2c = a^{-2}(a^2 - b^2c)$ lies in M. But then

$$a^{-2}b^2c = 1 - (1 - a^{-2}b^2c) \in 1 + M \subset \dot{F}^2,$$

whence $c \in \dot{F}^2$, contrary to assumption. Thus $a \in A$, as required.

THEOREM 4.4. *If $B = B(\dot{F}^2)$ then there is a square compatible valuation ring A of F, which is 2-henselian if* char $F \neq 2$, *with residue class field k and value group Γ such that*

(1) *B is a subgroup of UF^2 with $(UF^2 : B) \leqq 2$.*

(2) *$W(F)$ is isomorphic to the group algebra $W(k)[\Gamma/\Gamma^2]$.*

(3) (*Compare* [**3**, *Theorem* 3.1], [**4**, *Theorem* 2]) *If $B \neq \pm F^2$ (i.e., F is not a rigid field) then $B = U\dot{F}^2$, $W(F) \cong W(k)[\dot{F}/B]$, and k is basic (i.e., $B(\dot{k}^2) = \dot{k}$).*

(4) (*Compare* [**6**, *Proposition* 3.5], [**7**, *Corollary* 8], [**12**, § 2]). *If F is a rigid field then $(\dot{k} : \dot{k}^2) \leqq 4$ and $W(k)$ is isomorphic to one of the following rings: $\mathbf{Z}[G]$, $\mathbf{Z}/4\mathbf{Z}[G]$, $\mathbf{F}_2[G]$ with $|G| \leqq 2$ or $\mathbf{F}_2[H]$ with H the Klein 4-group.*

(5) *If* $(\dot{F} : \dot{F}^2) \geqq 8$ *then the following statements are equivalent:*
(a) Γ *is not* 2-*divisible.*
(*b*) *There exists a square compatible valuation ring on* F *whose value group is not* 2-*divisible.*
(c) $B \neq \dot{F}$.

Proof. By [**4**, Theorem 1] (or Example 2.5 (i)), B is a group. We take $A = O(B, \dot{F}^2)$ if $B \neq \pm\dot{F}^2$, while if $B = \pm\dot{F}^2$ we let A be the appropriate valuation ring constructed in the proof of Theorem 3.3. Then A is square compatible so if char $F \neq 2$, A is 2-henselian. By example 2.2 (ii), B is a subgroup of $U\dot{F}^2$ and by Theorem 2.16, $B = U\dot{F}^2$, if $B \neq \pm\dot{F}^2$. If $B = \pm\dot{F}^2$ then by Theorem 3.3, $(U\dot{F}^2 : \dot{F}^2) \leqq 4$ and $(U\dot{F}^2 : \dot{F}^2) \leqq 2$, if $-1 \in \dot{F}^2$, whence $(U\dot{F}^2 : B) \leqq 2$. Since A is square compatible, [**13,** § 12.2] yields the isomorphism $W(F) \cong W(k)[\Gamma/\Gamma^2]$ and if F is not rigid then $\Gamma/\Gamma^2 \cong \dot{F}/U\dot{F}^2 = \dot{F}/B$. Proposition 2.20 shows that k is basic when $B \neq \pm\dot{F}^2$ and we have thus proved statements (1), (2), and (3).

(4). The inequality $(\dot{k} : \dot{k}^2) \leqq 4$ is a consequence of Theorem 3.3. By Lemma 2.15, k is also rigid so if char $k \neq 2$ then by [**24**, Theorem 1.9] (or direct calculation), we see that $W(k)$ is isomorphic to one of the specified rings. If char $k = 2$ then by Proposition 2.8, $-1 \in \dot{F}^2$ and so, by Theorem 3.3, $(\dot{k} : \dot{k}^2) \leqq 2$. In this case it is obvious that either $W(k) \cong \mathbf{F}_2$ or $\mathbf{F}_2[G]$ with $|G| \leqq 2$.

(5). This follows from Theorem 3.8 and its proof.

Example 4.5. The assumption $(\dot{F} : \dot{F}^2) \geqq 8$ is necessary for (5). Let $<_1$, $<_2$ be the two orderings on $\mathbf{Q}(\sqrt{2})$, let F_i, $i = 1, 2$, be a real closure of $\mathbf{Q}(\sqrt{2})$ with respect to the ordering $<_i$, and let $F = F_1 \cap F_2$. Then F is pythagorean with exactly two orderings. Hence $(\dot{F} : \dot{F}^2) = 4$ and F is superpythagorean (see, for example, [**11**, § 4, 5]). Therefore, $B(F) = \pm\dot{F}^2 \neq \dot{F}$. However, because F is algebraic over $\mathbf{Q}$, there is no nontrivial square compatible valuation on F. Related to this we have

Proposition 4.6. *For a field* F *the following statements are equivalent:*
(1) F *is superpythagorean and for* $x \in F\backslash F^2$, $\sqrt{1+x} \in F$ *implies* $\sqrt{1-x} \in F$.
(2) F *carries a* 2-*henselian valuation with euclidean residue class field.*
(3) F *is formally real pythagorean, and for all* x *in* $F\backslash \pm F^2$, x *is either infinitely large or infinitely small over* $\mathbf{Q}$ *with respect to all orderings on* F.

Proof. Apply Proposition 3.6 and Remark 4.2.

Remark 4.7. Further conditions equivalent to those in Proposition 4.6 may be found in [**1**, Corollary 2, p. 68]. Among them: F is superpythagorean and $F^4 + F^4 = F^4$.

In conclusion, we use Theorem 4.4 and results from [**24**] to obtain another proof of Theorem 5 in [**4**] (compare the remarks following Theorem 5) and Theorem 2 in [**22**].

Let k be a field and I a totally ordered index set. We will denote by $k((I))$ the direct limit of all iterated formal Laurent series fields $k((t_{i_1}))\ldots((t_{i_r}))$, with $i_1 < i_2 < \ldots < i_r$ in I.

THEOREM 4.8. (cf. [**22**, Theorem 2], [**4**, Theorem 5]), [**3**, Corollary 3.2]). *If $B(\dot{F}^2) \neq \dot{F}$ then there is a basic field k and a nonempty (totally ordered) index set I, with cardinality equal to the $\mathbf{F}_2$-dimension of $\dot{F}/B(\dot{F}^2)$, such that $W(F) \cong W(k((I)))$.*

Proof. If $B(\dot{F}^2) \neq \pm\dot{F}^2$, this follows from Theorem 4.4 and [**24**, Lemma 1.6]; in case char $k = 2$, we can use [**14**, § 12] in the proof of [**24**, Lemma 1.6].

If $B(\dot{F}^2) = \pm\dot{F}^2$ then F is rigid and if char $F \neq 2$, [**24**, Theorem 1.9(2)] implies that $W(F)$ is isomorphic to a group ring $\mathbf{Z}/n\mathbf{Z}[G]$ with $n = 0, 2,$ or 4 and G isomorphic to the group $\dot{F}/\pm\dot{F}^2 = \dot{F}/B(\dot{F}^2)$. If char $F = 2$, then by Theorems 3.3 and 4.4, there is a valuation on F with residue field k_0 and group of units U such that

$$W(F) \cong W(k_0)[\dot{F}/U\dot{F}^2],\ |k_0/k_0{}^2| = (U\dot{F}^2 : B(\dot{F}^2)) \leqq 2, \text{ and } W(k_0) \cong \mathbf{F}_2[H],$$

with $|H| \leqq 2$. Thus if we let k be a field with $W(k) \cong \mathbf{Z}/n\mathbf{Z}$, when char $F \neq 2$ and $W(k) = F_2$, when char $F = 2$, then k will be a basic field and in all cases $W(F)$ will be isomorphic to $W(k)[G]$ with $G \cong \dot{F}/B(\dot{F}^2)$. Applying [**24**, Lemma 1.6] (plus [**14**, § 12], when char $F = 2$) completes the proof.

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Pennsylvania State University,
University Park, Pennsylvania