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ON LINEAR FUNCTIONALS AND SUMMABILITY FACTORS FOR STRONG SUMMABILITY

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0. Introduction. Let $A = (a_{nk})$ $(n, k = 0, 1, 2, \ldots)$ be an infinite matrix. We call a sequence $s = (s_k)$ $(k = 0, 1, 2, \ldots)$ *A-limitable* (denoted by $s \in (A)$) if the sequence $t = (t_n)$, $t_n = \sum_k a_{nk}s_k$ exists and converges. We call s *absolutely A-limitable* (denoted by $s \in |A|$), if t (defined as above) is of bounded variation, i.e. $\sum_{n=0}^{\infty} |t_n - t_{n-1}| < \infty$, $t_{-1} = 0$. Finally, if $A \geqq 0$ (i.e. $a_{nk} \geqq 0$ for all n, k), we call s *strongly A-limitable with exponent* p $(1 \leqq p < \infty)$ (denoted by $s \in [A]_p$) if there exists some number σ such that $\sum_k a_{nk}|s_k - \sigma|^p \to 0$ as $n \to \infty$. Furthermore, we call a formal series $\sum a_k$ *A-summable* (resp. *absolutely A-summable*, resp. *strongly A-summable with exponent* p) if the sequence $s = (s_k)$, $s_k = a_0 + \ldots + a_k$ belongs to (A) (resp. $|A|$, resp. $[A]_p$), and we write $\sum a_k \in (A)$ (resp. $\sum a_k \in |A|$, resp. $\sum a_k \in [A]_p$). Finally, we write $s \in o[A]_p$ if $\sum_k a_{nk}|s_k|^p \to 0$ $(n \to \infty)$. A matrix A is said to be *regular* (resp. *absolutely regular*) if $s_k \to \sigma$ implies $t_n = \sum_k a_{nk}s_k \to \sigma$ (resp. if $\sum_k |a_k| < \infty$ implies $\sum a_k \in |B|$ and

$$t_n = \sum_k a_{nk} \sum_{m=0}^{k} a_m \to \sum_k a_k \ (n \to \infty)).$$

The purpose of this paper is to characterize the sequences $\lambda = (\lambda_k)$ which have one of the following properties:

(0.1) $\quad s \in [A]_p$ implies $\sum \lambda_k s_k \in (B)$, resp. $|B|$, resp. $[B]_{\tilde{p}}$.

(0.2) $\quad \sum a_k \in [A]_p$ implies $\sum \lambda_k a_k \in (B)$, resp. $|B|$, resp. $[B]_{\tilde{p}}$.

(0.3) $\quad s \in [A]_p$ implies $\lambda s = (\lambda_k s_k) \in (B)$, resp. $|B|$, resp. $[B]_{\tilde{p}}$.

We call λ satisfying (0.1), resp. (0.2), resp. (0.3) a *sequence to series factor*, resp. *series to series factor*, resp. *sequence to sequence factor*, and we use the same notation for general sequence spaces X and Y instead of $[A]_p$ and (B) (resp. $|B|$, resp. $[B]_p$). In Section 1, we give an abstract functional analytic answer to the questions (0.1) and (0.2) which involves a condition in terms of continuous linear functionals on $[A]_p$ (called *functional condition*) and a condition of type (0.3).

Section 2 reduces the sequence to sequence factor problem from $o[A]_p$ to $|B|$ to a functional condition, and in Section 3 we show that (0.3) in case $[A]_p$ to (B) is equivalent to the case $[A]_p$ to $[C]_1$ with $c_{nk} = |b_{nk}|$. Section 4 discusses

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(0.3) in case $[A]_p$ to $[B]_{\tilde{p}}$, and in Section 5 we characterize continuous linear functionals on $o[A]_p$. In a final Section 6 we apply our results to weighted arithmetical means and to Nörlund means.

Part II of this paper, which will be published separately, contains an extended study of the continuous linear functionals on $o[A]_p$.

We use the following notations for special sequences resp. sequence spaces:

(a) $e = (1, 1, 1, \ldots), e^{(n)} = (\delta_{nk})$.

(b) $c_0 = \{s = (s_k) | s_k \to 0 \ (k \to \infty)\}$.

(c) $bv = \{s = (s_k) | \sum_k |s_k - s_{k-1}| < \infty\}$.

(d) $l_p = \{\alpha = (\alpha_k) | \sum_k |\alpha_k|^p < \infty\}, \quad 1 \leqq p < \infty$,
$l_\infty = \{b = (b_k) | \sup_k |b_k| < \infty\}$.

For a topological vector space X, we denote the space of all continuous linear functionals on X (the "dual" space) by X^*.

If the inverse of a matrix $A = (a_{nk})$ exists, we denote it by $A' = (a_{nk}')$. For a sequence (x_k) we write $x_k \uparrow$ (resp. $x_k \downarrow$) if x_k increases (resp. decreases) in the wider sense.

1. General functional-analytic results. Theorems 1 and 2 of this section are the basic theorems on series to series and sequence to series factors. Both theorems follow from some functional-analytic results which will be discussed first.

An FK-space X (see e.g. [**18**]) is called *solid*, if $x \in X$ implies $bx \in X$ for every $b \in l_\infty$, and X has property AK if $x = (x_k) \in X$ implies $x^{(n)} = (x_0, \ldots, x_n, 0, 0, \ldots) \in X$, $n \in \mathbf{N}$, and $x^{(n)} \to x$ as $n \to \infty$.

If X is a solid FK-space with AK, then the continuous linear functionals on X are $f(x) = \sum_k \epsilon_k x_k$ with $\sum_k |\epsilon_k x_k| < \infty$ for every $x \in X$, i.e. the dual space X^* is isomorphic with the space of all *absolute convergence factors* for X. (If k_0 is such that $x \in X$ implies $x_{k_0} = 0$, we choose $\epsilon_{k_0} = 0$.)

PROPOSITION 1. *Let X be a solid FK-space with AK. Let Y be an FK-space with the properties*

(i) *$bv \subset Y$, and*

(ii) *there is $f_0 \in Y^*$ such that $f_0(e^{(n)}) = 0$, $n = 0, 1, \ldots, f_0(e) = 1$.*

Then the following statements are equivalent.

(a) *λ is a sequence to series factor from X to Y, and*

(b) *λ is an absolute convergence factor for X.*

Proof. Condition (b) implies (a) by (i), and it remains to prove that (a) implies (b). Let λ be a sequence to series factor from X to Y. It follows from the closed graph theorem that

$$T(x) = (y_n), \quad y_n = \sum_{k=0}^{n} \lambda_k x_k$$

is a continuous linear operator from X to Y, hence

$$f_0(T(x)) = \sum_k \epsilon_k x_k$$

for some absolute convergence factor $\epsilon = (\epsilon_k)$. If $m \in \mathbf{N}$ is such that $x = (x_k) \in X$ exists with $x_m \neq 0$, then $e^{(m)} \in X$, and it follows that

$$\epsilon_m = f_0(T(e^{(m)})) = \lambda_m f_0(e - (e^{(0)} + \ldots + e^{(m-1)})) = \lambda_m.$$

COROLLARY. *If in addition,* $c_0 \subset X$, *and if* $\hat{X} = \{x + ce | x \in X, c \in \mathbf{R}\}$, *then Proposition* 1 *remains true when* X *in* (a) *is replaced by* $\hat{X}$.

This is an immediate consequence of $X^* \subset l_1$.

PROPOSITION 2. *Let* X *and* Y *be FK-spaces as in Proposition* 1. *Then the following statements are equivalent:*

(a) λ *is a series to series factor from* X *to* Y, *and*

(b) $(\Delta\lambda_k) = (\lambda_k - \lambda_{k+1})$ *is an absolute convergence factor for* X *and* λ *is a sequence to sequence factor from* X *to* Y.

Proof. Let $x = (x_k) \in X$, $x_k = a_0 + \ldots + a_k$. It follows from

$$\sum_{k=0}^{n} a_k \lambda_k = x_n \lambda_n + \sum_{k=0}^{n-1} x_k \Delta\lambda_k \quad (n \geqq 1) \tag{1.1}$$

that (b) implies (a) (since $bv \subset Y$.) In order to prove the necessity of (b) we proceed similarly as in the proof of Proposition 1: We introduce the continuous linear operator

$$\hat{T}(x) = (y_n), \quad y_n = \sum_{k=0}^{n} a_k \lambda_k$$

and obtain $f_0(\hat{T}(x)) = \sum_k \epsilon_k x_k$ for some absolute convergence factor $\epsilon = (\epsilon_k)$. If $x = e^{(m)}$, this implies

$$\epsilon_m = f_0(0, \ldots, 0, \lambda_m, \Delta\lambda_m, \Delta\lambda_m, \ldots) = \Delta\lambda_m,$$

which shows that the first condition in (b) is necessary, and the necessity of the second condition follows from $bv \subset Y$ and (1.1).

COROLLARIES. 1. *If* (a) *or* (b) *of Proposition* 2 *holds, then* $\hat{\hat{T}}(x) = \lambda x$ *is a continuous linear operator from* X *to* Y *and*

$$f_0(\lambda x) = \lim_{n \to \infty} f_0(\lambda x^{(n)}) = \lim_{n \to \infty} \sum_{k=0}^{n} \lambda_k x_k f_0(e^{(k)}) = 0.$$

2. *Proposition* 2 *remains true when* X *in* (a) *is replaced by*

$$\hat{X} = \{x + ce | x \in X, c \in \mathbf{R}\}.$$

The following results from summability theory (which are either well-known, see e.g. [**17**] and [**11**], or easily verified) are used to derive Theorems 1 and 2 from Propositions 1 and 2:

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(i) *Let* $A = (a_{nk})$ $(n, k = 0, 1, \ldots)$. *Then the spaces* $o[A]_p$, $1 \leqq p < \infty$ (*if* $A \geqq 0$), (A) *and* $|A|$ *are FK-spaces.* (*In case of* $o[A]_p$ *the seminorms are* $|s_k|$, $k = 0, 1, \ldots$ *and*

$$||s||_{A,p} = \sup_n \left(\sum_{k=0}^{\infty} a_{nk}|s_k|^p \right)^{1/p}.)$$

The space $[A]_p$ *is an FK-space when* A *is regular.*

(ii) *The space* $o[A]_p$ *is solid and has AK.*

(iii) *The spaces* $[B]_p$ (*when* $B \geqq 0$ *and regular*), (B) (*when* B *is regular*), *and* $|B|$ (*when* B *is absolutely regular*) *satisfy assumptions* (i) *and* (ii) *of Proposition* 1 *when*

$$f_0(s) = \lim_{n\to\infty} \sum_{k=0}^{\infty} b_{nk}s_k.$$

THEOREM 1. *Let* $A \geqq 0$ *and let* Y *be any of the spaces mentioned in* (iii). *Then the following statements are equivalent:*

(a) $s \in o[A]_p$ *implies* $\sum \lambda_k s_k \in Y$, *and*

(b) $s \in o[A]_p$ *implies* $\sum_k |\lambda_k s_k| < \infty$.

THEOREM 2. *Let* $A \geqq 0$ *and let* Y *be any of the spaces mentioned in* (iii). *Then the following statements are equivalent:*

(a) $s \in o[A]_p$ *implies* $\sum \lambda_k a_k \in Y$ $\quad (s_k = a_0 + \ldots + a_k)$, *and*

(b) $s \in o[A]_p$ *implies* $\sum_k |s_k \Delta\lambda_k| < \infty$ *and* $\lambda s \in Y$.

COROLLARY. *If* A *is regular, then in* (a) *of both theorems the space* $o[A]_p$ *may be replaced by* $[A]_p$.

Both theorems and the corollary follow from Propositions 1 and 2 and corollaries. We remark that the non-trivial part of Theorem 1, i.e. the conclusion from (a) to (b), is obvious if $B \geqq 0$ since then $\sum c_k \in Y$, $c_k \geqq 0$, implies $\sum_k c_k < \infty$.

Theorems 1 and 2 show that the determination of sequence to series and series to series factors requires the knowledge of the absolute convergence factors of $o[A]_p$, i.e. the knowledge of the space $o[A]_p^*$ of the continuous linear functionals in $o[A]_p$. The corresponding condition on λ will be called the *functional condition.* Section 5 of this paper will be devoted to the study of this condition, i.e. to $o[A]_p^*$. The additional condition $\lambda s \in Y$ of Theorem 2 will be discussed in Sections 2, 3, and 4.

For the corresponding situation in case of ordinary summability see e.g. [**15**, Theorem II.25].

2. The condition $\lambda s \in |B|$. We will show that this condition reduces to a functional condition if $|B|$ has a special absolute convergence factor — a condi-

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tion which is satisfied in many cases. The condition on $|B|$ is:

(2.1) B is absolutely regular, and $\sum a_k \in B$ implies $\sum_k |a_k h_k| < \infty$,

$$\text{where } h_k = \sum_{n=0}^{\infty} |b_{nk} - b_{n-1,k}|.$$

In particular, if B is triangular and $b_{nk} \downarrow$ as $n \uparrow$ $(n \geqq k)$, then $h_k = 2b_{kk}$, i.e. (2.1) requires that (b_{kk}) is an absolute convergence factor for $|B|$. This is true for all Cesàro methods (see, e.g. [**9**; **16**]).

In Section 6, we will discuss some other classes of matrices which satisfy (2.1).

The reduction of $\lambda s \in |B|$ to a functional condition follows from

THEOREM 3. *Let* $A \geqq 0$, *and let* B *satisfy* (2.1). *Then*

(2.2) $s \in o[A]_p$ *implies* $\lambda s \in |B|$

if and only if

(2.3) $s \in o[A]_p$ *implies* $\sum_k |h_k \lambda_k s_k| < \infty$.

Proof. If (2.2) holds, then $\sum_k h_k |\lambda_k s_k - \lambda_{k-1} s_{k-1}| < \infty$ whenever $s \in o[A]_p$. If we change s so that $s_{2k} = 0$ or $s_{2k+1} = 0$ $(k = 0, 1, \ldots)$ then we obtain $\sum_k |h_k \lambda_k s_k| < \infty$, i.e. (2.3) holds. Conversely, if (2.3) holds, then

$$\sum_n \left| \sum_k (b_{nk} - b_{n-1,k}) \lambda_k s_k \right| \leqq \sum_k |\lambda_k s_k| \sum_n |b_{nk} - b_{n-1,k}| < \infty,$$

i.e. (2.2) holds.

Theorem 3 shows that $\lambda s \in |B|$ is equivalent to the condition that $(h_k \lambda_k)$ is an absolute convergence factor for $o[A]_p$.

3. The condition $\lambda s \in (B)$.

We will show that this condition is equivalent to $\lambda s \in [C]_1$ for some C.

THEOREM 4. *Let* $B = (b_{nk})$ *be regular and let* $C = (|b_{nk}|)$. *Furthermore, let* $A \geqq 0$. *Then*

(3.1) $s \in o[A]_p$ *implies* $\lambda s \in (B)$

if and only if

(3.2) $s \in o[A]_p$ *implies* $\lambda s \in [C]_1$.

Proof. If (3.1) holds, then $f_n(s) = \sum_{k=0}^{\infty} b_{nk} \lambda_k s_k$ is a sequence of pointwise convergent, continuous linear functionals on $o[A]_p$, hence it is equicontinuous. The set $\{T_b\}$, $T_b(s) = b \cdot s$ with $b \in l_\infty$, $\|b\|_\infty \leqq 1$ is a set of equicontinuous linear transformations of $o[A]_p$ into itself, hence $\{f_n(T_b(s))\}$ is a set of equi-

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continuous linear functionals. But a subset of this set is $\{\tilde{f}_n(s) = \sum_{k=0}^{\infty} c_{nk}\lambda_k s_k\}$ $(n = 0, 1, \ldots)$ which is equicontinuous and pointwise convergent to zero on a dense subset, namely the set of all sequences with finitely many non-zero terms. Hence (3.2) follows. The converse direction is obvious.

It should be mentioned that (3.1) implies $\lambda s \in o(B)$ by Corollary 1 to Proposition 2.

4. The condition $\lambda s \in [B]_{\tilde{p}}$. In order to avoid minor complications it seems natural to introduce now the assumption that $A \geqq 0$ and that A has no zero-column, i.e.

$$(4.1) \qquad A \geqq 0 \quad \text{and} \quad a_k = \sup_n a_{nk} > 0.$$

If (4.1) holds, then $o[A]_p$ is a BK-space with norm

$$||s||_{A,p} = \sup_n \left(\sum_{k=0}^{\infty} a_{nk}|s_k|^p \right)^{1/p}.$$

If, in addition, A is regular, then $[A]_p$ is a BK-space. The following lemma yields information on the order of growth of sequences in $o[A]_p$ (compare [**8**]).

LEMMA 1. *Let A be regular and satisfy* (4.1).

(a) *If $s \in o[A]_p$ then $s_k a_k^{1/p} = o(1)$. This estimate is best possible, i.e. if $\eta_k \neq O(1)$ then there is $\tilde{s} \in o[A]_p$ such that $\eta_k \tilde{s}_k a_k^{1/p} \neq O(1)$.*

(b) *If $s \in o[A]_p$ implies $\lambda s \in o[B]_{\tilde{p}}$, where $B \geqq 0$, then*

$$(4.2) \qquad \lambda_k b_k^{1/p} = O(a_k^{1/\tilde{p}}).$$

Proof. (a) Let $s \in o[A]_p$, and choose $\epsilon > 0$. Then there are numbers $N(\epsilon)$, $K(\epsilon)$ such that

$$\sum_{k=0}^{\infty} a_{nk}|s_k|^p \leqq \epsilon \quad \text{for } n \geqq N(\epsilon), \quad a_{nk}|s_k|^p \leqq \epsilon \quad \text{for } k \geqq K(\epsilon), n < N(\epsilon).$$

Hence $a_{nk}|s_k|^p \leqq \epsilon$ for $k \geqq K(\epsilon)$ and all n, i.e. $a_k^{1/p}s_k \to 0$. (Note that this part of the proof does not require any assumption on A besides $A \geqq 0$.) Let $0 < \eta_k \neq O(1)$ and select a subsequence $k_i \uparrow \infty$ such that $\sum_i \eta_{k_i}^{-p/2} < \infty$. If $s_{k_i} = a_{k_i}^{-1/p}\eta_{k_i}^{-1/2}, s_k = 0$ otherwise, then

$$\sum_{k=0}^{\infty} a_{nk}|s_k|^p = \sum_{i=0}^{\infty} \frac{a_{nk_i}}{a_{k_i}} \eta_{k_i}^{-p/2} \to 0$$

(A is regular), but $\eta_{k_i}s_{k_i}a_{k_i}^{1/p} = \eta_{k_i}^{1/2} \to \infty$.

(b) This is an immediate consequence of (a).

Our next two theorems will discuss implications of the type

$$(4.3) \qquad \text{if } s \in o[A]_p \quad \text{then} \quad \lambda s \in [B]_{\tilde{p}}.$$

Assume that (4.3) holds, that A satisfies (4.1) and that $B \geqq 0$ is regular. Then

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$\|\lambda s\|_{B,\tilde{p}}$ is a continuous seminorm on $o[A]_p$, hence there is $K \geqq 0$ such that

(4.4) $\quad \|\lambda s\|_{B,\tilde{p}} \leqq K\|s\|_{A,p} \quad$ for all $s \in o[A]_p$.

THEOREM 5. *Let A be regular and satisfy* (4.1). *Then*

(4.5) $\quad s \in o[A]_p \quad$ *implies* $\quad \lambda s \in [A]_{\tilde{p}}$

if and only if

$$(4.6) \qquad \begin{cases} \sup_n \sum_k a_{nk}|\lambda_k|^{p\tilde{p}/(p-\tilde{p})} < \infty & \text{if } \tilde{p} < p, \\ \lambda_k = O(1)\, a_k^{(\tilde{p}-p)/p\tilde{p}} & \text{if } \tilde{p} \geqq p. \end{cases}$$

We omit the proof, since the following Theorem 6 is a partial generalization of Theorem 5, and a few obvious modifications of its proof lead to a proof of Theorem 5.

THEOREM 6. *Let A and B be normal, regular and assume that $a_{nk}, b_{nk} > 0$ if $k \leqq n$. Moreover, assume that $a_{nk} \downarrow$ and $b_{nk}/a_{nk} \downarrow$ as $n \uparrow$ $(n \geqq k)$. Then* (4.3) *holds if and only if*

$$(4.7) \qquad \begin{cases} \sup_n \sum_{k=0}^{n} |\lambda_k|^{p\tilde{p}/(p-\tilde{p})} b_{nk}^{\,p/(p-\tilde{p})}/a_{nk}^{\,\tilde{p}/(p-\tilde{p})} < \infty & \text{if } \tilde{p} < p, \\ \lambda_k = O(1)\,(a_{kk}^{\,1/p}/b_{kk}^{\,1/\tilde{p}}) & \text{if } \tilde{p} \geqq p. \end{cases}$$

Proof. Let $\tilde{p} < p$. Holder's inequality shows that $s \in o[A]_p$ and (4.7) imply $\lambda s \in o[B]_{\tilde{p}}$, hence (4.7) implies (4.3). Assume that (4.3) holds which implies (4.4). Let $s(m) = (s_0(m), \ldots, s_m(m), 0, 0, \ldots) \in o[A]_p$ where

$$s_k(m) = |\lambda_k|^{\tilde{p}/(p-\tilde{p})}(b_{mk}/a_{mk})^{1/(p-\tilde{p})} \qquad (k \leqq m).$$

The monotonicity of a_{nk} and b_{nk}/a_{nk} implies

$$\begin{aligned} \|\lambda s(m)\|_{B,\tilde{p}} &= \sup_{n \leqq m} \left\{ \sum_{k=0}^{n} a_{nk} \frac{b_{nk}}{a_{nk}} |\lambda_k|^{\tilde{p}p/(p-\tilde{p})} \left(\frac{b_{mk}}{a_{mk}}\right)^{\tilde{p}/(p-\tilde{p})} \right\}^{1/\tilde{p}} \\ &\geqq \sup_{n \leqq m} \left\{ \sum_{k=0}^{n} a_{nk}|\lambda_k|^{p\tilde{p}/(p-\tilde{p})} \left(\frac{b_{mk}}{a_{mk}}\right)^{p/(p-\tilde{p})} \right\}^{1/\tilde{p}} = (\|s(m)\|_{A,p})^{p/\tilde{p}} \\ &\geqq \|s(m)\|_{A,p} \left(\sum_{k=0}^{m} |\lambda_k|^{p\tilde{p}/(p-\tilde{p})} b_{mk}^{\,p/(p-\tilde{p})} a_{mk}^{\,-\tilde{p}/(p-\tilde{p})} \right)^{(p-\tilde{p})/\tilde{p}}, \end{aligned}$$

hence the first line of (4.7) follows from (4.4) and this estimate.

Let $\tilde{p} \geqq p$. If $s \in o[A]_p$, then it follows from Lemma 1, (4.7) and the monotonicity of a_{nk} and b_{nk}/a_{nk} that

$$\begin{aligned} \sum_{k=0}^{n} b_{nk}|\lambda_k s_k|^{\tilde{p}} &= \sum_{k=0}^{n} a_{nk}|s_k|^p \frac{b_{nk}}{a_{nk}} |\lambda_k|^{\tilde{p}}|s_k|^{\tilde{p}-p} \\ &= O(1) \sum_{k=0}^{n} a_{nk}|s_k|^p \frac{b_{kk}}{a_{kk}} \frac{a_{kk}^{\,\tilde{p}/p}}{a_{kk}} a_{kk}^{\,-(\tilde{p}-p)/p} = O(1) \sum_{k=0}^{n} a_{nk}|s_k|^p, \end{aligned}$$

hence (4.7) implies (4.3). The necessity of (4.7) follows from (4.2).

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Remarks. 1. The proof of Theorem 6 shows that the monotonicity of a_{nk} and b_{nk}/a_{nk} is not needed to prove that (4.7) is sufficient if $\tilde{p} < p$ resp. necessary if $\tilde{p} \geqq p$.

2. Hölder's inequality implies for regular matrices B

$$(4.8) \quad [B]_{\tilde{p}+\epsilon} \subset [B]_{\tilde{p}}, \quad \epsilon > 0.$$

It follows that for fixed p condition (4.7) becomes stronger as $\tilde{p}$ increases. This can also directly be shown by Hölder's inequality when $\tilde{p} < p$.

3. The assumptions of Theorem 6 are satisfied for $A = C_\alpha$, $B = C_\beta$, $0 < \beta \leqq \alpha \leqq 1$, hence $B \subset A$ in this case. In a sense, this relation is typical for the assumptions of Theorem 6: Let A, B be triangular, $a_{nk} > 0$, $b_{nk} > 0$ $(k \leqq n)$, $\sum_{k=0}^n a_{nk} = \sum_{k=0}^n b_{nk} = 1$. Then

$$(4.8) \quad b_{nk}/a_{nk} \downarrow \text{ as } n \uparrow \ (n \geqq k), \quad b_{nk}/a_{nk} \downarrow \text{ as } k \uparrow \quad (k \leqq n)$$

implies $A = B$, because for $n \geqq k$ we obtain

$$(4.9) \quad b_{kk}/a_{kk} \geqq b_{nk}/a_{nk} \geqq b_{nn}/a_{nn},$$

which implies

$$1 = \sum_{k=0}^n b_{nk} \geqq \frac{b_{nn}}{a_{nn}} \sum_{k=0}^n a_{nk} = \frac{b_{nn}}{a_{nn}},$$

and if $b_{nn}/a_{nn} = 1$ for all n, $A = B$ follows from (4.9). On the other hand, assume $b_{nn}/a_{nn} < 1$ for some n; then it follows from (4.9) that

$$\sum_{k=0}^n b_{nk} \leqq \sum_{k=0}^n \frac{b_{kk}}{a_{kk}} a_{nk} < \sum_{k=0}^n a_{nk} = 1,$$

which contradicts the assumptions on B. (Of course, the same conclusion holds when $b_{nk}/a_{nk} \uparrow$ in n and k.) Hence, if A and B satisfy in addition $M_1^*(A)$ or $M_1^*(B)$ (see e.g. [**15**, p. 34]), $b_{nk}/a_{nk} \downarrow$ as $n \uparrow$ $(n \geqq k)$, and if b_{nk}/a_{nk} is monotone in k (same kind of monotonicity for every n), then Theorem II.20 or Theorem II.21 [**15**] implies $B \subset A$.

4. In view of Remark 3 one might ask for the sequence to sequence factors in case $A \subset B$. In this case

$$(4.10) \quad [A]_p \subset [B]_p,$$

and (4.8) and (4.10) imply that *every* $\lambda \in l_\infty$ *is a sequence to sequence factor from* $o[A]_p$ *to* $o[B]_{\tilde{p}}$ *if* $\tilde{p} \leqq p$. This result will be helpful in our applications when $\lambda \in l_\infty$ follows from a functional condition.

The situation becomes considerably more difficult when $o[A]_p \subset o(B)_{\tilde{p}}$ does not hold. We do not have a general result in this case. For more special results see the end of Section 6 and Part II of this paper (which will be published separately).

5. Linear functionals on $o[A]_p$. In order to obtain $o[A]_p^*$ we embed $o[A]_p$ into a certain space $M_p^{(0)}$ which has a dual $M_q^{(1)}$ of comparatively simple structure. By the Hahn-Banach Theorem every element of $o[A]_p^*$ has an extension to $M_p^{(0)}$.

Let $1 \leqq p < \infty$, and let $M_p^{(0)}$ denote the space of all matrices $G = (g_{nk})$, $n, k = 0, 1, \ldots$, with $\sum_k |g_{nk}|^p = o(1)$ as $n \to \infty$. If $||G|| = \sup_n (\sum_k |g_{nk}|^p)^{1/p}$, then $M_p^{(0)}$ is a Banach space. Similarly, let $1 \leqq p \leqq \infty$, and let $M_p^{(1)}$ denote the Banach space of all matrices $H = (h_{nk})$ such that $||H|| < \infty$, where

$$||H|| = \begin{cases} \sum_n \left(\sum_k |h_{nk}|^p \right)^{1/p}, & 1 \leqq p < \infty, \\ \sum_n \sup_k |h_{nk}|, & p = \infty. \end{cases}$$

Let $1 \leqq p < \infty$, $(1/p) + (1/q) = 1$. Applying standard techniques one verifies that

$$(5.1) \quad f(G) = \sum_n \sum_k h_{nk} g_{nk}, \quad G \in M_p^{(0)}, H \in M_q^{(1)}$$

is the general form of a continuous linear functional on $M_p^{(0)}$, and it follows that $M_q^{(1)}$ is a representation of the dual of $M_p^{(0)}$ and $||f|| = ||H||$.

THEOREM 7. *Let $1 \leqq p < \infty$, $(1/p) + (1/q) = 1$, and assume that A is regular and satisfies* (4.1). *Then the followng statements are equivalent:*

(a) $\sum_k |\epsilon_k s_k| < \infty$ *whenever* $s \in o[A]_p$,

(b) $\epsilon_k = \sum_n h_{nk} a_{nk}^{1/p}$ *for some* $H \in M_q^{(1)}$,

(c) $\epsilon_k = \gamma_k \left(\sum_n \alpha_n a_{nk} \right)^{1/p}$ *for some* $\gamma \in l_q, \alpha \in l_1, \alpha_n \geqq 0$,

(d) $|\epsilon_k| \leqq \alpha_k^{1/q} \left(\sum_n \alpha_n a_{nk} \right)^{1/p}$ *for some* $\alpha \in l_1, \alpha_n \geqq 0$ $(\alpha_k^{1/q} = 1$ *for* $q = \infty)$.

Proof. (a) implies (b): The space $o[A]_p$ is norm-isomorphic with the (closed) subspace $\{g_{nk} = a_{nk}^{1/p} s_k | s \in o[A]_p\}$ of $M_p^{(0)}$. Since $f(s) = \sum_k \epsilon_k s_k$ is in $o[A]_p^*$ (by (a)), it follows from (5.1) that $H \in M_q^{(1)}$ exists such that

$$\sum_k \epsilon_k s_k = \sum_n \sum_k h_{nk} a_{nk}^{1/p} s_k$$

and this implies (b) (take $s = e^{(m)}$).

(b) implies (c): Let $p > 1$, and define $\alpha_n = (\sum_k |h_{nk}|^q)^{1/q}$. Then $\sum_n \alpha_n < \infty$, and if $\alpha_n > 0$ for all n, then by (b)

$$|\epsilon_k| = \left| \sum_n \frac{h_{nk}}{\alpha_n^{1/p}} \alpha_n^{1/p} a_{nk}^{1/p} \right| \leqq \left(\sum_n \frac{|h_{nk}|^q}{\alpha_n^{q/p}} \right)^{1/q} \left(\sum_n \alpha_n a_{nk} \right)^{1/p}$$
$$= \tilde{\gamma}_k \left(\sum_n \alpha_n a_{nk} \right)^{1/p}.$$

But

$$\sum_k \tilde{\gamma}_k{}^q = \sum_n \frac{1}{\alpha_n{}^{q/p}} \sum_k |h_{nk}|^q = \sum_n \alpha_n < \infty .$$

This shows that (c) holds in this case. Some obvious modifications of the proof when some $\alpha_n = 0$ or $p = 1$ show that (c) holds in general.

(c) implies (d): Since the case $p = 1$ is trivial, assume $p > 1$. If $\beta_k = \alpha_k + |\gamma_k|^q$, then $\beta_k{}^{1/q} \geqq |\gamma_k|$ and (c) implies (d) with β in place of α.

(d) implies (a): Let $s \in o[A]_p$. If $p > 1$ (the case $p = 1$ is obvious) then (d) implies

$$\sum_k |\epsilon_k s_k| \leqq \left(\sum_k \alpha_k\right)^{1/q} \left(\sum_k |s_k|^p \sum_n \alpha_n a_{nk}\right)^{1/p} < \infty .$$

Remark. Conditions (b)-(d) of Theorem 7 are of a "two parameter type", and the two parameters may be "essentially" different. (For instance, the right hand side of (c) is the same when $A = C_1, p = 1$ and $\alpha_n = (n + 1)\Delta(n + 1)^{-3}$, $\gamma_n = 1$ or $\alpha_n = (n + 1)\Delta(n + 1)^{-2}$, $\gamma_n = (n + 1)^{-1}$.) It may be difficult to decide whether a given sequence ϵ satisfies one of the conditions (b)-(d). Our next theorem shows that for weighted means M_p also a "one parameter condition" exists.

6. Applications. We first give a characterization of $o[A]_p{}^*$ when A is a weighted mean:

THEOREM 8. *Let* $p_k > 0$, $P_n = p_0 + \ldots + p_n$, $a_{nk} = p_k P_n{}^{-1} (k \leqq n)$, $a_{nk} = 0$ $(k > n)$. *Then* (d) *of Theorem* 7 *is equivalent to*

$$(6.1) \quad \begin{cases} \displaystyle\sum_n p_n \sup_{k \geqq n} \frac{|\epsilon_k|}{p_k} < \infty & \text{if } p = 1, \\ \displaystyle\sum_n p_n \left(\sum_{k=n}^{\infty} \frac{p_k}{P_k} \left|\frac{\epsilon_k}{p_k}\right|^q\right)^{1/q} < \infty & \text{if } p > 1 ((1/p) + (1/q) = 1). \end{cases}$$

Proof. We give the proof for $p > 1$, the case $p = 1$ follows after some obvious modifications. Condition (d) is in the present case equivalent to

$$(6.2) \quad \frac{|\epsilon_k|^q}{P_k p_k{}^{q-1}} \leqq \left(\frac{\alpha_k}{P_k}\right) \left(\sum_{n \geqq k} \frac{\alpha_n}{P_n}\right)^{q-1} .$$

In view of the inequality $(a - b)a^{q-1} \leqq a^q - b^q \leqq q(a - b)a^{q-1}$ $(a > b > 0$, $q \geqq 1)$, condition (6.2) is equivalent to

$$(6.3) \quad \frac{|\epsilon_k|^q}{P_k p_k{}^{q-1}} \leqq \left(\sum_{n \geqq k} \frac{\alpha_n}{P_n}\right)^q - \left(\sum_{n \geqq k+1} \frac{\alpha_n}{P_n}\right)^q$$

(in the sense that (6.2) implies (6.3) and (6.3) implies (6.2) with e.g. $q\alpha_k$ instead of α_k). The second condition (6.1) is an immediate consequence of

(6.3). Conversely, let (6.1) hold, and let

$$\alpha_n = P_n(\delta_n - \delta_{n+1}), \quad \delta_n = \left(\sum_{k=n}^{\infty} \frac{p_k}{P_k} \left| \frac{\epsilon_k}{p_k} \right|^q \right)^{1/q}.$$

We have $\alpha_n \geqq 0$ and

$$\sum_0^N \alpha_n = \sum_0^N p_n \delta_n - P_N \delta_{N+1} \leqq \sum_0^N p_n \delta_n,$$

hence $\alpha \in l_1$ by (6.1). This choice of α leads immediately to $\delta_k = \sum_{n \geqq k} \alpha_n/P_n$, and this shows that (6.3) holds. But (6.3) is equivalent to (d) of Theorem 7.

Remark. For $A = C_1$, Borwein [**2**] gave equivalent conditions characterizing $o[A]_p^*$.

We are in a position now to discuss special theorems concerning sequence to sequence factors of the type

(6.4) $[M_p]_\alpha \rightarrow |B|$

(6.5) $[M_p]_\alpha \rightarrow [B]_\beta$

where M_p denotes a weighted mean. We first discuss (6.4).

Let B be triangular, and let $\hat{B} = \Delta B S$, where $S = (s_{nk})$, $\Delta = (\Delta_{nk})$ and $s_{nk} = 1$ $(k \leqq n)$, $s_{nk} = 0 (k > n)$, $\Delta_{nn} = 1$, $\Delta_{n,n-1} = -1$, $\Delta_{nk} = 0$ otherwise. $\hat{B}$ is the series-to-series form of B, i.e. $\sum a_k \in |B|$ if and only if $\sum_n |\alpha_n| < \infty$ where $\alpha_n = \sum_{k=0}^n \hat{b}_{nk} a_k$.

LEMMA 2. *Let B be normal, and let $\sum_{n=k}^\infty |\hat{b}_{nk}' b_{nn}| = O(1)$. Then $\sum a_k \in |B|$ implies $\sum_n |a_n b_{nn}| < \infty$.*

Proof.

$$\sum_n |a_n b_{nn}| = \sum_n |b_{nn}| \left| \sum_{k=0}^n \hat{b}_{nk}' \alpha_k \right| \leqq \sum_k |\alpha_k| \sum_{n=k}^\infty |\hat{b}_{nk}' b_{nn}|.$$

In what follows, we use the formula $\hat{B}' = \Delta B' S$, i.e.

(6.6) $$\hat{b}_{nk}' = \sum_{m=k}^n (b_{nm}' - b_{n-1,m}'), \quad (k \leqq n).$$

LEMMA 3. *Let M_p be a weighted mean with $p_n > 0$, $p_n/P_n \downarrow$. Then $\sum a_k \in |M_p|$ implies $\sum_n |a_n \, p_n/P_n| < \infty$.*

Proof. (See [**7**]). We have $\hat{b}_{nn}' = P_n/p_n$, $\hat{b}_{n,n-1}' = -P_{n-2}/p_{n-1}$ $(n \geqq 1$, $P_{-1} = 0)$, $\hat{b}_{nk}' = 0$ otherwise. It follows that

$$\sum_{n=k}^\infty |\hat{b}_{nk}' b_{nn}| = 1 + \frac{P_{k-1}}{p_k} \frac{p_{k+1}}{P_{k+1}} \leqq 1 + \frac{p_{k+1}/P_{k+1}}{p_k/P_k} \leqq 2,$$

and Lemma 3 follows from Lemma 2.

LEMMA 4. *Let N_p be a Nörlund mean with $0 < p_n \downarrow$, $p_{n+1}/p_n \uparrow$ as $n \uparrow$. Then $\sum a_k \in |N_p|$ implies $\sum_n |a_n/P_n| < \infty$.*

For a proof, see [**4**].

Our first theorem is of the type $[M_p]_\alpha \to |M_q|$ $(1 \leqq \alpha < \infty)$.

THEOREM 9. *Let M_p, M_q be regular weighted means and let $p_n > 0$, $q_n > 0$, $q_n/Q_n\downarrow$. Then λ is a series to series factor from $[M_p]_\alpha$ to $|M_q|$ if and only if*

$$(6.7)\quad \begin{cases} \sum_n p_n \sup_{k \geqq n} \left|\dfrac{\Delta\lambda_k}{p_k}\right| < \infty & \textit{for } \alpha = 1, \\ \sum_n p_n \left(\sum_{k=n}^{\infty} \dfrac{p_k}{P_k}\left|\dfrac{\Delta\lambda_k}{p_k}\right|^{\tilde{\alpha}}\right)^{1/\tilde{\alpha}} < \infty & \textit{for } \alpha > 1,\ (1/\alpha) + (1/\tilde{\alpha}) = 1, \end{cases}$$

and

$$(6.8)\quad \begin{cases} \sum_n p_n \sup_{k \geqq n} \left(\dfrac{q_k|\lambda_k|}{Q_k p_k}\right) < \infty & \textit{for } \alpha = 1, \\ \sum_n p_n \left(\sum_{k=n}^{\infty} \dfrac{p_k}{P_k}\left|\dfrac{q_k\lambda_k}{Q_k p_k}\right|^{\tilde{\alpha}}\right)^{1/\tilde{\alpha}} < \infty & \textit{for } \alpha > 1,\ (1/\alpha) + (1/\tilde{\alpha}) = 1. \end{cases}$$

A special case of this theorem is due to Pati [**13**; **14**] (who gives sufficient conditions only when $\alpha = 1$, $p_n = 1$ or $p_n = 1/(n+1)$, $q_n = 1$). The theorem follows from Theorems 2, 3, 8, and Lemma 3.

Our next theorem is of the type $[M_p]_\alpha \to |N_q|$.

THEOREM 10. *Let M_p be regular, $p_n > 0$ and let N_q be regular, $0 < q_n\downarrow$, $q_{n+1}/q_n\downarrow$ as $n\uparrow$. Then λ is a series to series factor from $[M_p]_\alpha$ to $|N_q|$ if and only if* (6.7) *and*

$$(6.9)\quad \begin{cases} \sum_n p_n \sup_{k \geqq n} \left|\dfrac{\lambda_k}{Q_k p_k}\right| < \infty & \textit{for } \alpha = 1, \\ \sum_n p_n \left(\sum_{k=n}^{\infty} \dfrac{p_k}{P_k}\left|\dfrac{\lambda_k}{Q_k p_k}\right|^{\tilde{\alpha}}\right)^{1/\tilde{\alpha}} < \infty & \textit{for } \alpha > 1,\ (1/\alpha) + (1/\tilde{\alpha}) = 1. \end{cases}$$

A special case of this theorem is due to Lal [**10**] (who gives sufficient conditions only when $\alpha = 1$, $p_n = q_n = 1/(n+1)$. See also Daniel [**3**]. The proof follows from Theorems 2, 3, 8, and Lemma 4.

Next we discuss (6.5).

THEOREM 11. *Let M_p, M_q be regular weighted means and let $p_n > 0$, $q_n > 0$, $P_n/Q_n\downarrow$ as $n\uparrow$. Then λ is a series to series factor from $[M_p]_\alpha$ to $[M_q]_\beta$ if and only if* (6.7) *and*

$$(6.10)\quad \begin{cases} \sum_{k=0}^{n} \left(\dfrac{q_k^{\alpha}}{p_k^{\beta}}|\lambda_k|^{\alpha\beta}\right)^{1/(\alpha-\beta)} = O(1)\left(\dfrac{Q_n^{\alpha}}{P_n^{\beta}}\right)^{1/(\alpha-\beta)} & \textit{for } 1 \leqq \beta < \alpha, \\ \lambda_k = O(1)\left(\dfrac{p_k}{P_k}\right)^{1/\alpha}\left(\dfrac{Q_k}{q_k}\right)^{1/\beta} & \textit{for } 1 \leqq \alpha \leqq \beta. \end{cases}$$

The case $p_k = q_k = 1$, $\alpha = \beta$ is essentially due to Borwein [**1**] ((6.10) is a consequence of (6.7) in this case). The proof follows from Theorems 2, 5, and 8.

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A consequence of Remark 4 in Section 4 is

THEOREM 12. *Let M_p, M_q be regular weighted means and p_n, $q_n > 0$, $p_n/q_n \uparrow$ as $n \uparrow$. Then λ is a series to series factor from $[M_p]_\alpha$ to $[M_q]_\beta$, $\beta \leqq \alpha$, if and only if* (6.7) *holds.*

For a proof note that $M_p \subset M_q$ under the stated conditions and that $\lambda_n = O(1)$ follows from (d) in Theorem 7, which is equivalent to (6.7).

THEOREM 13. *Let M_p be a regular weighted mean, $p_n > 0$, and let $B \geqq 0$ be regular. Then λ is a series to series factor from $[M_p]_\alpha$ to $[B]_\beta$, $\beta \leqq \alpha$, if and only if* (6.7) *and*

$$(6.11)\quad \begin{cases} \sum_n p_n \sup_{k \geqq n} \dfrac{b_{mk}|\lambda_k|^\beta}{p_k} = O(1) & \text{for } \alpha = \beta, \\ \sum_n p_n \left(\sum_{k=n}^\infty \dfrac{p_k}{P_k} \left| \dfrac{b_{mk}}{p_k} \right|^{\alpha/(\alpha-\beta)} |\lambda_k|^{\alpha\beta/(\alpha-\beta)} \right)^{(\alpha-\beta)/\alpha} = O(1) & \text{for } \alpha > \beta. \end{cases}$$

Proof. It is easy to derive from the definition of sequence to sequence factors that λ is such a factor from $[M_p]_\alpha$ to $[B]_\beta$ if and only if $\tilde{\lambda} = (|\lambda_k|^\beta)$ is a sequence to sequence factor from $[M_p]_{\alpha/\beta}$ to $[B]_1$ if $\beta \leqq \alpha$. The latter condition is equivalent to the statement, that

$$f_m(s) = \sum_{k=0}^m b_{mk}|\lambda_k|^\beta |s_k|$$

defines a pointwise, hence uniformly bounded sequence of continuous linear functionals on $[M_p]_{\alpha/\beta}$. Using the fact that (6.1) with $p = \alpha/\beta$ defines a norm on $[M_p]_{\alpha/\beta}^*$ such that it becomes a BK-space, this is equivalent to (5.11). Hence Theorem 13 follows, using Theorem 2.

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