Ergod. Th. & Dynam. Sys. (1985), **5**, 123–143
Printed in Great Britain

A Morse equation in Conley's index theory for semiflows on metric spaces

KRZYSZTOF P. RYBAKOWSKI AND EDUARD ZEHNDER
Albert-Ludwigs Universität, Institut für Angewandte Mathematik, Hermann-Herder-Str. 10, 7800 *Freiburg i. Br., West Germany*; *Ruhr-Universität, Bochum, Institut für Mathematik, Universitätsstr.* 150, *D*-4630 *Bochum* 1, *West Germany*

(*Received* 21 *September* 1982)

Abstract. Given a compact (two-sided) flow, an isolated invariant set S and a Morse-decomposition $(M_1, \ldots, M_n)$ of S, there is a generalized Morse equation, proved by Conley and Zehnder, which relates the Alexander–Spanier cohomology groups of the Conley indices of the sets M_i and S with each other. Recently, Rybakowski developed the technique of isolating blocks and extended Conley's index theory to a class of one-sided semiflows on non-necessarily compact spaces, including e.g. semiflows generated by parabolic equations. Using these results, we discuss in this paper Morse decompositions and prove the above-mentioned Morse equation not only for arbitrary homology and cohomology groups, but also in this more general semiflow setting.

0. *Introduction*

In his CBMS notes **[5]** C. Conley develops concepts and ideas which are designed for the qualitative study of stable phenomena of flows. In order to outline briefly in particular the index theory for flows we consider a continuous flow on a locally compact and metric space X. A compact and invariant subset $S \subset X$ is called isolated, if it admits a compact neighbourhood N such that S is the maximal invariant subset which is contained in N. With such an isolated invariant set S a pair (N_1, N_2) of compact spaces can be associated, where $N_2 \subset N_1$ is roughly the 'exit set' of N_1 and where $S \subset \operatorname{int}(N_1 \setminus N_2)$ is the maximal invariant set contained in N_1. For a precise definition of such an index pair we refer to § 3 below. The homotopy type of the pointed space $(N_1/N_2, [N_2])$ then does not depend on the particular choice of the index pair for S, and is called the index of S. It is denoted by $h(S) := [(N_1/N_2, [N_2])]$. Therefore, to an isolated invariant set S we can assign the algebraic invariant $p(t, h(S))$, which is the power series in t, whose coefficients are the ranks of the Alexander–Spanier cohomology modules of any index pair (N_1, N_2) for S. We assume that the isolated invariant set admits a Morse decomposition, that is, there is an ordered family $(M_1, \ldots, M_n)$ of finitely many subsets of S, which are disjoint, compact and invariant. Moreover, for every $x \in S \setminus \{\bigcup M_j, 1 \le j \le n\}$ there is a pair of indices $i < j$ such that for the limit sets of x we conclude $\omega(x) \subset M_i$ and $\omega^*(x) \subset M_j$. The algebraic invariants of S are related to the algebraic invariants of the elements

M_j of its Morse decomposition. More precisely, the following identity is proved in [6]:

$$\sum_{j=1}^{n} p(t, h(M_j)) = p(t, h(S)) + (1+t)Q(t). \tag{1}$$

$Q(t)$ is a power series in t having only non-negative integer coefficients. The terms in Q measure the number of cohomologically non-trivial connections between pairs M_i and M_j of the decomposition.

The identity can be viewed as a generalization of the classical Morse inequalities for the gradient flow on a manifold. In fact let $X = S = M^d$ be a d-dimensional compact manifold, and let f be a C^2 function on M and consider the gradient flow $\dot{x} = -\nabla f(x)$ on M. Assume the critical points to be isolated, then the family $(x_1, \ldots, x_n)$ of all critical points is a Morse decomposition of the manifold M if we order them in such a way that $f(x_i) \le f(x_j)$ for $i \le j$. This is an immediate consequence of the gradient structure of the flow. Since the critical points $\{x_j\}$ are compact and isolated invariant sets we conclude from (1) the equation:

$$\sum_{j=1}^{n} p(t, h(\{x_j\})) = p(t, h(M)) + (1+t)Q(t). \tag{2}$$

As $(M, \emptyset)$ is an index pair for the invariant set M, the first term on the right hand side is the Poincaré polynomial

$$p(t, M) = \sum_{k=0}^{d} \beta_k t^k,$$

the β_k being the Betti numbers of the manifold M. If we assume now that, in addition, the critical points are non-degenerate, then the manifold M is the union of the stable and unstable invariant manifolds of the critical points and their indices can easily be computed. Observe that in this case the only local topological invariant of a critical point x_j, which is a hyperbolic equilibrium point of the flow, is the dimension of the unstable invariant manifold, which is equal to the Morse index d_j of the critical point x_j. It is easy to show (see [6], for instance), that the Conley index of the set $\{x_j\}$ is given by $h(\{x_j\}) = [(S^{d_j}, p)]$ where p is a distinguished point of the d_j-dimensional sphere S^{d_j}. Therefore, $p(t, h(\{x_j\})) = t^{d_j}$. Summarizing we find for the Morse decomposition of the manifold M indeed the clasical equation of Morse theory:

$$\sum_{j=1}^{n} t^{d_j} = \sum_{k=0}^{d} \beta_k t^k + (1+t)Q(t), \tag{3}$$

$Q(t)$ being a polynomial having non-negative integer coefficients only. (See [4] about the work of Marston Morse).

Hence the index theory outlined above can be viewed as a generalization of the classical index theory for flows other than gradient flows on spaces other than manifolds. An index is associated not only with critical points but also with every compact and isolated invariant set of the flow. It is the homotopy type of a pair of compact spaces. In addition, an analogue of the 'Homotopy Axiom' of the Leray–Schauder degree theory is possible in this generalized Morse theory, see [**5**]. With this addition the theory becomes a useful tool in problems of non-linear functional analysis and we should mention that it has already allowed many applications in differential equations. The index is used for instance to find special shocks [**12**], it is used to prove existence and multiplicity results for systems of non-linear elliptic boundary value problems [**2**]. In [**3**] and [**6**] the index theory allows one to find periodic solutions of time-dependent Hamiltonian equations. Also, the existence of heteroclinic orbits for semilinear parabolic equations is based on the index [**11**].

Recently, the first author extended in [**9**] and in [**10**] the outlined index theory to continuous local semiflows on metric spaces, which are not assumed to be locally compact. Of course, some compactness condition has still to be imposed and we refer to § 1 below. This extension allows direct applications to partial differential equations of parabolic type, and even to some hyperbolic equations, see [**11**].

It is the aim of this paper to prove the above Morse equation (1) for a Morse decomposition of a compact invariant set in the general setting of a local semiflow on a metric space (theorem 2). Moreover, the equation will be proved for every homology and cohomology theory, not only for the Alexander–Spanier cohomology theory (theorem 3). The crucial step is the construction of an index triple for a repeller–attractor pair, which consists of isolating blocks rather than general index pairs (theorem 2). This result might be of interest in its own right.

The organisation of the paper is as follows. In the first two sections the concept of a Morse decomposition of a compact invariant set is extended to the general setting described in § 1. In § 3 a special index triple is constructed for a repeller–attractor pair. The construction relies on the existence theory of isolating blocks as presented in [**9**]. This result is then used for the proof of the Morse equation in §§ 4, 5.

1. *Set-up and definitions*

We shall consider on a metric space X a continuous local semiflow. This is a continuous map $\phi: D \to X$ where D is an open subset of $\mathbb{R}^+ \times X$ with the property that for every $x \in X$ there is an $\omega_x, 0 < \omega_x \leq \infty$ such that $(t, x) \in D$ if and only if $0 \leq t < \omega_x$. Moreover, abbreviating $\phi(t, x) =: x \cdot t$, a semiflow is required to satisfy:

$$x \cdot 0 = x \qquad \text{for every } x \in X,$$

$$x \cdot (t+s) = (x \cdot t) \cdot s,$$

whenever (t, x), $(s, x \cdot t)$ and $(t+s, x) \in D$.

(1) *Solutions.* Let $x \in X$, then a *left solution through* x is a continuous map $\sigma: I \to X$, with $I = (a, 0]$ for some a in $-\infty \leq a < 0$ such that

(i) $\sigma(0) = x$, and

(ii) for all $t \in I$ and $s > 0$ with $s + t \leq 0$ it follows that $s < \omega_{\sigma(t)}$ and $\sigma(t) \cdot s = \sigma(t+s)$.

If $a = -\infty$ then we call σ a *full left solution.* We can extend a left solution through x onto $I \cup [0, \omega_x)$ by setting $\sigma(t) = x \cdot t$ for $0 \leq t < \omega_x$. The extended σ is then called a *solution* through x, and if $a = -\infty$ and $\omega_x = +\infty$, it is called a *full solution.*

(2) *Invariant sets, isolating neighbourhoods, limit sets.* For a subset $Y \subset X$ we set $A^+(Y) = \{x \in Y | x \cdot [0, \omega_x) \subset Y\}$ and $A^-(Y) = \{x \in Y |$there exists at least one full left solution σ through x satisfying $\sigma(\mathbb{R}^-) \subset Y\}$. Y is then called *positively invariant* if $Y = A^+(Y)$, *negatively invariant* if $Y = A^-(Y)$ and *invariant* if $Y = A^-(Y) = A^+(Y)$.

The following concept is crucial for Conley's index theory. If $N \subset X$ is a closed subset, such that the largest invariant set K contained in N is disjoint from the boundary of N, $K \cap \partial N = \emptyset$, then N is called an *isolating neighbourhood* (of K). K may of course be the empty set. If, on the other hand, K is a closed invariant set for which there is a neighbourhood U of K such that K is the largest invariant set in U, then K is called an *isolated invariant set.*

For a subset $Y \subset X$ satisfying $\omega_x = \infty$ for all $x \in Y$, the ω-limit set of Y is defined to be the set

$$\omega(Y) := \bigcap_{t \geq 0} \mathrm{cl}\,\{Y \cdot [t, \infty)\}.$$

The ω^*-limit set is defined for a full left solution σ through $x \in X$ as

$$\omega^*(\sigma) := \bigcap_{t \geq 0} \mathrm{cl}\,\{\sigma((-\infty, -t])\}.$$

We point out that left solutions are not necessarily unique for one-sided semiflows, so $\omega^*(\sigma)$ depends on the whole left solution σ rather than just on the point x. This contrasts with the situation for two-sided flows, where the set $\omega^*(x)$ is well defined.

(3) *Compactness condition: admissible sets.* The existence statements later on require a compactness condition, which restricts the class of subsets under consideration or the class of flows. A closed subset $N \subset X$ is called *admissible* if

(i) for every two sequences $\{x_n\}$ in N and $\{t_n\}$ in $\mathbb{R}^+$ satisfying $x_n \cdot [0, t_n] \subset N$ and $t_n \to \infty$, it follows that the sequence $\{x_n \cdot t_n\}$ is pre-compact;

(ii) for $x \in N$ with $\omega_x < \infty$ we have $x \cdot [0, \omega_x) \not\subset N$.

(4) *Example.* In order to illustrate the concepts we consider the equation

$$\frac{\partial}{\partial t} u - \Delta u = f(u) \qquad \text{on } \Omega$$

$$u = 0 \qquad \text{on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^n$ having a smooth boundary $\partial\Omega$. In order to formulate the problem as an abstract evolution equation in a Banach space we set

$$W := \{u \in W_p^2(\Omega) | u = 0 \text{ on } \partial\Omega\}$$

for some $p > n$. The operator $A := -\Delta$ with domain $D(A) := W$ generates an analytic semigroup in $L_p(\Omega)$. Set $X^0 = L_p(\Omega)$ and denote its norm by $|\ \ |$. Define for $0 < \alpha \leq 1$ the scale of Banach spaces $X^\alpha := D(A^\alpha)$ with norms $|u|_\alpha := |A^\alpha u|$. Then $\beta > \alpha$ implies that $X^\beta \subset X^\alpha$ continuously and densely. Moreover, as the resolvent of the operator A is compact, this embedding is compact. In addition, $X^\alpha \subset C(\bar{\Omega})$ continuously, provided $\alpha \geq \frac{1}{2}$. For details and references we refer to [1] and [7, chap. 1-3].

Now set $X = X^{\alpha^*}$ for some fixed $\alpha^* > \frac{1}{2}$ and assume the function $f:\mathbb{R}\to\mathbb{R}$ to be locally Lipschitz continuous. It can then be shown by means of the integral equation

$$u(t) = e^{-At}u + \int_0^t e^{-A(t-s)} f(u(s))\, ds$$

and by means of the standard estimates for linear analytic semigroups, that the equation defines a continuous local semiflow on X. We claim that the closed set

$$N := \{u \in X \mid |u|_{C(\bar{\Omega})} \leq C\}$$

for some $C > 0$ is an admissible subset for this local semiflow. Indeed, assume $x_n \in N$ and $t_n \in \mathbb{R}^+$ with $t_n \to \infty$ and with $x_n \cdot [0, t_n] \subset N$. Since the pieces of solutions under consideration are contained in N we may assume that f is bounded. Observe that there is a constant $C_1 > 0$ such that $|u| \leq C_1$ for every $u \in N$. It then follows by means of the integral equation that for every $\beta > 0$ there is a $C_\beta > 0$ such that $|u(t)|_\beta \leq C_\beta$ for all $t \geq 1$ and all initial conditions $u \in N$. Now choose $\beta > \alpha^*$, then in particular

$$|x_n \cdot t_n|_\beta \leq C_\beta \qquad \text{for every } n,$$

hence there is a subsequence converging in $X = X^{\alpha^*}$ as X^β is compactly embedded in X^{α^*}. This proves part (i) of the definition of admissibility. A similar argument proves part (ii).

2. *Morse decompositions*

We shall next extend some concepts and results given in [**5**] for flows to our setting of a semiflow on a metric space X.

Definition 1. Let $S \subset X$ be a compact and invariant subset with $\omega_x = \infty$ for every $x \in S$. A subset $A \subset S$ is called an *attractor* (in S) if there is a neighbourhood U of A such that $\omega(U \cap S) = A$. If A is an attractor, then the set

$$A^* := \{x \in S \mid \omega(x) \cap A = \emptyset\}$$

is called the *repeller dual to* A (relative to S), and the pair (A^*, A) is called a *repeller–attractor pair* in S.

PROPOSITION 1. *Let (A^*, A) be a repeller–attractor pair in S. Then*

(i) *A and A^* are disjoint, compact and invariant*;

(ii) *if $\sigma:\mathbb{R}\to S$ is a full solution through $y \in S$, then the following holds true*:

(a) *if $y \in A^*$ or if $\omega(y) \cap A^* \neq \emptyset$, then $\sigma(\mathbb{R}) \subset A^*$*;

(b) *if $\omega^*(\sigma) \cap A \neq \emptyset$ then $\sigma(\mathbb{R}) \subset A$*;

(c) *if $y \notin A^* \cup A$, then $\omega^*(\sigma) \subset A^*$ and $\omega(y) \subset A$.*

Proof. (i) Clearly $A \cap A^* = \emptyset$, and it follows from definition 1 and from the compactness of S that A is compact and invariant. Hence A^* too is invariant. Assume $x_n \in A^*$ and $x_n \to x \in S$, then $\omega(x) \cap A = \emptyset$. In fact otherwise $x \cdot \tau \in U$ for some $\tau > 0$, with U as in definition 1. Hence $x_n \cdot \tau \in U$ for large n and so $\omega(x_n) = \omega(x_n \cdot \tau) \subset \omega(U \cap S) = A$, contradicting the assumption $\omega(x_n) \cap A = \emptyset$. This proves that A^* is closed, hence compact. In order to prove (ii) we make use of the following simple fact (see [**10**, Lemma 3.1]).

LEMMA 1. *Let (A^*, A) be an attractor-repeller pair in S. Let B be closed and $B \cap A = \emptyset$. Then for every $\varepsilon > 0$ there is a $\tau = \tau(\varepsilon)$ such that for $x \in S$ and $t \geq \tau$ we conclude from $x \cdot t \in B$ that $d(x, A^*) < \varepsilon$, where d denotes distance in the metric space X.*

To prove (a) let $y \in A^*$ or $\omega(y) \cap A^* \neq \emptyset$ and pick a closed neighbourhood B of A^* with $B \cap A = \emptyset$. Then there is a sequence $t_n \to \infty$ such that $\sigma(t_n) \in B$. Let $t \in \mathbb{R}$ and let $\varepsilon > 0$, then $t_n - t \geq \tau(\varepsilon)$ for n large with $\tau(\varepsilon)$ as in the lemma. Since $\sigma(t) \cdot (t_n - t) = \sigma(t_n) \in B$ we conclude $d(\sigma(t), A^*) < \varepsilon$. This holds true for every $\varepsilon > 0$ hence $\sigma(t) \in A^*$. To prove (b) assume $\omega^*(\sigma) \cap A \neq \emptyset$, so that there is a sequence $t_n \to \infty$ with $\sigma(-t_n) \in U \cap S$, U being as in definition 1. Pick $t \in \mathbb{R}$, then for n large $t_n + t \geq 0$, and so $\sigma(t) = \sigma(-t_n) \cdot (t_n + t)$. Therefore $\sigma(t) \in \omega(U \cap S)$, i.e. $\sigma(t) \in A$. Finally, to prove (c) assume that $y \notin A^* \cup A$, and let $x \in \omega^*(\sigma)$, so that $\sigma(-t_n) \to x$ for some sequence $t_n \to \infty$. It then follows from lemma 1 applied to $B = \{y\}$, that for $\varepsilon > 0$, $d(\sigma(-t_n), A^*) < \varepsilon$ if n is sufficiently large, hence $x \in A^*$. If on the other hand $x \in \omega(y)$, then $\sigma(t_n) \to x$ for a sequence $t_n \to \infty$. We claim that for some t_{n_0} we have $\sigma(t_{n_0}) \in U$ with U as in definition 1. In fact otherwise $\sigma(t_n) \in S \setminus U$ for all n, and we conclude by lemma 1, choosing $B = \operatorname{cl}(S \setminus U)$, that $y \in A^*$ contradicting the assumption on y. From $\sigma(t_{n_0}) \in U \cap S$ and $\omega(U \cap S) = A$ we conclude $x \in A$. This completes the proof of proposition 1. □

Definition 2. Let S be a compact and invariant subset of X with $\omega_x = \infty$ for every $x \in S$. An ordered collection $(M_1, \ldots, M_n)$ of subsets $M_j \subset S$ is called a *Morse decomposition* of S, if there exists an increasing sequence

$$\emptyset = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n = S$$

of attractors (in S), such that

$$M_j = A_j \cap A_{j-1}^*, \qquad 1 \leq j \leq n.$$

For example, if A is an attractor in S, then (A, A^*) is a Morse decomposition of S. In fact, set $A_0 = \emptyset$, $A_1 = A$ and $A_2 = S$, then $M_1 = A$ and $M_2 = A^*$.

PROPOSITION 2. *If $(M_1, \ldots, M_n)$ is a Morse decomposition of S, then M_j is compact and invariant, $1 \leq j \leq n$. Moreover, if S has an isolating neighbourhood which is admissible, then the same holds true for the subsets M_j, $1 \leq j \leq n$.*

Proof. Since $M_j = A_j \cap A_{j-1}^*$ we conclude from proposition 1 that M_j is compact and invariant. To prove the second part, let N be an isolating neighbourhood of S. Choose some j, $1 \leq j \leq n$. Since $A_j \cap A_j^* = \emptyset$ there is an $\varepsilon > 0$ such that $d(x, y) \geq \varepsilon$ for $x \in A_j$ and $y \in A_j^*$, $1 \leq j \leq n$. Choose some j, $1 \leq j \leq n$, and choose $0 < \delta \leq \varepsilon/2$ so that $\hat{N} := \operatorname{cl}(U_\delta(M_j)) \subset \operatorname{int}(N)$. Clearly $M_j \subset \operatorname{int}(\hat{N})$. Let K be the largest invariant set contained in $\hat{N}$, then $M_j \subset K \subset S$. Suppose $K \setminus M_j \neq \emptyset$ and choose $y \in K \setminus M_j$ and let $\sigma: \mathbb{R} \to \hat{N}$ be a full solution through y. Since $y \notin M_j$ we have $y \notin A_j$ or $y \notin A_{j-1}^*$. If $y \notin A_j$ then by proposition 1 $\omega^*(\sigma) \subset A_j^*$ and so $A_j^* \cap \hat{N} \neq \emptyset$. Therefore there are $x \in A_j^*$ and $x_0 \in M_1 = A_j \cap A_{j-1}^*$ with $d(x, x_0) \leq \delta$ contradicting $d(x, x_0) \geq \varepsilon$. If on the other hand $y \notin A_{j-1}^*$, then by proposition 1 we have $\omega(y) \in A_{j-1}$ and so $A_{j-1} \cap \hat{N} \neq \emptyset$. It follows that there are $x \in A_{j-1}$ and $x_0 \in M_1 = A_j \cap A_{j-1}^*$ with $d(x, x_0) \leq \delta$, again a contradiction. We conclude that $\hat{N}$ is an isolating neighbourhood of M_j, which, in addition, is admissible if N is admissible. □

PROPOSITION 3. *Let* $(M_1, \ldots, M_n)$ *be a Morse decomposition of* S *and let* $\emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = S$ *be an associated sequence of attractors. Then*:

(i) *The sets* M_j *are pairwise disjoint.*

(ii) *Let* $y \in S$ *and let* $\sigma: \mathbb{R} \to S$ *be any full solution through* y, *then either* $\sigma(\mathbb{R}) \subset M_j$ *for some* j *or there are indices* $i < j$ *such that* $\omega^*(\sigma) \subset M_j$ *and* $\omega(y) \subset M_i$.

(iii) *The attractors are uniquely determined by* $(M_1, \ldots, M_n)$, *namely* $A_k = \{y \in S \mid$ *there is a full solution* $\sigma: \mathbb{R} \to S$ *through* y *with* $\omega^*(\sigma) \subset M_1 \cup \cdots \cup M_k\}$ *for* $1 \leq k \leq n$.

Proof. (i) Let $i < j$, then

$$\begin{aligned} M_i \cap M_j &= A_i \cap A^*_{i-1} \cap A_j \cap A^*_{j-1} \\ &= A_i \cap A^*_{j-1} \subset A_{j-1} \cap A^*_{j-1} = \emptyset, \end{aligned}$$

hence the sets M_j are pairwise disjoint.

(ii) Let $y \in S$ and let $\sigma: \mathbb{R} \to S$ be any full solution through y. Since $A_n = S$ and $A^*_0 = S$ there is a smallest integer, i, such that $\omega(y) \subset A_i$, and there is a largest integer, j, such that $\omega^*(\sigma) \subset A^*_j$. Clearly $i > 0$ and $j < n$. Now $\omega(y) \not\subset A_{i-1}$ hence $y \notin A_{i-1}$ and also $y \in A^*_{i-1}$. In fact, if $y \notin A^*_{i-1}$ then $y \notin A_{i-1} \cup A^*_{i-1}$ and by proposition 1(ii(c)) we conclude $\omega(y) \subset A_{i-1}$, a contradiction. Therefore $\sigma(\mathbb{R}) \subset A^*_{i-1}$ and so $\omega(y) \subset A_i \cap A^*_{i-1}$ since by proposition 1(i) the set A^*_{i-1} is closed. On the other hand $\omega^*(\sigma) \not\subset A^*_{j+1}$ and we claim that $\sigma(\mathbb{R}) \subset A_{j+1}$. In fact, otherwise $\sigma(t) \notin A_{j+1}$ for some $t \in \mathbb{R}$. If now $\sigma(t) \notin A^*_{j+1}$ then by proposition 1(ii(c)) we conclude that $\omega^*(\sigma) \subset A^*_{j+1}$ a contradiction, hence $\sigma(t) \in A^*_{j+1}$ and by proposition 1(ii(a)) we have $\sigma(\mathbb{R}) \subset A^*_{j+1}$ hence $\omega(\sigma) \subset A^*_{j+1}$, again a contradiction. Hence indeed $\sigma(\mathbb{R}) \subset A_{j+1}$. Now $j \geq i-1$, in fact otherwise $j+1 \leq i-1$ and thus $A_{j+1} \subset A_{i-1}$ and therefore $\sigma(\mathbb{R}) \subset A_{i-1} \cap A^*_{i-1} = \emptyset$. If $j = i-1$, then

$$\sigma(\mathbb{R}) \subset A^*_{i-1} \cap A_i = M_i.$$

If $j > i-1$, then

$$\omega(y) \subset A^*_{i-1} \cap A_i = M_i \quad \text{and} \quad \omega^*(\sigma) \subset A^*_j \cap A_{j+1} = M_{j+1}.$$

(iii) Let $y \in A_k$. Since A_k is invariant, there is a full solution $\sigma: \mathbb{R} \to A_k$ through y and so $\omega^*(\sigma) \subset A_k$. Let $i \leq k$ be the smallest integer such that $\omega^*(\sigma) \subset A_i$. Then $i > 0$ and $\omega^*(\sigma) \not\subset A_{i-1}$ and hence $\omega^*(\sigma) \subset A^*_{i-1}$. Therefore

$$\omega^*(\sigma) \subset A_i \cap A^*_{i-1} = M_i \subset (M_1 \cup \cdots \cup M_k).$$

Conversely, suppose that there is a solution $\sigma: \mathbb{R} \to S$ through y such that $\omega^*(\sigma) \subset M_1 \cup \cdots \cup M_k$. Then $\omega^*(\sigma) \subset M_j$ for some $j \leq k$, hence $\omega^*(\sigma) \subset A_j \subset A_k$ and so $\sigma(\mathbb{R}) \subset A_k$ by proposition 1(ii(b)). This completes the proof of proposition 3. □

The above proposition admits the following converse.

PROPOSITION 4. *Let* S *be as in proposition* 3 *and let* $(M_1, \ldots, M_n)$ *be an ordered collection of pairwise disjoint compact and invariant subsets of* S. *Suppose that for* *every $y \in S$ and every full solution $\sigma: \mathbb{R} \to S$ through y either $\sigma(\mathbb{R}) \subset M_j$ for some j or else there are indices $i < j$ such that $\omega^*(\sigma) \subset M_j$ and $\omega(\sigma) \subset M_i$. Then $(M_1, \ldots, M_n)$ is a Morse decomposition of S.*

Proof. Set $A_0 := \emptyset$ and for $1 \le k \le n$ $A_k := \{y \in S \mid \text{there is a full solution } \sigma: \mathbb{R} \to S$ through y satisfying $\omega^*(\sigma) \subset (M_1 \cup \cdots \cup M_k)\}$. We shall show that $A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n = S$ is a sequence of attractors in S such that $A_i \cap A_{i-1}^* = M_i$, thus proving the statement.

Step 1. The sets A_k, $1 \le k \le n$ are closed. Since by definition $A_n = S$, the set A_n is closed. We now proceed inductively and assume A_{k+1} to be closed for some $1 \le k \le n-1$. Let $y_m \in A_k$ with $y_m \to y \in S$. Then $y \in A_{k+1}$ since $A_k \subset A_{k+1}$ and A_{k+1} is closed. There are full solutions $\sigma_m: \mathbb{R} \to S$ with $\sigma_m(0) = y_m$ and $\omega^*(\sigma_m) \subset M_1 \cup \cdots \cup M_k$. Using the compactness of S and the properties of a full solution one finds a subsequence, again denoted by $\{\sigma_m\}$, which converges pointwise to a solution $\sigma: \mathbb{R} \to S$ through y and it remains to prove that $\omega^*(\sigma) \subset (M_1 \cup \cdots \cup M_k)$. Indeed, since $\sigma_m(\mathbb{R}) \subset A_k \subset A_{k+1}$ and A_{k+1} is closed, we conclude $\sigma(\mathbb{R}) \subset A_{k+1}$ and so $\omega^*(\sigma) \subset A_{k+1}$. Observe that $M_j \cap A_{k+1} = \emptyset$ for $j > k+1$ since M_j is invariant. On the other hand $\omega^*(\sigma) \subset M_j$ for some j by our assumptions and therefore

$$\omega^*(\sigma) \subset M_1 \cup \cdots \cup M_k \cup M_{k+1}.$$

Hence either $\omega^*(\sigma) \subset M_1 \cup \cdots \cup M_k$ in which case we are done, or else $\omega^*(\sigma) \subset M_{k+1}$. In the latter case, let $V \supset M_{k+1}$ be an open neighbourhood of M_{k+1} such that $\mathrm{cl}(V) \cap M_j = \emptyset$ for $j \ne k+1$. There is a sequence $t_\nu \to \infty$ and a $z \in M_{k+1}$ such that $\sigma(-t_\nu) \in V$ and $d(\sigma(-t_\nu), z) \le \nu^{-1}$ for all $\nu \ge 1$. Therefore, for every ν there is an $m_\nu \ge \nu$ such that $\sigma_{m_\nu}(-t_\nu) \in V$ and

$$d(\sigma_{m_\nu}(-t_\nu), z) \le 2 \cdot \nu^{-1}.$$

Since $\omega^*(\sigma_m) \cup \omega(\sigma_m) \subset (M_1 \cup \cdots \cup M_k)$ for every m, there are $\tau_\nu < t_\nu < s_\nu$ such that $\sigma_{m_\nu}(-s_\nu)$ and $\sigma_{m_\nu}(-\tau_\nu) \in \partial V$ and $\sigma_{m_\nu}(-t) \in \mathrm{cl}(V)$ for $\tau_\nu \le t \le s_\nu$. The invariance of M_{k+1} now implies that $t_\nu - \tau_\nu \to \infty$. Let $x_\nu := \sigma_{m_\nu}(-s_\nu)$, then $x_\nu \in S$ and since S is compact we may assume $x_\nu \to x \in \partial V$. It then follows that $x \cdot t \in \mathrm{cl}(V)$ for all $t \ge 0$ and so $\omega(x) \in \mathrm{cl}(V)$ which implies by our hypothesis that $\omega(x) \subset M_{k+1}$. Since A_{k+1} is closed we have $x \in A_{k+1}$ and so there is a full solution $\tilde{\sigma}: \mathbb{R} \to S$ through x with $\omega^*(\tilde{\sigma}) \subset M_1 \cup \cdots \cup M_{k+1}$. From the ordering of the sets M_j it follows that $\tilde{\sigma}(\mathbb{R}) \subset M_{k+1}$ hence $x \in M_{k+1}$. This however contradicts $x \in \partial V$, since $M_{k+1} \cap \partial V = \emptyset$. Hence step 1 is proved.

Step 2. A_k is an attractor in S, $1 \le k \le n$. This, in fact, is true for $k = n$. We proceed by induction and assume A_{k+1} to be an attractor in S for some $k \le n-1$. Choose a neighbourhood $U_{k+1} \supset A_{k+1}$ of A_{k+1} such that $\omega(U_{k+1} \cap S) = A_{k+1}$. Since A_k is closed, $M_{k+1} \cup A_k \subset A_{k+1}$ and $M_{k+1} \cap A_k = \emptyset$, we can choose a neighbourhood U_k of A_k and a neighbourhood V of M_{k+1} both contained in U_{k+1} such that $\mathrm{cl}(U_k) \cap \mathrm{cl}(V) = \emptyset$. Since A_k is invariant and contained in U_k we have $A_k \subset \omega(U_k \cap S)$. Suppose $\omega(U_k \cap S) \setminus A_k \ne \emptyset$, and choose $y \in \omega(U_k \cap S) \setminus A_k$, then there are sequences $x_n \in U_k \cap S$ and $t_n \to \infty$ such that $x_n \cdot t_n \to y$. We may assume that $x_n \cdot (t_n + t) \to \sigma(t)$

for every $t\in\mathbb{R}$ with a solution $\sigma:\mathbb{R}\to S$ through y. With

$$\omega(U_k\cap S)\subset\omega(U_{k+1}\cap S)=A_{k+1}$$

we conclude $\sigma(\mathbb{R})\subset A_{k+1}$, hence, by step 1, $\omega^*(\sigma)\subset A_{k+1}$ and so $\omega^*(\sigma)\subset(M_1\cup\cdots\cup M_{k+1})$. But $y\notin A_k$ and hence $\omega^*(\sigma)\subset M_{k+1}$. There is a sequence $\rho_\nu\to\infty$ and a $z\in M_{k+1}$ such that $\sigma(-\rho_\nu)\in V$ and $d(\sigma(-\rho_\nu), z)\le\nu^{-1}$ for every ν. Therefore for every ν there is $n_\nu\ge\nu$ such that $t_{n_\nu}>\rho_\nu$ and $x_{n_\nu}\cdot(t_{n_\nu}-\rho_\nu)\in V$ and $d(x_{n_\nu}\cdot(t_{n_\nu}-\rho_\nu), z)\le 2\nu^{-1}$. We will show that by choosing U_k small enough, we can arrange that $\omega(U_k\cap S)=A_k$. In fact, if this is not true, then there is a sequence $\delta_\nu\to 0$ such that $\mathrm{cl}\,(U_{\delta_\nu}(A_k))\cap\mathrm{cl}\,(V)=\varnothing$, and $U_{\delta_\nu}(A_k)\subset U_{k+1}$ and $\omega(U_{\delta_\nu}(A_k)\cap S)\setminus A_k\neq\varnothing$, where $U_{\delta_\nu}(A_k)$ is the δ_ν-neighbourhood of A_k. Using what we have proved thus far, it is easily seen that there are sequences $x_\nu\in U_{\delta_\nu}(A_k)$, $s_\nu>0$ such that $x_\nu\cdot s_\nu\in V$ and $d(x_\nu\cdot s_\nu, M_{k+1})\le 2\nu^{-1}$. There are sequences $\tau_\nu<s_\nu<\tilde\tau_\nu\le\infty$ such that $x_\nu\cdot\tau_\nu\in\partial V$, $x_\nu\cdot[\tau_\nu,\tilde\tau_\nu)\subset\mathrm{cl}\,(V)$ and either $\tilde\tau_\nu=\infty$ or $x_\nu\cdot\tilde\tau_\nu\in\partial V$. Set $\hat x_\nu=x_\nu\cdot\tau_\nu$, then we may assume $\hat x_\nu\to\hat x\in S$. The invariance of A_k and $x_\nu\to A_k$ easily imply $\tau_\nu\to\infty$, hence $\hat x\in\omega(U_{k+1}\cap S)=A_{k+1}$. On the other hand, $x_\nu\cdot s_\nu\to M_{k+1}$ and the invariance of M_{k+1} imply $\tilde\tau_\nu\to\infty$, hence $\hat x\cdot[0,\infty)\subset\mathrm{cl}\,(V)$. Therefore $\omega(\hat x)\subset M_{k+1}$ and $\hat x\in A_{k+1}$. Now this obviously implies $\hat x\in M_{k+1}$, a contradiction since $\hat x\in\partial V$. Hence, indeed, U_k can be chosen such that $\omega(U_k\cap S)=A_k$, i.e. A_k is an attractor.

Step 3. $M_j=A_j\cap A_{j-1}^*$. Indeed, if $y\in M_j$, then there is a solution $\sigma:\mathbb{R}\to M_j$ through y and therefore $y\in A_j$. Suppose $y\notin A_{j-1}^*$, then $\omega(y)\subset A_{j-1}$ and therefore $\omega(y)\subset M_k$ for some $k\le j-1$. Since $\omega(y)\subset M_j$ we find $\omega(y)\subset M_k\cap M_j=\varnothing$, a contradiction. Hence $M_j\subset A_j\cap A_{j-1}^*$. If $y\in A_j\cap A_{j-1}^*$, then there is a solution $\sigma:\mathbb{R}\to S$ through y such that $\omega^*(\sigma)\subset M_1\cup\cdots\cup M_j$. From $y\in A_{j-1}^*$ we conclude $\omega(y)\cap(M_1\cup\cdots\cup M_{j-1})=\varnothing$ and hence $\omega(y)\subset M_k$ for some $k\ge j$. But then by the ordering $k=j$ and $\sigma(\mathbb{R})\subset M_j$, hence in particular $y\in M_j$, completing the proof of the proposition. □

Proposition 5. *Let $(M_1,\dots,M_n)$ be a Morse decomposition of a compact isolated invariant set S, and let $\varnothing=A_0\subset A_1\subset\cdots\subset A_n=S$ be the associated sequence of attractors. Let $1\le j\le n$, then A_j is a compact invariant and isolated set and (M_j, A_{j-1}) is a repeller–attractor pair in A_j. If S admits an admissible isolating neighbourhood, then so does A_j.*

Proof. A_j is clearly invariant and compact, and since $M_j=A_j\cap A_{j-1}^*$ the pair (M_j, A_{j-1}) is a repeller-attractor pair in A_j. To show that A_j is isolated let N be an isolating neighbourhood of S. Since $A_j\cap A_j^*=\varnothing$ we can choose a closed neighbourhood $\hat N$ of A_j such that $\hat N\subset N$ and $\hat N\cap A_j^*=\varnothing$. Let K be the largest invariant set contained in $\hat N$, then $A_j\subset K\subset S$. We claim that $A_j=K$. In fact suppose $K\setminus A_j\neq\varnothing$ and pick $y\in K\setminus A_j$ and let $\sigma:\mathbb{R}\to K$ be a full solution through y. Since $y\notin A_j\cup A_j^*$ we conclude from proposition 1 that $\omega^*(\sigma)\subset A_j^*$ and hence $A_j^*\cap\hat N\neq\varnothing$ in contradiction to the choice of $\hat N$. Since $\hat N\subset N$, $\hat N$ is admissible if N is admissible. □

3. *Isolating blocks and index triples*

The aim of this section is the construction of a very special index triple for an attractor-repeller pair. It will be built by means of isolating blocks rather than

https://doi.org/10.1017/S0143385700002790 Published online by Cambridge University Press

general index pairs as it is done for our setting in [**10**]. This special construction will allow us later on to formulate the Morse theory for arbitrary (co)homology modules and not only for the Alexander–Spanier cohomology theory. We first recall some of the relevant concepts from [**9**].

Let $B \subset X$ be a closed set and $x \in \partial B$ a boundary point. Then x is called a *strict egress* (resp. *strict ingress*, resp. *bounce-off*) *point of* B, if for every solution $\sigma:[-\delta_1, \delta_2] \to X$ through $x = \sigma(0)$, with $\delta_1 \geq 0$ and $\delta_2 > 0$ there are $0 \leq \varepsilon_1 \leq \delta_1$ and $0 < \varepsilon_2 \leq \delta_2$ such that for $0 < t \leq \varepsilon_2$:

$$\sigma(t) \notin B \qquad (\text{resp. } \sigma(t) \in \operatorname{int}(B), \text{resp. } \sigma(t) \notin B),$$

and for $-\varepsilon_1 \leq t < 0$:

$$\sigma(t) \in \operatorname{int}(B) \qquad (\text{resp. } \sigma(t) \notin B, \text{resp. } \sigma(t) \notin B).$$

We use B^e (resp. B^i, resp. B^b) to denote the set of all strict egress (resp. strict ingress, resp. bounce-off) points of the closed set B. We finally set $B^+ = B^i \cup B^b$ and $B^- = B^e \cup B^b$.

Definition 3. A closed set $B \subset X$ is called an *isolating block* if

(i) $\partial B = B^e \cup B^i \cup B^b$

(ii) B^- is closed.

Remark. We point out that in the definition of an isolating block in [**9**, definition 2.1] only condition (i) is required. However, the isolating block constructed in [**9**, theorem 2.1] meets conditions (i) and (ii), and actually only such blocks occur in [**9**].

If $S \subset B$ is the largest invariant set in the isolating block B, then clearly S is an isolated invariant set and B is an isolating neighbourhood of S. Therefore B is called an *isolating block for S*. If B is an isolating block for S, then the pair (B, B^-) of closed spaces is an example of a so-called index pair for S. This concept, which is stable under perturbation of the semiflow, is defined as follows.

Definition 4. If $S \subset X$ is an isolated invariant set and if N_1 and N_2 with $N_2 \subset N_1$ are two closed sets in X, then the pair (N_1, N_2) of closed spaces is called an *index pair for* S (in N_1) if the following properties are satisfied:

(1) N_1 is an isolating neighbourhood of S;

(2) $S \subset \operatorname{int}(N_1 \setminus N_2)$;

(3) N_2 is positively invariant relative to N_1, i.e. if $x \in N_1$ and $x \cdot [0, t] \subset N_2$ for some $t > 0$, then $x \cdot [0, t] \subset N_1$;

(4) if $x \in N_1$ and $x \cdot \tau \notin N_1$ for some $0 < \tau < \omega_x$, then there is a $t < \tau$ such that $x \cdot [0, t] \subset N_1$ and $x \cdot t \in N_2$.

We point out that there is always an admissible index pair for S in fact even an isolating block for S, provided there exists an admissible isolating neighbourhood for S (see [**9**, theorem 2.1]). We call the index pair (N_1, N_2) *admissible*, if N_1 is an admissible set.

Definition 5. Let S be an isolated invariant set and let (A^*, A) be a repeller–attractor pair in S. Let N_1, N_2 and N_3 be three closed sets satisfying $N_1 \supset N_2 \supset N_3$. Then (N_1, N_2, N_3) is called an *index triple for* (A^*, A) *relative to* S, if the following properties hold:

(1) (N_1, N_3) is an index pair for S;

(2) (N_2, N_3) is an index pair for A;

(3) If U is a relatively open set in N_1 satisfying $A \subset U \subset N_2$ then $(N_1 \setminus U, N_2 \setminus U)$ is an index pair for A^*.

We are ready to formulate the result of this section.

THEOREM 1. *Let S be an isolated compact and invariant subset of X and let (A^*, A) be a repeller-attractor pair in S. Assume there exists an isolating neighbourhood N of S which is admissible. Then there are two sets B_1 and B_2 in N such that*:

(i) *B_1 is an isolating block for A^*, B_2 is an isolating block for A and $B := B_1 \cup B_2$ is an isolating block for S.*

(ii) *$(B, B_2 \cup B^-, B^-)$ is an index triple for (A^*, A) relative to S.*

(iii) $B_1 \cap B_2 \subset B_1^- \cap B_2^+$.

Schematically:

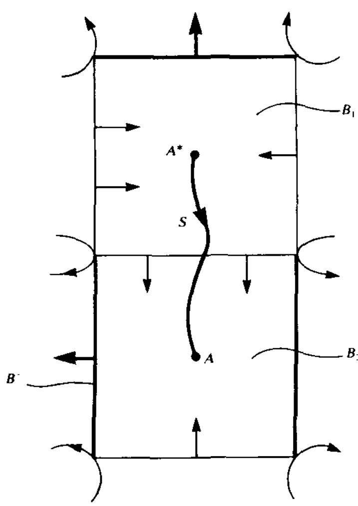

FIGURE 1

Proof. We may assume $A \neq \varnothing$ and $A^* \neq \varnothing$, otherwise the statement is an immediate consequence of [9, theorem 2.1]. Since $A \cap A^* = \varnothing$ we may choose a closed isolating neighbourhood $N_1 \subset N$ of A^* satisfying $N_1 \cap A = \varnothing$. We set $U = \operatorname{int}(N_1)$ so that $A^* \subset U \subset N_1$.

We shall need the function $g = g_U^+ : U \to \mathbb{R}^+$ introduced in [9]. It is defined as follows. Let

$$G(x) = d(x, A^*) \cdot \{d(x, A^*) + d(x, X \setminus N_1)\}^{-1}$$

and define $t^+ : U \to \mathbb{R}^+ \cup \{\infty\}$ by

$$t^+(x) = \sup \{t \mid 0 \leq t < \omega_x \text{ and } x \cdot [0, t] \subset N_1\}.$$

Set for $x \in U$:

$$g(x) = \inf \{(1+t)^{-1} G(x \cdot t) \mid 0 \leq t < t^+(x)\}.$$

By [9, lemma 2.1] we can find open sets V and W satisfying $A^* \subset V \subset \text{cl}(V) \subset W \subset \text{cl}(W) \subset U$, such that $g|\text{cl}(W)$ is continuous. For $\varepsilon > 0$ we define

$$G_\varepsilon = \text{cl}\{y \in V | g(y) < \varepsilon\}.$$

Clearly $A^* \subset \text{int}(G_\varepsilon)$. We claim.

LEMMA 2. *There is an isolating neighbourhood $\hat{N} \subset N$ of S such that $\hat{N} \cap \partial G_\varepsilon \subset V$ for some $\varepsilon > 0$.*

Proof of lemma 2. We first claim that there is an $\varepsilon > 0$, such that

$$(S \setminus A) \cap \partial G_\varepsilon \subset V.$$

Indeed, otherwise there are sequences $\varepsilon_n \to 0$ and $x_n \to x$ with $x_n \in (S \setminus A) \cap \partial G_{\varepsilon_n} \cap \partial V$. It then follows from the continuity of $g|\text{cl}(W)$ that $x \in (S \setminus A) \cap \partial V$ and $g(x) = 0$. Therefore by [9, lemma 2.1] $\omega(x) \subset A^*$ and hence, by proposition 1, $x \in A^*$, contradicting $A^* \cap \partial V = \varnothing$. Now let $\varepsilon > 0$ be as in the above claim and assume the lemma not to be true. Then there is a sequence $x_n \in \partial G_\varepsilon \cap \partial V$ such that $x_n \to x \in S$ and so $x \in \partial G_\varepsilon \cap \partial V \cap S$. Since $A \cap \partial V = \varnothing$ we conclude $x \notin A$ and hence $x \in S \setminus A$, which contradicts the claim. □

In view of lemma 2 we conclude from [9, theorem 2.1]:

LEMMA 3. *There exists an isolating block $B \subset N$ for S satisfying $(B \cap \partial G_\varepsilon) \subset V$ for some $\varepsilon > 0$.*

We now set

$$B_1 := B \cap G_\varepsilon \quad \text{and} \quad B_2 := \text{cl}(B \setminus G_\varepsilon)$$

and verify that for these two sets all the requirements of theorem 1 are satisfied.

LEMMA 4. *B_1 is an isolating block for A^*, and $B_1^- = \partial B_1 \cap (B^- \cup \partial G_\varepsilon)$.*

Proof. To prove that B_1 is an isolating neighbourhood of A^* assume K_1 to be the largest invariant set in B_1. Since $A^* \subset \text{int}(B) \cap \text{int}(G_\varepsilon) \subset B_1$ we have $A^* \subset K_1$. On the other hand, G_ε is an isolating neighbourhood of A^* hence $K_1 \subset A^*$ and so $K_1 = A^*$. In order to verify that B_1 is an isolating block, let $x \in \partial B_1$ and let $\sigma: [-\delta_1, \delta_2] \to X$ be a solution through x, where $-\delta_1 \leq 0 < \delta_2$.

(a) Assume $x \in \text{int}(G_\varepsilon)$. If δ_1 and δ_2 are small, and $-\delta_1 \leq t \leq \delta_2$, then $\sigma(t) \in \text{int}(B)$ implies $\sigma(t) \in \text{int}(B_1)$ and $\sigma(t) \in X \setminus B$ implies $\sigma(t) \in X \setminus B_1$. Therefore $x \in B^i$ (resp. $x \in B^e$, resp. $x \in B^b$) implies $x \in B_1^i$ (resp. $x \in B_1^e$, resp. $x \in B_1^b$).

(b) Assume $x \in \partial G_\varepsilon$. If δ_1 and δ_2 are sufficiently small, then $\sigma(t) \in V$ for $-\delta_1 \leq t \leq \delta_2$ by lemma 3, hence $\sigma(t)$ is in the domain of definition of g. It follows from [9, lemma 2.1], that $g(\sigma(t)) < \varepsilon$ if $t < 0$, and $g(\sigma(t)) = \varepsilon$ if $t = 0$, and $g(\sigma(t)) > \varepsilon$ if $t > 0$. Consequently $\sigma([-\delta_1, 0)) \subset \text{int}(G_\varepsilon)$ and $\sigma((0, \delta_2]) \subset X \setminus G_\varepsilon$. Hence $x \in \text{int}(B) \cup B^e$ (resp. $x \in B^i \cup B^b$) implies $x \in B_1^e$ (resp. $x \in B_1^b$). We conclude that B_1 is an isolating block for A^*, and that

$$B_1^- = \partial B_1 \cap (B^- \cup \partial G_\varepsilon).$$

The proof of the lemma is finished. □

LEMMA 5. *B_2 is an isolating block for A and $B_2^- = B^- \cap \partial B_2$.*

Proof. To prove that B_2 is an isolating neighbourhood of A, let K_2 be the largest invariant set in B_2. From $A \subset \operatorname{int}(B) \cap (X \setminus G_\varepsilon) \subset B_2$ we find $A \subset K_2$. Since (A, A^*) is a Morse decomposition of S we have $K_2 \subset A$, hence $K_2 = A$. To verify that B_2 is an isolating block, choose $x \in \partial B_2$ and a solution $\sigma: [-\delta_1, \delta_2] \to X$ through x, with $-\delta_1 \le 0 < \delta_2$.

(a) Assume $x \in (X \setminus G_\varepsilon)$. Then $x \in \partial B$. If δ_1, δ_2 are small then $\sigma(t) \in \operatorname{int}(B)$ implies $\sigma(t) \in \operatorname{int}(B_2)$ and $\sigma(t) \in X \setminus B$ implies $\sigma(t) \in X \setminus B_2$. Therefore $x \in B^i$ (resp. $x \in B^e$, resp. $x \in B^b$) implies that $x \in B_2^i$ (resp. $x \in B_2^e$, resp. $x \in B_2^b$).

(b) Assume $x \in G_\varepsilon$. Then $x \in \partial G_\varepsilon$ and as in the proof of the previous lemma we conclude $\sigma([-\delta_1, 0)) \subset \operatorname{int}(G_\varepsilon)$ and $\sigma((0, \delta_1]) \subset (X \setminus G_\varepsilon)$. Therefore $x \in \operatorname{int}(B) \cup B^i$ (resp. $x \in B^e \cup B^b$) implies that $x \in B_2^i$ (resp. $x \in B_2^b$). It follows that B_2 is an isolating block with $B_2^- = B^- \cap \partial B_2$. □

LEMMA 6. $B_1 \cap B_2 \subset B_1^- \cap B_2^+$.

Proof. Clearly $B_1 \cap B_2 \subset B \cap \partial G_\varepsilon$, we show that $B_1 \cap B_2 \subset \partial B_1 \cap \partial B_2$. In fact assume $x \in (B_1 \cap B_2)$ and $x \in \operatorname{int} B_1$. Then there is a sequence $x_n \in B \setminus G_\varepsilon$ with $x_n \to x$, hence for n large $x_n \in \operatorname{int} B_1$ which is contained in $\operatorname{int}(G_\varepsilon)$ a contradiction, hence $B_1 \cap B_2 \subset \partial B_1$. If on the other hand $x \in \operatorname{int} B_2$, then $x \in \partial G_\varepsilon$ and there is a sequence $x_n \to x$ with $x_n \in \operatorname{int}(G_\varepsilon)$, therefore if n is large $x_n \in \operatorname{int} B_2 \cap \operatorname{int} G_\varepsilon$ and so $x_n \notin \operatorname{cl}(B \setminus G_\varepsilon) = B_2$ a contradiction, hence $B_1 \cap B_2 \subset \partial B_2$. The statement now follows from the parts (b) in the proofs of lemmas 4 and 5. □

LEMMA 7. $(B, B_2 \cup B^-, B^-)$ *is an index triple for* (A^*, A) *relative to* S.

Proof. Clearly (B, B^-) is an index pair for S. We prove that $(B_2 \cup B^-, B^-)$ is an index pair for A. Clearly B^- is positively invariant relative to $B_2 \cup B^-$, also $A \subset \operatorname{int}((B_2 \cup B^-) \setminus B^-)$. Now, let $x \in B_2 \cup B^-$ and $x \cdot t \notin B_2 \cup B^-$ for some $t > 0$ and set

$$\tau = \sup\{s \mid x \cdot [0, s] \subset B_2 \cup B^-\}.$$

Then $x \cdot \tau =: y \in \partial(B_2 \cup B^-)$. We have to show that $y \in B^-$. Assume $y \notin B^-$. Then obviously $y \in \partial B_2$ and since B_2 is an isolating block we conclude $y \in B_2^- = B^- \cap \partial B_2$, a contradiction. We have verified that $(B_2 \cup B^-, B^-)$ is an index pair for A.

Finally, let U be any open set with $A \subset (U \cap B) \subset B_2 \cup B_-$. It remains to prove that $(B \setminus U, (B_2, \cup B^-) \setminus U)$ is an index pair for A^*. We claim that $B \setminus U$ is an isolating neighbourhood of A^*. Indeed, by proposition 1, the ω-limit set of every trajectory not in A^* is in A, and since $A^* \subset (\operatorname{int} G_\varepsilon) \cap \operatorname{int} B$ and

$$\operatorname{int} G_\varepsilon \cap \operatorname{int} B \cap U \subset \operatorname{int} G_\varepsilon \cap B_2 = \varnothing$$

the claim follows. We claim next that $(B_2 \cup B^-) \setminus U$ is positively invariant relative to $B \setminus U$. In fact, let $x \in (B_2 \cup B^-) \setminus U$ and $x \cdot [0, t] \subset B \setminus U$. Suppose $x \cdot t \notin B_2 \cup B^-$ and define τ as above. We conclude that $\tau < t$ and $x \cdot \tau \in B^-$ contradicting the fact that $x \cdot [0, t] \subset B$. Hence the claim is proved. Clearly:

$$A^* \subset \operatorname{int}(B \setminus (B_2 \cup B^-)) = \operatorname{int}\{(B \setminus U) \setminus ((B_2 \cup B^-) \setminus U)\}.$$

In order to verify the last condition of an index pair, let $x \in B \setminus U$ and $x \cdot \tau \notin B \setminus U$ for some $\tau > 0$. Set

$$s := \sup\{t \mid x \cdot [0, t] \subset B \setminus U\},$$

https://doi.org/10.1017/S0143385700002790 Published online by Cambridge University Press

and put $y = x \cdot s$. Then $y \in \partial(B \setminus U)$ and we have to show that $y \in (B_2 \cup B^-) \setminus U$. We distinguish three cases.

(1) $y \in \operatorname{int} B \cap \partial U$. Then there is a sequence $x_n \to y$ with $x_n \in U$ and $x_n \in \operatorname{int} B$. Hence $x_n \in (U \cap B) \subset B_2 \cup B^-$ and so $y \in (B_2 \cup B^-) \setminus U$.

(2) $y \in \partial B \setminus \operatorname{cl}(U)$. Then $y \in B^-$, for otherwise $y \in B^i$ and therefore there is a $t > 0$ such that $y \cdot [0, t] \subset B \setminus U$, contradicting the definition of y. Therefore $y \in B^- \setminus U$.

(3) $y \in \partial B \cap \partial U$. Then either $y \in B^-$ and hence $y \in B^- \setminus U$ or $y \in B^i$. In the latter case $y \cdot [0, t^*] \subset B$ for some $t^* > 0$. Hence by the definition of y there is a sequence $s_n \to 0$ such that

$$y \cdot s_n \in (B \cap U) \subset B_2 \cup B^-$$

which implies $y \in B_2$ and since $y \in \partial U$ we again find $y \in (B_2 \cup B^-) \setminus U$. This concludes the proof of the lemma. □

The proof of theorem 1 is complete.

4. *The Morse equation for the Alexander–Spanier cohomology*

On a metric space X we consider a continuous local semi-flow. Following [**9**] we single out the following family $\mathcal{S}$ of isolated invariant sets:

$$\mathcal{S} := \{S \subset X \mid S \text{ is a compact, isolated invariant set, for which there exists an admissible isolating neighbourhood}\}$$

For $S \in \mathcal{S}$ an index can be defined, which is the homotopy type of a pointed topological space. This is done as follows. If $S \in \mathcal{S}$ then there exists an admissible index pair (N_1, N_2) for S (see [**9**, theorem 2.1]). Moreover, if $(\bar{N}_1, \bar{N}_2)$ is any other admissible index pair for S, then, by [**9**, theorem 4.1], the pairs

$$(N_1/N_2, [N_2]) \simeq (\bar{N}_1/\bar{N}_2, [\bar{N}_2]) \tag{4.1}$$

are homotopically equivalent. We recall that if (A, B) is a pair of closed sets in X with $A \supset B$, then A/B is the quotient space of A obtained by collapsing all points of B to one point. More precisely A/B is the quotient space $A/\sim$ where the equivalence relation identifies all points in B: if $x_1, x_2 \in A$ then $x_1 \sim x_2$ if either $x_1 = x_2 \in A \setminus B$, or x_1 and x_2 are in B. As a set A/B is the disjoint union of $A \setminus B$ and a distinguished point $[B]$.

Now, by (4.1) we can associate to $S \in \mathcal{S}$ the homotopy type, denoted by $h(S)$, of a pair $(N_1/N_2, [N_2])$, where (N_1, N_2) is any admissible index pair for S:

$$h(S) := [(N_1/N_2, [N_2])]. \tag{4.2}$$

The algebraic invariants of S to be considered in this section are the Alexander–Spanier cohomology modules. For a closed pair (A, B) with $A \supset B$ in X we denote by $H^q(A, B) = H^q(A, B; G)$ the cohomology modules, where G is an R-module, R being an integral domain. (See [**13**, chap. 8].) Assuming the ranks of the modules to be finite we set

$$r_q(A, B) = \operatorname{rank} H^q(A, B). \tag{4.3}$$

The algebraic invariants of $S \in \mathcal{S}$ can then be represented by the following formal power series having non-negative integer coefficients

$$p(t, h(S)) = \sum_{q \geq 0} r_q t^q, \tag{4.4}$$

where $r_q = \operatorname{rank} H^q(N_1/N_2, [N_2])$, (N_1, N_2) being any admissible index pair for S. We point out that by (4.1) the right hand side of (4.4) does not depend on the particular index pair chosen.

As an illustration we consider the flow of the linear equation $\dot{x} = Ax$ in a Banach space E, where $A \in \mathcal{L}(E)$ is continuous. We shall assume the spectrum of A to be bounded away from the imaginary axis. Then $E = E_+ \oplus E_-$ with two closed subspaces E_+ and E_- which are invariant under A and which have the property that the real part of the spectrum of $A|E_-$ (resp. $A|E_+$) is positive (resp. negative). It is easily seen that $0 \in E$ is an isolated invariant set for the flow $\phi(t, x) = e^{tA}x$. If $\dim E_- < \infty$, then every closed and bounded neighbourhood of 0 is an admissible isolating neighbourhood. It is then easy to see that with $d := \dim E_- < \infty$, the homotopy type of 0 is given by

$$h(\{0\}) = [(S^d, p)],$$

where S^d is the d-dimensional sphere and $p \in S^d$ a distinguished point. Therefore $p(t, h(\{0\})) = t^d$.

Now let $S \in \mathcal{S}$ and let $(M_1, \dots, M_n)$ be a Morse decomposition of S with associated sequence of attractors $\emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = S$. We have seen in proposition 2 and proposition 5, that $A_j \in \mathcal{S}$ and also $M_j \in \mathcal{S}$. Hence $h(S)$, $h(M_j)$ and $h(A_j)$ are defined. The Morse equation now relates the algebraic invariants of S to the algebraic invariants of the elements M_j of the Morse decomposition. The result is the following:

THEOREM 2. *Let $S \in \mathcal{S}$ and let $(M_1, \dots, M_n)$ be a Morse decomposition of S with associated sequence $\emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = S$ of attractors. Assume the modules $H^q(h(S))$, $H^q(h(A_j))$ and $H^q(h(M_j))$ to be of finite rank for $q \geq 0$ and $1 \leq j \leq n$. Then*
(i)

$$\sum_{j=1}^{n} p(t, h(M_j)) = p(t, h(S)) + (1+t)Q(t),$$

where $Q(t)$ is a formal power series in t having only non-negative integer coefficients. $Q(t) = \sum_{j=1}^{n} Q_j(t)$ where:

$$(1+T)Q_j(t) = p(t, h(M_j)) + p(t, h(A_{j-1})) - p(t, h(A_j)).$$

(ii) *If $Q_j(t) \neq 0$, then there is a solution $\sigma: \mathbb{R} \to S$ such that $\omega^*(\sigma) \subset M_j$ and $\omega(\sigma) \subset M_i$ for some $i < j$.*

In view of the last statement, the terms Q measure the number of cohomologically non-trivial connections between pairs M_i and M_j, $i \neq j$.

Proof. We make use of some cohomology theory. If $A \supset B \supset C$ are closed subsets of a topological space X we have an exact sequence of R-modules and

https://doi.org/10.1017/S0143385700002790 Published online by Cambridge University Press

homomorphisms

$$\cdots \xrightarrow{\gamma_{q-1}} H^q(A, B) \xrightarrow{\alpha_q} H^q(A, C) \xrightarrow{\beta_q} H^q(B, C) \xrightarrow{\gamma_q} H^{q+1}(A, B) \longrightarrow \cdots \quad (4.5)$$

where α_q and β_q are induced by the corresponding inclusion maps of pairs. We also set $H^q = 0$ for $q < 0$. If the ranks of the modules are finite we set $r_q(A, B) = \operatorname{rank} H^q(A, B)$ and $d_q(A, B, C) = \operatorname{rank}(\operatorname{im} \gamma_q)$, and introduce the formal power series:

$$\begin{aligned} p(t, A, B) &= \sum_{q\geq 0} r_q(A, B)t^q \\ Q(t, A, B, C) &= \sum_{q\geq 0} d_q(A, B, C)t^q. \end{aligned} \quad (4.6)$$

Lemma 8. *Let $A \supset B \supset C$ be closed subsets. Assume the cohomology modules to be of finite rank, then*:

$$p(t, A, B) + p(t, B, C) = p(t, A, C) + (1+t)Q(t, A, B, C).$$

Proof of lemma 8. We first make a simple observation. Consider any sequence of R-modules and homomorphisms:

$$\cdots \xrightarrow{f_1} E_2 \xrightarrow{f_2} E_3 \xrightarrow{f_3} E_4 \xrightarrow{f_4} \cdots$$

and assume the sequence to be exact and the modules E_k, $2 \leq k \leq 4$ of finite rank, then

$$\operatorname{rank}(\operatorname{im} f_1) + \operatorname{rank}(\operatorname{im} f_4) = \operatorname{rank}(E_2) - \operatorname{rank}(E_3) + \operatorname{rank}(E_4). \quad (4.7)$$

In fact if E is of finite rank and $F \subset E$ is a submodule then both F and E/F, the quotient module, are of finite rank, and $\operatorname{rank}(E) = \operatorname{rank}(F) + \operatorname{rank}(E/F)$. From this remark (4.7) follows easily. In order to prove the lemma we conclude by this observation from the exactness of the sequence (4.5) that

$$r_q(A, B) - r_q(A, C) + r_q(B, C) = d_q(A, B, C) + d_{q-1}(A, B, C).$$

If we multiply by t^q and sum over $q \geq 0$ and observe that $H^q = \{0\}$ for $q < 0$, we obtain the assertion. □

We now use the fact that we are dealing with the Alexander-Spanier cohomology theory and recall (see [**8**, theorem 8.7]):

Lemma 9. *Let (A, B) with $A \supset B$ be a pair of closed subsets of a metric space. Then the projection map $p:(A, B) \to (A/B, [B])$ induces the isomorphisms*:

$$H^q(A, B) \cong H^q(A/B, [B]), \qquad q \geq 0$$

for the Alexander-Spanier cohomology.

This lemma allows us to express the algebraic invariants of $h(S)$ up to isomorphisms in terms of the invariants of any admissible index pair (N_1, N_2) of S. In particular we conclude from (4.4) that

$$p(t, h(S)) = \sum_{q\geq 0} \operatorname{rank} H^q(N_1, N_2)t^q, \quad (4.8)$$

with an index pair (N_1, N_2) for S.

Now let $(M_1, \ldots, M_n)$ be the Morse decomposition of S with associated sequence A_j, $0 \le j \le n$, of attractors as in the statement of the theorem. Then, by proposition 5, $A_j \in \mathcal{S}$ and (M_j, A_{j-1}) is a repeller-attractor pair in A_j. By theorem 1 we find an index triple (N_1^j, N_2^j, N_3^j) for this attractor-repeller pair in A_j. We conclude that (N_1^j, N_3^j) is an index pair for A_j and (N_2^j, N_3^j) is an index pair for A_{j-1}. Moreover, if U is an open set with $A_{j-1} \subset U \subset \mathrm{cl}\,(U) \subset \mathrm{int}\,(N_2^j)$, then $(N_1^j \setminus U, N_2^j \setminus U)$ is an index pair for M_j. In view of lemma 9 and using the excision axiom of cohomology theory we find

LEMMA 10. *For the repeller-attractor pair (M_j, A_{j-1}) in A_j there exists a triple $N_1^j \supset N_2^j \supset N_3^j$ of closed sets, such that:*

$$H^q(h(M_j)) \cong H^q(N_1^j, N_2^j) \cong H^q(N_1^j \setminus U, N_2^j \setminus U)$$
$$H^q(h(A_j)) \cong H^q(N_1^j, N_3^j)$$
$$H^q(h(A_{j-1})) \cong H^q(N_2^j, N_3^j).$$

From lemma 8, lemma 10 and (4.8) we conclude that

$$p(t, h(M_j)) + p(t, h(A_{j-1})) = p(t, h(A_j)) + (1+t)Q(T, N_1^j, N_2^j, N_3^j). \tag{4.9}$$

Summation over $1 \le j \le n$ gives:

$$p(t, h(A_0)) + \sum_{j=1}^{n} p(t, h(M_j)) = p(t, h(A_n)) + (1+t)Q(t),$$

with $Q(t) = \sum_{j=1}^n Q(t, N_1^j, N_2^j, N_3^j)$. Now $A_n = S$ and hence $p(t, h(A_n)) = p(t, h(S))$. Moreover $A_0 = \emptyset$ and hence $h(A_0)$ is the homotopy type of a pointed one point space $(\{p\}, p)$, with p being any point. Hence $H^q(h(A_0)) = 0$ for all q. This proves the first part of theorem 2.

Now assume $Q(t, N_1^j, N_2^j, N_3^j) \ne 0$, and assume that there is no solution $\sigma: \mathbb{R} \to S$ satisfying $\omega^*(\sigma) \subset M_j$ and $\omega(\sigma) \subset M_i$ for some $i < j$. We shall obtain a contradiction. We conclude from proposition 1 that A_j is the disjoint union of A_{j-1} and M_j, i.e. $A_j = A_{j-1} \dot\cup M_j$. Now according to (4.9) $Q(t, N_1^j, N_2^j, N_3^j)$ depends only on $h(M_j)$, $h(A_{j-1})$ and $h(A_j)$. We can therefore choose the index triple for the repeller-attractor pair (M_j, A_{j-1}) in A_j in such a way that

$$(N_1^j, N_2^j, N_3^j) = (B, B_2 \cup B^-, B^-)$$

where $B = B_1 \cup B_2$ and $B_1 \cap B_2 = \emptyset$. B is an isolating block for A_j, B_1 is an isolating block for M_j and B_2 an isolating block for A_{j-1}. But $B = B_1 \dot\cup B_2$ and $B^- = B_1^- \dot\cup B_2^-$, and therefore

$$(N_1^j, N_2^j, N_3^j) = (B_1 \dot\cup B_2, B_2 \dot\cup B_1^-, B_2^- \dot\cup B_1^-).$$

Application of the sequence (4.5) to this triple gives:

$$\cdots \xrightarrow{\gamma_{q-1}} H^q(B_2 \dot\cup B_1, B_2 \dot\cup B_1^-) \xrightarrow{\alpha_q} H^q(B_2 \dot\cup B_1, B_2^- \dot\cup B_1^-) \longrightarrow \cdots.$$

Recall that the map α_q is the map i^* induced by the inclusion $i: (B_2 \cup B_1, B_2^- \cup B_1^-) \to (B_2 \cup B_1, B_2 \cup B_1^-)$. Let e be the inclusion map $e: (B_1, B_1^-) \to (B_2 \cup B_1, B_2 \cup B_1^-)$. Since $B_1 \cap B_2 = \emptyset$, the induced map e^* is an isomorphism. Suppose first $B_2^- \cup B_1^- \ne \emptyset$ and let $r: (B_2 \dot\cup B_1, B_2 \dot\cup B_1^-) \to (B_2 \dot\cup B_1, B_2^- \dot\cup B_1^-)$ be any continuous map satisfying

$r(x)=x$ for all $x\in B_1$. Then clearly $i\circ r\circ e=e$ and so $e^*\circ r^*\circ i^*=e^*$ and therefore $r^*\circ\alpha_q=\mathrm{id}$, as e^* is an isomorphism. Now if $B_1^-\cup B_2^-=\varnothing$, then let $h:(B_1,\varnothing)\to(B_2\cup B_1,\varnothing)$ and $j:(B_1,\varnothing)\to(B_2\cup B_1, B_2)$ be inclusion maps. Then, by excision, j^* is an isomorphism, and obviously $i\circ h\circ j=j$. Hence as before $h^*\circ\alpha_q=\mathrm{id}$. It follows in both cases that $\ker(\alpha_q)=0$, and since the sequence is exact we find $\mathrm{im}\,(\gamma_{q-1})=\ker(\alpha_q)=0$. This holds true for all q and hence $Q(t, N_1^j, N_2^j, N_3^j)=0$ in contradiction to the assumption. This finishes the proof of theorem 2. □

5. *The Morse equation for an arbitrary homology and cohomology theory*

Let $\{H_q\}$ (resp. $\{H^q\}$) be any (unreduced) homology (resp. cohomology) theory with coefficients in an R-module G, where R is an integral domain [**13**, chapter 9]. The aim is to prove that the statement in theorem 2 holds true for any homology and cohomology theory, not just for the Alexander-Spanier cohomology. We point out that the latter theory was used only in order to prove lemma 9. We shall prove below that for the special index pairs constructed in theorem 1 and used in theorem 2 the statement of lemma 9 holds true with respect to any cohomology or homology theory.

If (A, B) with $A\supset B$ is a pair of topological spaces, then B is called a *strong deformation retract* of a neighbourhood of itself, if there is an open neighbourhood $V\subset A$ of B, and a continuous map $H: V\times[0,1]\to A$ such that

$$\begin{aligned} &H(x,1)\in B && \text{for all } x\in V,\\ &H(x,0)=x && \text{for all } x\in A,\\ &H(y,t)=y && \text{for all } y\in B.\end{aligned}$$

We shall make use of the following well known result.

LEMMA 11. *Let A be a metric space and $B\subset A$ a closed subset. Assume B is a strong deformation retract of some neighbourhood of itself. Then the projection map $p:(A,B)\to(A/B,[B])$ induces an isomorphism*

$$H_q(A,B)\cong H_q(A/B,[B]),$$

resp. $H^q(A,B)\cong H^q(A/B,[B])$, of the homology, resp. cohomology modules.

We shall show that the special index pairs of theorem 1 meet the assumptions of lemma 11.

LEMMA 12. *Let $S\in\mathcal{S}$ and (A^*, A) be a repeller-attractor pair in S. Let $(B, B_2\cup B^-, B^-)$ be the index triple of theorem 1 for (A^*, A) relative to S. Let U be an open set such that*

$$A\subset U\subset \mathrm{cl}\,(U)\subset \mathrm{int}\,(B_2\cup B^-).$$

Then the pairs (B, B^-) and $(B_2\cup B^-, B^-)$ and $(B\setminus U, (B_2\cup B^-)\setminus U)$ meet the requirements of lemma 11, i.e. the second space is a strong deformation retract of some open neighbourhood in the first space.

Postponing the proof of lemma 12, we first prove the main result:

THEOREM 3. *Let $S \in \mathcal{S}$ and let $(M_1, \ldots, M_n)$ be a Morse decomposition of S with associated sequence $\emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = S$ of attractors. Assume the cohomology modules $H^q(h(S))$, $H^q(h(A_j))$ and $H^q(h(M_j))$, (resp. the homology modules $H_q(h(S))$, $H_q(h(A_j))$ and $H_q(h(M_j))$) are of finite rank for all $q \geq 0$ and $1 \leq j \leq n$. Then the statement of theorem 2 holds true with respect to the cohomology theory $\{H^q\}$, (resp. with respect to the homology theory $\{H_q\}$).*

Proof of theorem 3. In view of (4.1) the algebraic invariants of $K \in \mathcal{S}$ can be computed up to isomorphisms with respect to any index pair for K. Therefore, if we use for the repeller-attractor pair (M_j, A_{j-1}) in A_j the special index triple of theorem 1, the required statement of lemma 10 follows in view of lemma 11 and lemma 12. The proof is then identical to the proof of theorem 2. □

It remains to prove lemma 12. For an admissible set $B \subset X$ we define the function $s_B: B \to \mathbb{R}^+ \cup \{\infty\}$ by

$$s_B(x) = \sup\{t \geq 0 | x \cdot [0, t] \subset B\}.$$

LEMMA 13. *The function s_B is continuous if B is admissible and an isolating block.*

Proof. Let $x_n, x \in B$ and $x_n \to x$. Assume first that $s_B(x) \neq \infty$. Since B is admissible and an isolating block we conclude $s_B(x) < \omega_x$ and $x \cdot s_B(x) \in B^-$. If $\varepsilon > 0$ is small then $x \cdot (s_B(x) + \varepsilon) \in X \setminus B$ and therefore $x_n \cdot (s_B(x) + \varepsilon) \in X \setminus B$, by the continuity of the semiflow, and hence $s_B(x_n) < s_B(x) + \varepsilon$, if n is large. Similarly one proves $s_B(x_n) > s_B(x) - \varepsilon$ for large n, hence s_B is continuous at x. Suppose now that $s_B(x) = \infty$. Then $\omega_x = \infty$ and $x \cdot [0, \infty) \subset$ B. Hence $x \cdot (0, \infty) \subset \operatorname{int}(B)$ as B is an isolating block. Therefore, for any $\varepsilon, T > 0$ there is an integer $n(\varepsilon, T)$ such that $x_n \cdot [\varepsilon, T] \subset \operatorname{int}(B)$ for $n \geq n(\varepsilon, T)$. If s_B is not continuous at x we can therefore find a sequence $\delta_n \downarrow 0$ such that $x_n \cdot \delta_n \notin B$. Since solutions can leave B only through B^- we find another sequence $0 \leq \varepsilon_n < \delta_n$ satisfying $x_n \cdot \varepsilon_n \in B^-$. Since B^- is closed we conclude from $x_n \cdot \varepsilon_n \to x$ that $x \in B^-$. But then $s_B(x) = 0$, a contradiction. Hence, indeed, s_B is continuous. □

Proof of lemma 12. (1) (B, B^-): Let $V = B \setminus A^+(B)$ and define $H: V \times [0, 1] \to B$ by

$$H(x, t) = x \cdot (t \cdot s_B(x)).$$

By lemma 13, s_B is continuous. Moreover $s_B(x) = 0$ for $x \in B^-$, since B is an isolating block. It follows that H is the required retraction map onto B^-, since V is a neighbourhood of B^- in B.

(2) $(B_2 \cup B^-, B^-)$: Observe that $B_2 \cup B^-$ is positively invariant relative to B. Therefore, with $V = (B_2 \cup B^-) \setminus A^+(B)$, the above map $H: V \times [0, 1] \to V$ is the required retraction in this case.

(3) $(B \setminus U, (B_2 \cup B^-) \setminus U)$: Let $V := (B \setminus U) \setminus A^+(B_1)$ and define $H: V \times [0, 1] \to (B \setminus U)$ by

$$H(x, t) = \begin{cases} x \cdot (t \cdot s_{B_1}(x)) & \text{if } x \in B_1, \\ x & \text{otherwise.} \end{cases}$$

Clearly V is a neighbourhood of $(B_2 \cup B^-) \setminus U$ in $B \setminus U$. H is well defined and $H(x, 0) = x$ for all $x \in V$. We claim that $H(x, t) = x$ for all $x \in (B_2 \cup B^-) \setminus U$ and all $0 \le t \le 1$. Indeed, let $x \in (B_2 \cup B^-) \setminus U$ and $x \in B_1$. If $x \in B_2$ it follows from theorem 1(iii) that $x \in B_1^-$ and so $s_{B_1}(x) = 0$ implying that $H(x, t) = x$ for $0 \le t \le 1$ in this case. If however $x \notin B_2$, then $x \in B^-$ and hence for some $\tau > 0$ we have

$$x \cdot (0, \tau] \subset (X \setminus B) \subset X \setminus B_1.$$

But this again implies $s_{B_1}(x) = 0$, and the claim is proved.

Next we claim that $H(x, 1) \in (B_2 \cup B^-) \setminus U$ for all $x \in V$. If $x \in B_1 \cap V$ then

$$H(x, 1) = x \cdot s_{B_1}^+(x) \in B_1^- \setminus \mathrm{cl}\,(U)$$

since $\mathrm{cl}\,(U) \subset \mathrm{int}\, B_2$ and $B_1 \cap \mathrm{int}\, B_2 = \emptyset$. Set $y := x \cdot s_{B_1}(x)$ and assume $y \notin B^-$; then for some $\tau > 0$

$$y \cdot (0, \tau] \subset \mathrm{int}\,(B) \cap (X \setminus B_1) \subset B_2.$$

Hence $y \in B_2$. We have shown that $H(x, 1) \in (B_2 \cup B^-) \setminus U$ if $x \in B_1 \cap V$. Assume now $x \in V \setminus B_1$, then $x \in B_2 \setminus U$ and $H(x, 1) = x$ by definition of H. This proves the claim.

It remains to prove that H is continuous. We only have to consider the case $x_n \in V \setminus B_1$ with $x_n \to x \in B_1$ and $t_n \to t$. In this case $H(x_n, t_n) = x_n$, we have to show that $x \in B_1^-$ in order to complete the proof. Indeed, since $x_n \in V \setminus B_1$ it follows that $x_n \in B_2$ as $B = B_1 \cup B_2$. Therefore $x \in B_1 \cap B_2$ and hence by theorem 1(iii) $x \in B_1^- \cap B_2^+$. The proof of the lemma is complete. □

Remark. The continuity of s_B was used (without proof) in the sketch of a proof of proposition 2.1 in [**9**]. We point out that an assumption is missing in the statement of that proposition, namely that the semiflow does not explode in B, i.e. if $x \in B$ and $\omega_x < \infty$ then $x \cdot t \notin B$ for some $t < \omega_x$. Without this assumption our proof of the continuity is incorrect, since it may happen that $s_B(x) = \omega_x$ such that $x \cdot s_B(x)$ is not defined. The hypothesis, however, that the semiflow does not explode in B is part of our definition of admissibility.

REFERENCES

[1] H. Amann. Invariant sets and existence theorems for semi-linear parabolic and elliptic systems. *J. Math. Anal. Appl.* **65** (1978), 432–467.

[2] H. Amann & E. Zehnder. Non-trivial solutions for a class of non-resonance problems and applications to non-linear differential equations. *Ann. Scuola Norm. sup. Pisa Cl. Sci.*, (IV) **VII**, Nr. 4 (1980), 539–603.

[3] H. Amann & E. Zehnder. Periodic solutions of asymptotically linear Hamiltonian systems. *Manuscripta Math.* **32** (1980), 149–189.

[4] R. Bott. Marston Morse and his mathematical works. *Bull. Amer. Math. Soc.* **3** (1980), 907–950.

[5] C. C. Conley. Isolated invariant sets and the Morse index. *CBMS Regional Conf. Ser. in Math.* **38**, Amer. Math. Soc.: Providence, 1978.

[6] C. C. Conley & E. Zehnder. Morse type index theory for flows and periodic solutions for Hamiltonian systems. *Comm. Pure and Appl. Math.* **XXXVII** (1984), 207–253.

[7] Dan Henry. *Geometric Theory of Semilinear Parabolic Equations.* Springer Lecture Notes in Math. 840. Springer Verlag: Berlin-Heidelberg-New York, 1981.

[8] W. S. Massey. *Homology and Cohomology Theory.* M. Dekker Inc: New York and Basel, 1978.
[9] K. P. Rybakowski. On the homotopy index for infinite-dimensional semiflows. *Trans. Amer. Math. Soc.* **269** (1982), 351–382.
[10] K. P. Rybakowski. The Morse-index, repeller–attractor pairs and the connection index for semiflows on noncompact spaces. *J. Diff. Eq.* **47** (1983), 66–98.
[11] K. P. Rybakowski. Trajectories joining points of nonlinear parabolic and hyperbolic partial differential equations, *J. Diff. Eq.* **51** (1984), 182–212.
[12] J. A. Smoller & C. C. Conley. On the structure of magneto-hydrodynamic shock waves. *Comm. Pure and Appl. Math.* **27** (1974), 367–375.
[13] E. H. Spanier. *Algebraic Topology.* McGraw Hill: New York, 1966.

https://doi.org/10.1017/S0143385700002790 Published online by Cambridge University Press