A NOTE ON FREE PRODUCTS WITH A NORMAL AMALGAMATION

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1. Introduction

It is a consequence of the Kurosh subgroup theorem for free products that if a group has two decompositions

$$G = \Pi^*\{A_i : i \in I\} = \Pi^*\{B_j : j \in J\}$$

where each A_i and each B_j is indecomposable, then I and J can be placed in one-to-one correspondence so that corresponding groups if not conjugate are infinite cycles. We prove here a corresponding result for free products with a normal amalgamation.

(1.1) THEOREM. *If G is a group with two decompositions as a free product with normal amalgamation*:

$$G = \Pi^*(\{A_i : i \in I\}; H) = \Pi^*(\{B_j : j \in J\}; K)$$

where H, K are normal in G and where for each $i \in I$ and each $j \in J$, $A_i/A_i \cap K$ and $B_j/B_j \cap H$ are indecomposable then

(i) $H = K$,

and

ii) *there exists a one-to-one correspondence between I and J such that if i, j correspond then either A_i and B_j are conjugate in G or each is a (splitting) extension of H by an infinite cycle. (We assume that for each $i \in I$ and each $j \in J$, $H < A_i$ and $K < B_j$.)*

Of course it is easy to construct groups with two different free decompositions which do not satisfy the conclusions of the theorem. For example if $X = A * B$ and if $\lambda : A \to A'$, $\mu : B \to B'$ are epimorphisms, at least one of which is proper, then λ, μ can be extended simultaneously to an epimorphism $\alpha : A * B \to A' * B'$ and, by (2.5)

$$X = \Pi^*(\{A \cdot \ker \alpha, B \cdot \ker \alpha\}; \ker \alpha).$$

[1] This work formed part of an M.Sc. thesis presented to the University of Queensland in 1964. My thanks are due to my supervisor, Mr M. P. O'Donnell and to Professor Hanna Neumann who suggested this problem to me.

The hypotheses of the theorem are not satisfied here since $A < A \cdot \ker \alpha$ and a simple application of the Kurosh subgroup theorem and (2.1) yields that $A \cdot \ker \alpha$ is properly decomposable. So far as I am aware, it is unknown whether a free product can have a proper factor group decomposable in an essentially different way to that just described.

2. Preliminary results

First we state the well-known Schreier theorem for free products with a single amalgamation.

(2.0) THEOREM (cf. (2.4) and (3.3) on pp. 510—1 in [2]). *A group P which embeds the amalgam $\mathfrak{A} = (\{X_i : i \in I\}; L)$ is the generalized free product of $\mathfrak{A}$ if and only if every element $p \in P$ can be written uniquely in the form*

$$p = s_1 s_2 \cdots s_n l$$

where each s_j is not the identity and belongs to some arbitrary but fixed left transversal S_k of L in X_k (chosen so that $1 \in S_i$, $i \in I$), where $l \in L$ and where s_j and s_{j+1} belong to different S_k for $1 \leqq j \leqq n-1$. The number n is, as usual, the length $\lambda(p)$ of p; elements of L have zero length.

The following lemmas are more or less trivial modifications of well-known results in the literature.

(2.1) LEMMA (cf. Theorem 5.1 on p. 514 of [2]). *If $P = \Pi^*(\{X_i : i \in I\}; L)$ and $p \in P$ has finite order u modulo some X_i then p is conjugate to an element of some X_k. If $p^u \neq 1$ and L is normal in P then p is conjugate to an element of X_i.*

PROOF. Suppose that p has normal form

$$p = s_1 s_2 \cdots s_n l$$

where we may assume $n > 1$. By hypothesis $p^u \in X_i$. If $s_n l s_1 \notin L$ then for all non-zero integers m, $\lambda(p^m) > 1$; and hence $s_n l s_1 = l_1 \in L$. Similarly we deduce that

$$s_{n-t} l_t s_{t+1} = l_{t+1} \in L, \qquad t = 0, 1, \cdots, [n/2]-1.$$

If n is even, say $n = 2r$, then $s_{r+1} l_{r-1} s_r \in L$ which implies that s_r, s_{r+1} belong to the same X_k; hence n is odd, say $n = 2m+1$ and we have

$$s_m^{-1} s_{m-1}^{-1} \cdots s_1^{-1} p s_1 s_2 \cdots s_m = s_{m+1} l_m;$$

that is, p is conjugate to an element of some X_k. When L is normal in P elements of different X_k cannot be conjugate unless they belong to L; the last part of the Lemma follows from this.

(2.2) Lemma. *If* $P = \Pi^*(\{X_i : i \in I\}; L)$ *and* N *is a normal subgroup of* P *contained in some* X_i *then* $N \leqq L$.

Proof. Suppose $N \cap L = M$. Choose a left transversal of M in N, say T, with $1 \in T$. The elements of T are left coset representatives of L in X_i, since $t_1, t_2 \in T$ and $t_1^{-1}t_2 \in L$ implies $t_1^{-1}t_2 \in L \cap N = M$ so that $t_1 = t_2$. A system of left coset representatives of L in X_i can now be chosen, S_i say, with $T \subseteqq S_i$. If $j \neq i$, $1 \neq s_j \in S_j$ and $1 \neq t \in T$ then since N is normal in P

$$s_j^{-1} t s_j = t'm, \quad t' \in T, \quad m \in M.$$

But the left-side has length three and the right-side length one at most. Hence $N \leqq L$.

(2.3) Lemma (cf. IV on p. 16 of [1]). *If* $P = \Pi^*(\{X_i : i \in I\}; L)$ *and* $J \subseteqq I$, *write* N_J *for the normal closure of the set* $\{X_j : j \in J\}$ *in* P. *If* P_0 *is the subgroup generated by* $\{X_i : i \in I - J\}$ *and* N_0 *the normal closure of* L *in* P_0 *then*

$$P/N_J \cong P_0/N_0.$$

Proof. The passage from P to P/N_J involves putting equal to 1 all the elements of X_j, $j \in J$. Hence P/N_J is generated by X_i, $i \in I - J$ with defining relations those of X_i, $i \in I - J$ together with the relations $l = 1$, $l \in L$. In other words $P/N_J \cong P_0/N_0$.

(2.4) Theorem. (Theorem 13.4 in [3]; also provable direct from the Kurosh subgroup theorem). *If* $P = \Pi^*(\{X_i : i \in I\}; L)$, *if* L *is normal in* P *and if* $U \nless L$ *is a subgroup of* P, *then* U *contains subgroups* F, U_{ij} $(i \in I, j \in J_i)$ *such that*

$$U = \Pi^*(\{F, U_{ij} : i \in I, j \in J_i\}; M)$$

where $M = L \cap U$, F/M *is free and* U_{ij} *is conjugate to a subgroup of* X_i. *Any or all of* F, U_{ij} *may be* M.

(2.5) Lemma. *Let* P *be a group,* L *a normal subgroup of* P *and* X_i, $i \in I$, *subgroups of* P *containing* L. *Then* $P = \Pi^*(\{X_i : i \in I\}; L)$ *if and only if* $P/L = \Pi^*\{X_i/L; i \in I\}$.

Proof. See, for example, [4].

3. Proof of Theorem (1.1)

Using (2.4), (2.5) and the fact that $A_i/A_i \cap K$ is indecomposable for all $i \in I$ we deduce that either

(3.1) $A_i = gp(f_i)K_i$ or A_i *is conjugate to a subgroup of some* B_j

where $K_i = K \cap A_i$, $f_i \in A_i$ has infinite order and $gp(f_i) \cap K_i = 1$. Similarly

(3.1) $B_j = gp(g_j)H_j$ or B_j *is conjugate to a subgroup of some* A_i

where $H_j = H \cap B_j$, $g_j \in B_j$ has infinite order and $gp(g_j) \cap H_j = 1$.

(3.2) $A_i \nleqq K$ *and* $B_j \nleqq H$ *for* $i \in I$, $j \in J$.

For if for some i, j, $A_i \leqq K$ and $B_j \leqq H$, we would have at once $H = A_i$, which we have ruled out. Hence if $A_i \leqq K$, $B_j \nleqq H$ for any $j \in J$. If B_j is conjugate to a subgroup of some A_t, $t \in I$, then A_i is conjugate to a subgroup of A_t which implies $t = i$, and then $B_j \leqq K$ since K is normal; if $B_j = gp(g_j)H_j$ then since $H_j < A_i < B_j$, $A_i = gp(g_j^\alpha)H_j$ for some integer $\alpha \neq 0$. Lemma 2.1 then shows that g_j is conjugate to an element of A_i and hence B_j is conjugate to a subgroup of A_i, giving a contradiction in either case.

(3.3) *If* A_i *is conjugate to a subgroup of* B_j, *then* A_i *is conjugate to* B_j.

By (3.1), if B_j/H_j is not free, then B_j is conjugate to a subgroup of A_t, $t \in I$, say

(*) $A_i^g \leqq B_j$ and $B_j^h \leqq A_t$.

Then

(**) $$A_i^{gh} \leqq A_t$$

which implies $t = i$; and strict inequality in (**) would imply $gh \notin A_i$ contradicting Schreier's theorem; whilst strict inequality in either place in (*) would mean strict inequality in (**). Hence A_i and B_j are conjugate.

If B_j/H_j is free then $B_j = gp(g_j)H_j$ as in (3.1). Since here $H < B_j$, $H = H_j$. If $A_i^g \leqq B_j$ then since $H < A_i^g$,

$$A_i^g = gp(g_j^\alpha)H$$

for some integer $\alpha \neq 0$. Therefore B_j is conjugate to a subgroup of A_i by (2.1) and the preceding part gives A_i and B_j conjugate.

(3.4) *If* $S_1 = \{A_i : A_i$ *conjugate to some* $B_j\}$ *and* $S_2 = \{B_j : B_j$ *conjugate to some* $A_i\}$, *then* S_1 *and* S_2 *can be placed in one-to-one correspondence; and if* $A_i \notin S_1$ *then* $A_i = gp(f_i)K_i$ *and if* $B_j \notin S_2$, $B_j = gp(g_j)H_j$.

This is just (3.1) and (3.2) and the fact that no two different factors of a free product can be conjugate, by Schreier's theorem.

(3.5) *If* S_1 *is not empty then the theorem is true.*

For if $A_i \in S_1$, A_i is conjugate to some B_j and therefore $H \leqq B_j$ and $K \leqq A_i$ which by (2.2) gives $H \leqq K \leqq H$; that is $H = K$. Next put N

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for the normal closure in G of S_1; N is then also the normal closure in G of S_2. Put $G_1 = gp\{A_i : A_i \notin S_1\}$, $G_2 = gp\{B_j : B_j \notin S_2\}$. By (2.3)

$$G/N \cong G_1/H \text{ and } G/N \cong G_2/H.$$

G_1/H and G_2/H are free groups by (2.5) and therefore have the same rank.

(3.6) *If S_1 is empty, then $A_i \cap B_j = H \cap K = L$, $i \in I$, $j \in J$.*

Since $K_i = K \cap A_i \leqq A_i \cap B_j$ and $H_j = H \cap B_j \leqq A_i \cap B_j$,

$$A_i \cap B_j = gp(f_i^{u_i})K_i = gp(g_j^{v_j})H_j$$

for integers u_i, v_j. If $u_i \neq 0$ for some i, then $1 \neq f_i^{u_i} \in B_j$. Using (2.1) we deduce that A_i is conjugate to a subgroup of B_j contradicting (3.3) and the hypothesis that S_1 is empty. Hence $u_i = 0$, all $i \in I$; similarly $v_j = 0$, all $j \in J$. Finally

$$L = H \cap K \leqq K_i = H_j \leqq L,$$

so that $A_i \cap B_j = L$.

(3.7) *If S_1 is empty then the theorem is true.*

Suppose $H \neq K$; then H say, contains L properly, and there are integers $u_i \neq 0$ with

$$H = gp(f_i^{u_i})L, \qquad i \in I.$$

We have $f_i^{u_i} = f_t^{\pm u_t} \pmod L$; for there exist integers α, β such that

$$f_t^{u_i} = f_t^{\alpha u_t} \pmod L \text{ and } f_t^{u_t} = f_i^{\beta u_i} \pmod L$$

and therefore

$$f_t^{u_t} = f_t^{\alpha\beta u_t} \pmod L$$

so that $u_t(1-\alpha\beta) = 0$ and $\alpha\beta = 1$.

It follows that $f_i^{u_i}$ commutes, modulo L, with f_t, $t \in I$, and therefore with every element of G. In particular

$$f_i^{u_i} g_j = g_j f_i^{u_i} \pmod L$$

which, if i, j are chosen so that $f_i^{u_i}$ as a normal word with respect to the B-amalgam does not begin with g_j, provides a contradiction. Hence $H = K$ and mapping onto G/H we have two sets of free generators for it, giving $|I| = |J|$.

(3.8) Corollary. *The hypothesis of the theorem that $A_i/A_i \cap K$ and $B_j/B_j \cap H$ are indecomposable is realized if A_i and B_j are all completely indecomposable (that is, no factor group of A_i or B_j is decomposable). In this case H is a characteristic subgroup of G.*

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4. An example

That any attempt to generalize this theorem to the case of non-normal amalgamation will fail is almost obvious. In fact we will show that there exists a group with two decompositions satisfying the conditions of (3.8), except that one amalgamation is not normal, but not the conclusions.

Let

$$G = \Pi^*(\{A, B, C\}; H)$$

where A is non-abelian of order 6, B and C are non-cyclic of order 4 and H is of order 2. Write $N = A^G$, the normal closure of A in G. Then by (2.3) and (2.5)

$$G = \Pi^*(\{BN, CN\}; N) = \Pi^*(\{A, B, C\}; H).$$

Clearly A, B, C are all completely indecomposable and we have to show that the same is true of BN and CN. To this end note first that

$$N = H^{BN} = H^{CN}.$$

This is true since $A = H^A$ and therefore $A^N = H^N$ which in turn means that H^N is normalized by A, B, C and is therefore normal in G. That is

$$N = A^G \leqq A^N = H^N \leqq N$$

which gives $N = H^N = H^{BN} = H^{CN}$ as required.

Finally suppose that BN, say, has a decomposition

$$BN = \Pi^*(\{X, Y\}; M)$$

where M is normal in BN. Applying Theorem 2.4 we deduce that A is conjugate in BN to a subgroup of X, say; and therefore without loss of generality we may suppose $A \leqq X$. Two cases arise; first, if $H \leqq M$ then $N = H^{BN} \leqq M$ and since $BN/N \cong B/B \cap N = B/H$, a group of order 2, BN/M cannot be decomposable. Hence $H \cap M = 1$ and applying Schreier's Theorem we deduce that $B \leqq X$, which yields

$$BN = B \cdot H^{BN} \leqq B \cdot X^{BN} = X^{BN}$$

and hence $Y = M$. We have thus shown that BN (and in a similar fashion CN) is completely indecomposable.

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