SUMMABILITY-(Z, p) AND SEQUENCES OF PERIODIC TYPE

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1. Background. We shall say that the sequence $x = \{s_k\}$ is summable-(Z, p) to the value s if

$$\lim_n Z^p_{n+p}(x) = s,$$

where p is a positive integer and

$$Z^p_{n+p}(x) \equiv (s_{n+1} + s_{n+2} + \ldots + s_{n+p})/p \qquad (n \geqslant 1 - p;\, s_k \equiv 0 \text{ if } k \leqslant 0).$$

Ignoring the values of n, $1 - p \leqslant n < 0$, which are clearly irrelevant, the transformation (Z, p) coincides with the regular Nörlund transformation defined by the sequence $(1, 1, \ldots, 1, 0, 0, \ldots)$ containing p initial 1's. This class of methods was first studied systematically by Silverman and Szász **(8)**. As our point of departure we quote the following results for reference.

(1.1) *Summability-(Z, p) implies summability-$(C, 1)$ to the same value for every $p = 1, 2, 3, \ldots$* **(8**, Th. 11**)**.

(1.2) *If p is a divisor of q, then the convergence field of (Z, p) is contained in that of (Z, q)* **(8**, Th. 14**)**.

(1.3) *If d is the greatest common divisor of p and q, then the convergence fields of (Z, p) and (Z, q) intersect in that of (Z, d)* **(8**, Th. 15**)**.

Note that $(Z, 1)$ is the identity transformation and that each method $(Z, p > 1)$ evaluates a bounded divergent sequence; see the proof of (2.1)–(i). Moreover, the consistency of the methods (Z, p) is implied by (1.1) although it follows as well from known properties of Nörlund methods.

With few exceptions our interest is focused on certain subsets of the Banach space (m) of real bounded sequences $x = \{s_k\}$ for which $\|x\| \equiv \sup_k |s_k|$. We denote by Z_p the *bounded* convergence field of (Z, p), by Z the set $\bigcup_p Z_p$, and by $\bar{Z}$ the closure of Z in (m). For each $p \geqslant 1$ the following equation can be verified easily by induction on n:

$$Z^p_{n+p}(x) = Z^p_p(x) - \frac{1}{p} \sum_{k=1}^{n} (s_k - s_{k+p}) \qquad (p, n = 1, 2, 3, \ldots).$$

The sequence $x = \{s_k\}$, bounded or not, is therefore summable-(Z, p) if and only if the series

(1.4) $$\sum_{k=1}^{\infty} (s_k - s_{k+p})$$

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is convergent. We observe that each $x \in Z$ possesses for some p the approximate type of periodicity suggested by (1.4). We propose to study the nature of this periodic behaviour together with that of certain other classes which we proceed to describe.

For each $p \geqslant 1$ we denote by $|Z_p|$ that subset of Z_p of all $x = \{s_k\}$ for which

$$\sum_{k=1}^{\infty} |s_k - s_{k+p}| < \infty; \tag{1.5}$$

and by $|Z|$ the set $\bigcup_p |Z_p|$. Note that $|Z_1|$ is the class of all sequences of bounded variation.

Berg and Wilansky **(3)** introduced the class (sp) of *semiperiodic* sequences as the closure in (m) of the class (p) of *periodic* sequences, and gave a precise characterization. Somewhat more generally we consider the class (usp) of *ultimately semiperiodic* sequences obtained as the closure in (m) of the class (up) of *ultimately periodic* sequences, namely, those sequences $x = \{s_k\}$ for which there exist two integers p and K, depending on x, such that $s_{k+p} = s_k$ for all $k > K$. It may be noted that all convergent sequences belong to (usp) but not to (sp). The Berg-Wilansky characterization now takes the following form:

(1.6) *A sequence $x = \{s_k\}$ belongs to* (usp) *if and only if corresponding to $\epsilon > 0$ there exist positive integers p and K such that $|s_k - s_{k+\lambda p}| < \epsilon$ for all $k > K$ and all $\lambda = 1, 2, 3, \ldots$.*

As the closure of a linear set, (usp) is evidently closed and linear.

A sequence $x = \{s_k\}$ belongs to the class (ap) of *almost-periodic* sequences in the sense of Lorentz **(6)** if and only if corresponding to $\epsilon > 0$ there exist positive integers K and L having the property that every interval $(h, h + L)$ for integral $h \geqslant 0$ contains an integer q such that $|s_k - s_{k+q}| < \epsilon$ for all $k > K$. It follows at once from this definition that (ap) is closed in (m). That (ap) is a linear subset of (m) can be proved by paralleling the corresponding argument for almost-periodic functions.

Finally, a sequence $x = \{s_k\}$ is said to be *almost-convergent* to s if and only if *all* Banach limits have the value s at x **(6**, p. 169**)**. The set (ac) of all such sequences is a linear closed subset of (m); it is characterized by the fact that

$$\lim_p Z^p_{n+p}(x) = s \text{ uniformly}$$

for $n = 1, 2, 3, \ldots$ **(6**, Th. 1**)**. It is apparent that (ac) is a subset of C_1, the bounded convergence field of $(C, 1)$. The inclusion is proper since the sequence $\{\alpha_k\}$ defined by $\alpha_k = 1$ if $k = n^3 + j$ $(j = 1, 2, \ldots, n;\ n = 1, 2, 3, \ldots)$; $\alpha_k = 0$ otherwise, is summable-$(C, 1)$ to zero but is not almost-convergent. Furthermore, it is known that (ap) is a proper subset of (ac) **(6**, p. 173**)**.

Before proceeding to investigate further relationships between these classes of periodic-type sequences we add a few remarks concerning the methods (Z, p).

2. Remarks concerning summability-(Z, p). Let Z_p^* denote the entire convergence field of (Z, p) for $p \geqslant 1$. Some questions left open in (1.2) are answered in the following proposition.

(2.1) (i) *If p is a proper divisor of q, then Z_p^* is a proper subset of Z_q^*.* (ii) *If $Z_p^* \subset Z_q^*$, then p is a divisor of q.*

Proof of (i). It suffices to observe that a sequence of 0's and 1's of period q containing a single "1" per period is in Z_q^* but not in Z_p^*.

Proof of (ii). First of all the condition $Z_p^* \subset Z_q^*$ implies that $p \leqslant q$. For if $p > q$, the sequence of 0's and 1's just described, with p in place of q, is summable-(Z, p) but not summable-(Z, q). Suppose then that $q = rp + d$ $(0 < d < p)$ and consider the equation

$$Z^q_{n+q} = \frac{p}{q}\,(Z^p_{n+p} + Z^p_{n+2p} + \ldots + Z^p_{n+rp}) + \frac{d}{q}\,Z^d_{n+rp+d},$$

where for brevity

$$Z^q_{n+q} \equiv Z^q_{n+q}(x), \quad \text{etc.}$$

This relation shows that if $Z_p^* \subset Z_q^*$, then $Z_p^* \subset Z_d^*$, where $p > d$. This contradiction implies $d = 0$ or $q = rp$.

As a consequence of (2.1)–(i) the special set of convergence fields $Z_{p!}^*$ for $p \geqslant 1$ form a strictly increasing sequence. That is to say, the summability methods $(Z, p!)$ steadily increase in strength. More precisely, the convergence fields Z_p^* themselves have the interesting property expressed in the following observation.

(2.2) *With appropriate definitions the convergence fields Z_p^* $(p = 1, 2, 3, \ldots)$ form a lattice isomorphic to the familiar* g.c.d.–l.c.m. *lattice of the positive integers.*

For set inclusion yields the required partial ordering, and we then define l.u.b.(Z_p^*, Z_q^*) as Z_m^*, where $m \equiv$ l.c.m.(p, q); and g.l.b.(Z_p^*, Z_q^*) as Z_d^*, where $d \equiv$ g.c.d.(p, q). These definitions possess the requisite properties since if $Z_p^* \subset Z_\mu^*$ and $Z_q^* \subset Z_\mu^*$, it follows from (2.1)–(ii) that $p|\mu$ and $q|\mu$ so that $m \leqslant \mu$. Also, if $Z_\delta^* \subset Z_p^*$ and $Z_\delta^* \subset Z_q^*$, then $Z_\delta^* \subset Z_p^* \cap Z_q^* = Z_d$ by (1.3), and thus $\delta \leqslant d$ by (2.1)–(ii).

We remark in passing that the sets $|Z_p|$, as well as Z_p, form lattices in the same fashion.

We show next that the result (1.1) remains true in the case of bounded sequences if "summability-$(C, 1)$" is strengthened to "almost-convergence." In fact, we shall prove that

(2.3) *The set $\tilde{Z}$ is a proper subset of* (ac).

Proof. Let p be given and let $Z^p x$ be the (Z, p)-transform of $x = \{s_k\}$ as defined in §1. For each $r > 0$ let S^{-r} be the inverse shift operator that prefixes r zeros to the sequence x:

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$$S^{-r}x \equiv (0, \ldots, 0, s_1, s_2, s_3, \ldots);$$

then

$$Z^p x = (x + S^{-1}x + \ldots + S^{-p+1}x)/p.$$

Let $L_1(x)$ and $L_2(x)$ be any two Banach limits over (m), and assume that x is summable-(Z, p), i.e., that $Z^p x \in$ (c), the space of convergent sequences. By the properties of a Banach limit, $L_i(Z^p x) = L_i(x)$ for $i = 1, 2$. Since $L_1(Z^p x) = L_2(Z^p x)$, it follows that $L_1(x) = L_2(x)$, which proves that $x \in$ (ac). Consequently, since p is arbitrary, $Z \equiv \bigcup_p Z_p \subset$ (ac), and therefore $\bar{Z} \subset$ (ac), since (ac) is closed in (m).

It remains to show that (ac)$\setminus \bar{Z} \neq \emptyset$. For this purpose we define $V(x)$, the value set of $x = \{s_k\}$, as the set $\{v | s_k = v$ for some $k\}$, and prove the following lemma.

(2.4) *If* $y = \{t_k\} \in \bar{Z}$ *and the set* $V(y)$ *is finite, then* $y \in$ (up).

Proof. Define ϵ as the positive number $\min_{i \neq j} |v_i - v_j|$, where $v_1, v_2, \ldots$ are the different elements of $V(y)$. Since $y \in \bar{Z}$, there exists an $x = \{s_k\}$ in Z such that

$$||y - x|| \equiv \sup_k |t_k - s_k| < \epsilon/3.$$

For some integer p we have $x \in Z_p$ so that Condition (1.4) assures the existence of an index K for which $|s_k - s_{k+p}| < \epsilon/3$ for all $k > K$. Then

$$|t_k - t_{k+p}| \leqslant |t_k - s_k| + |s_k - s_{k+p}| + |s_{k+p} - t_{k+p}| < \epsilon$$

for all $k > K$. Therefore, $\{t_k\}$ is ultimately periodic and this completes the proof of the lemma.

To finish the proof of (2.3) we introduce the set of sequences $\{\alpha_k\}$ of cardinal c in which each pair $\alpha_{2i-1}, \alpha_{2i}$ is a permutation of 0, 1. Each such $\{\alpha_k\}$ is almost-convergent to 1/2, has a finite value set, and only a countable number of them are ultimately periodic. The conclusion now follows from the lemma.

The functionals $Z^p_{n+p}(x)$ for $x \in$ (m) also play an important role in the theory of Banach limits. If L is the class of these generalized limits and $\tau(x) \equiv \sup_{\psi \in L} \psi(x)$, then Jerison (**5**, Th. 5) evaluated $\tau(x)$ as

$$\lim_p \limsup_n Z^p_{n+p}(x).$$

In this connection we state without proof the following related facts which appear to have escaped observation. The functional

$$P(x) \equiv \inf_p \sup_n Z^p_{n+p}(x),$$

where "sup" appears in place of the usual "lim sup," is positively homogeneous and convex, and has the property that any linear functional $L(x) \leqslant P(x)$ is a Banach limit. Moreover, it can be shown that

$$\lim_p \sup_n Z^p_{n+p}(x)$$

exists, and is equal to both $P(x)$ and $\tau(x)$.

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As a concluding remark we recall the existence of Banach-Hausdorff limits established by Eberlein **(4)**. These are Banach limits $L(x)$ with the additional property that $L(Hx) = L(x)$ for every $x \in (\mathrm{m})$ and every regular Hausdorff matrix H. It is interesting, but less profound, that a similar fact holds for the transformations (Z, p). Thus, $L(Z^p x) = L(x)$ for every $x \in (\mathrm{m})$, every $p \geqslant 1$, and every $L(x) \in L$, since $Z^p x$ has the simple form in terms of the shift operator used in the proof of (2.3).

3. Sequences of periodic type. We proceed now to develop some of the internal properties of sequences in the various classes of §1 and to describe some of the interrelationships between these classes. All classes under consideration are linear subsets of (m) and it will appear in due course that they are all proper subsets of (ac), a fact already shown for $\tilde{Z}$ in (2.3), and mentioned for (ap) in §1. To avoid tiresome circumlocution we introduce the following terminology. If a sequence $\{s_k\}$ is visualized as the composition of the sequences

$$X_i \equiv \{s_{i+\lambda p}\}_{\lambda=0}^{\infty} \qquad \text{for } i = 1, 2, \ldots, p,$$

we shall say that $\{s_k\}$ is the *periodic union* of the p sequences X_i. For example, if each X_i is a convergent sequence we refer to $\{s_k\}$ as the periodic union of p convergent sequences. Also, for brevity, statements involving the index p will hold for all positive integers p unless otherwise specified. Finally, in several instances where the form of a subscript k is typographically cumbersome, we use the notation $s(k)$ in place of s_k.

(3.1) *A sequence $\{s_k\}$ belongs to $|Z_p|$ if and only if it is the periodic union of p sequences of bounded variation.*

This is obvious from (1.5), the defining property of $|Z_p|$.

(3.2) *A sequence $\{s_k\}$ belongs to $\overline{|Z_p|}$ if and only if it is the periodic union of p convergent sequences.*

This follows directly from the fact that $|Z_1|$, the set of all sequences of bounded variation, is dense in the space (c) of convergent sequences.

(3.3) *None of the sets $|Z_p|$ is closed.*

From (3.1) and (3.2) it is clear that $\overline{|Z_p|} \setminus |Z_p| \neq \emptyset$.

(3.4) *The set $|Z| \equiv \bigcup_p |Z_p|$ is separable but not closed.*

Proof. Let $X_n \equiv \{s_k^n\}$ for $n = 1, 2, 3, \ldots$ be a dense subset of $|Z_1|$. Then the set S_p of all periodic unions of order p, formed from the X_n, is denumerable and dense in $|Z_p|$. Hence, $\bigcup_p S_p$ is denumerable and dense in $|Z|$.

To show that $|Z|$ is not closed we construct an example that will serve other purposes as well. For $n = 1, 2, 3, \ldots$ we define by induction a periodic sequence $y_n \equiv \{t_k^n\}$, of period 2^n, as follows: $y_1 = (0, 1, 0, 1, 0, 1, \ldots)$; if

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y_{n-1} of period 2^{n-1} has already been defined, then the initial period of y_n is obtained by writing in succession two periods of y_{n-1} and replacing the last entry of the second period by half of it. For $1 \leqslant k \leqslant 2^n$ we have $t_k^n = 0$ if k is odd, $=1$ if k is an odd multiple of 2, $=1/2$ if k is an odd multiple of 4, $\ldots$, $=1/2^{n-2}$ if k is an odd multiple of 2^{n-1}, $=1/2^{n-1}$ if $k = 2^n$. We define $y_0 \equiv \{t_k^0\}$ by setting $t_k^0 = 0$ if k is odd, $=1/2^{j-1}$ if k has the form $(2\alpha + 1)\cdot 2^j$. Then we find that $|t_k^0 - t_k^n| < 2^{1-n}$ for $k = 1, 2, 3, \ldots$, which implies $||y_0 - y_n|| \to 0$. Since $y_n \in |Z_{2^n}| \subset |Z|$ it follows that y_0 is a limit element of $|Z|$. But $y_0 \notin |Z|$ since by (3.1) each element of $|Z|$ has but a finite number of limit points on the real line, whereas y_0 has the infinite set of limit points $(0, 1, 1/2, 1/4, \ldots, 1/2^j, \ldots)$. Thus $|Z|$ is not closed.

(3.5) *If* $x = \{s_k\}$ *is the periodic union of* p *convergent sequences, then* $x \in Z_p$.

For, in this case Condition (1.4) is evidently satisfied.

(3.6) *Every sequence* $\{s_k\} \in Z_p$ *is either* (i) *the periodic union of* p *convergent sequences; or* (ii) *its set of limit points contains an interval.*

Proof. Assume that $\{s_k\} \in Z_p$ is not of the form (i). Then at least one (in fact, at least two) of the component sequences, say $\{s_{i+\lambda p}\} \equiv \{t_\lambda\}$, must be divergent. It is, moreover, bounded and satisfies the condition $\lim_\lambda (t_\lambda - t_{\lambda+1}) = 0$. It follows easily that every point of the interval $[\liminf t_\lambda, \limsup t_\lambda]$ is a limit point of $\{t_\lambda\}$, and therefore of $\{s_k\}$.

(3.7) *Each of the sets* Z_p *is closed.*

For each Z_p is the bounded convergence field of the regular matrix method (Z, p).

(3.8) *The set* $Z \equiv \bigcup_p Z_p$ *is not separable and not closed.*

Proof. By a result of Agnew (**1**, p. 99) the convergence field Z_p $(p > 1)$ is not separable since it contains a divergent sequence; see proof of (2.1)–(i). Hence Z is not separable. To see that Z is not closed we refer to the example under (3.4). Since $y_n \in Z_{2^n} \subset Z$, then $y_0 \in \bar{Z}$. But $y_0 \notin Z$ by (3.6).

(3.9) *For each* $p \geqslant 2$, $\overline{|Z_p|} \subset Z_p$ *and* $Z_p \setminus \overline{|Z_p|} \neq \emptyset$.

Proof. The first follows from $|Z_p| \subset Z_p$ and (3.7). To establish the second we consider the following example. For $j = 1, 2, 3, \ldots$ let $t_j \equiv a_1 + a_2 + \ldots + a_j$ be such that $\{t_j\}$ is bounded and divergent, with $a_j \to 0$. For $\lambda = 0, 1, 2, \ldots$ let $s_{1+\lambda p} \equiv t_{\lambda+1}$, $s_{2+\lambda p} \equiv -t_{\lambda+1}$, and $s_{i+\lambda p} \equiv 0$ for $3 \leqslant i \leqslant p$. Then $\{s_k\} \in Z_p$ since (1.4) holds, but $\{s_k\} \notin \overline{|Z_p|}$ since (3.2) is violated.

(3.10) *The set* $|Z|$ *coincides with* (usp).

Proof. Since (up) $\subset \bigcup_p |Z_p| \equiv |Z|$, we have $\overline{\text{(up)}} \equiv$ (usp) $\subset \overline{|Z|}$. To reverse the inclusion it will suffice to show that $|Z|$ is contained in the closed set

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(usp). Hence, let $\{s_k\} \in |Z|$ and let $\epsilon > 0$ be given. Then there exist integers p and K such that

$$S_k^p \equiv \sum_{j \geqslant k} |s_j - s_{j+p}| < \epsilon \qquad \text{for all } k > K.$$

For $\lambda = 1, 2, 3, \ldots$ and any $k > K$ we have

$$|s_k - s_{k+\lambda p}| \leqslant |s_k - s_{k+p}| + |s_{k+p} - s_{k+2p}| + \ldots$$
$$+ |s_{k+(\lambda-1)p} - s_{k+\lambda p}| \leqslant S_k^p < \epsilon.$$

Thus $\{s_k\} \in$ (usp) by (1.6).

(3.11) *The set* $|Z|$ *is a proper subset of* Z.

This is implied by the example in the proof of (3.9).

(3.12) *The set* $\overline{|Z|}$ *is a proper subset of* $\bar{Z}$.

Proof. To prove that the inclusion is proper, we construct the sequence $z \equiv \{u_k\}$ whose first three groups of terms are as follows:

$$|1/2, -1/2|_1 3/4, -3/4, 2/4, -2/4, 1/4, -1/4|_2$$
$$1/8, -1/8, 2/8, -2/8, \ldots, 7/8, -7/8|_3.$$

The nth group, containing $2^{n+1} - 2$ terms, is composed of all positive proper fractions $m/2^n$, each followed by its negative, and arranged so that $|u_k|$ is non-decreasing if n is odd, and non-increasing if n is even. It is clear that $(u_{n+1} + u_{n+2})/2 \to 0$ as $n \to \infty$ so that $z \in Z_2 \subset Z \subset \bar{Z}$.

To obtain a contradiction suppose that $z \in \overline{|Z|} =$ (usp), and let $0 < \epsilon < 1/4$. Then there exist integers p and K satisfying the condition of (1.6) for $z = \{u_k\}$. We can fix $K_1 > K$ so large that if u_k is the *first* term of any *odd*-numbered group with $k > K_1$, then one or more values of λ can be found for which $u_{k+\lambda p}$ falls in, but beyond the middle of, that group. Then $0 < u_k < 1/4$ and either $u_{k+\lambda p} > 1/2$ or $u_{k+\lambda p} < -1/2$. In either case we arrive at the inequality $|u_k - u_{k+\lambda p}| > 1/4 > \epsilon$ for infinitely many k and certain λ for each k. Since this is contrary to the condition of (1.6), it follows that $z \notin$ (usp), and the proof is complete.

We add here some remarks concerning the structure of the set Z. From (3.6) we see that $Z = Z_F \cup Z_I$, where $Z_F \equiv \{x \in Z | \, x \text{ is of the form (3.6)–(i)}\}$, $Z_I \equiv \{x \in Z | \, x \text{ is of the form (3.6)–(ii)}\}$, and $Z_F \cap Z_I = \emptyset$. By (3.2), $Z_F = \bigcup_p \overline{|Z_p|}$ so that $|Z| \subset Z_F$ and hence (usp) $= \overline{|Z|} \subset \bar{Z}_F$. On the other hand, since

$$Z_F = \bigcup_p \overline{|Z_p|} \subset \overline{\bigcup_p |Z_p|} = \overline{|Z|} = \text{(usp)},$$

we have $\bar{Z}_F \subset$ (usp). Consequently, $\bar{Z}_F =$ (usp), and since (usp) is separable, the set Z_F is non-dense in the non-separable set Z. Therefore, the complementary set Z_I is everywhere dense in Z.

(3.13) *The set* (usp) *is a proper subset of* (ap).

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The inclusion follows in a routine fashion from the definitions. Moreover, as indicated in (**3**, p. 364), the sequence $\{\cos k\}$ belongs to (ap) but not to (usp).

It is clear from what precedes that

$$Z_F \subset Z \subset \overline{Z}, \quad \text{and} \quad Z_F \subset \overline{|Z|} = (\text{usp}) \subset (\text{ap}).$$

The overlapping of these sets is clarified in the next proposition.

(3.14) *The following sets are non-empty:*

$$\text{(i) } Z\diagdown\overline{|Z|}; \qquad \text{(ii) } \overline{|Z|}\diagdown Z; \qquad \text{(iii) } Z\diagdown(\text{ap}); \qquad \text{(iv) } (\text{ap})\diagdown\overline{Z}.$$

Proof. (i) The sequence z of (3.12) belongs to Z but not to $\overline{|Z|}$.

(ii) The sequence y_0 defined in (3.4) belongs to $\overline{|Z|}$ but not to Z.

(iii) Referring again to the sequence $z \in Z$ defined in (3.12), let $v_1, v_2, \ldots, v_k, \ldots$ $(k = 1, 2, \ldots, 2^{n+1} - 2)$ denote the terms of the nth group for odd $n > 1$. Assume that z satisfies the definition of almost-periodicity (given in §1) for a given $\epsilon < 1/4$. For every odd n sufficiently large we can determine h_n so that

$$2^n < h_n < h_n + L < 2^{n+1} - 2.$$

Then for all q, $h_n < q < h_n + L$, we have

$$|v_1 - v_{1+q}| \geqslant |v_{1+q}| - v_1 > \tfrac{1}{2} - \tfrac{1}{8} > \tfrac{1}{4} > \epsilon.$$

This contradiction shows that $z \notin (\text{ap})$.

(iv) Since $\{\cos k\} \in (\text{ap})$ we assume that $\{s_k\}$ in a certain Z_p exists such that $|\cos k - s_k| < 1/4$ for all k. We now make use of the known fact that the sequence $\{e^{ipn}\}_{n=1}^{\infty}$ of complex numbers is everywhere dense on the unit circle. As a consequence there exist a sequence of positive integers $n_k \to \infty$, and an integer $m > 0$, such that $e^{ipn_k} \to 1$, and such that $e^{ipm} = e^{i\alpha}$ for some α $(2\pi/3 < \alpha < 4\pi/3)$. Since $e^{ip(n_k+m)} \to e^{i\alpha}$, we infer the existence of an integer $K > 0$ such that

$$|\cos pn_k - \cos p(n_k + m)| > 1$$

for all $k > K$. Moreover, in view of (1.4), $s_k - s_{k+p} \to 0$ as $k \to \infty$, so that an integer $K_0 \geqslant K$ exists for which

$$D \equiv |s(pn_k) - s(pn_k + pm)| \leqslant |s(pn_k) - s(pn_k + p)| + \ldots + |s(pn_k + mp - p) - s(pn_k + mp)| < 1/2$$

for all $k > K_0$. On the other hand, the foregoing inequalities imply that

$$D \geqslant |\cos pn_k - \cos p(n_k + m)| - |s(pn_k) - \cos pn_k| - |s(pn_k + pm) - \cos p(n_k + m)| > 1/2$$

for all $k > K_0$. This contradiction shows that $\{\cos k\}$ cannot be approximated by elements of Z, and the proof is complete.

In the light of the information assembled above concerning the various sets in question, their mutual relationship on the whole turns out to be rather more chaotic than one might desire. Consequently, the following interesting relations are somewhat unexpected.

$$Z_F = Z \cap (\text{usp}) = Z \cap (\text{ap}). \tag{3.15}$$

In the language of summability the set equations (3.15) may be phrased as follows.

(3.16) *An ultimately semiperiodic (or almost-periodic) sequence is summable by a method (Z, p) if and only if it is the periodic union of p convergent sequences.*

Proof. In view of $Z_F \subset Z \cap (\text{usp}) \subset Z \cap (\text{ap})$ it will suffice to show that $Z \cap (\text{ap}) \subset Z_F$. The latter inclusion is equivalent to the following assertion which we proceed to establish: *If* $x = \{s_k\} \in Z_p \setminus \overline{|Z_p|}$ *for some* p, *then* $x \notin (\text{ap})$. Since $\{s_k\} \in Z_p \setminus \overline{|Z_p|}$, there exists an index i $(1 \leqslant i \leqslant p)$ such that $\{s_{i+\lambda p}\}_{\lambda=0}^{\infty}$ diverges, and such that

$$\lim_{\lambda\to\infty} (s_{i+\lambda p} - s_{i+\lambda p+p}) = 0. \tag{3.17}$$

We introduce an auxiliary sequence $y \equiv \{t_k\} \in \overline{|Z_p|}$ by setting $t_k = M$ if $k \neq i + \lambda p$; $=0$ if $k = i + \lambda p$, for all $\lambda \geqslant 0$, where M is a constant at our disposal. Defining

$$z \equiv \{u_k\} \equiv \{s_k + t_k\} = x + y,$$

we note that z is likewise an element of $Z_p \setminus \overline{|Z_p|}$. We now fix M so large that

$$\limsup_\lambda u_{i+\lambda p} < -1 + \liminf_\lambda u_{j+\lambda p}$$

for all $j \neq i$ $(1 \leqslant j \leqslant p)$; and observe that the divergence of $\{s_{i+\lambda p}\} = \{u_{i+\lambda p}\}$ implies

$$a \equiv \liminf_\lambda u_{i+\lambda p} < \limsup_\lambda u_{i+\lambda p} \equiv b.$$

Then an integer N_1 exists such that

$$u_{j+\lambda p} > b + 1 \qquad \text{for all } j \neq i, \text{ all } \lambda > N_1. \tag{3.18}$$

The object of this manoeuvring is to isolate the relevant terms $s_{i+\lambda p} = u_{i+\lambda p}$ in an interval by translating all of the troublesome terms to the right a suitable distance M.

In order to arrive at a contradiction we assume now that $z \in (\text{ap})$ so that for a given $\epsilon < (b - a)/2$ there exist positive integers K and L satisfying the definition in §1 for the sequence $z = \{u_k\}$. Since $u_{i+\lambda p} = s_{i+\lambda p}$, we conclude from (3.17) that an integer $N > \max(K, N_1)$ exists such that

$$|u_{i+\lambda p} - u_{i+\lambda p+p}| < \epsilon/2L^2 \qquad \text{for all } \lambda > N. \tag{3.19}$$

We now fix $\lambda_1 > N$ and $\lambda_2 > \lambda_1 + L$ so that

$$u(i + \lambda_1 p) < a + \epsilon/4L, \qquad u(i + \lambda_2 p) > b - \epsilon/4L. \tag{3.20}$$

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Let the interval $(h, h+L)$ be defined by $h = (\lambda_2 - \lambda_1)p - L$, and let q assume all integral values $(\lambda_2 - \lambda_1)p - \mu$ $(1 \leqslant \mu \leqslant L-1)$ in this interval. If $q \not\equiv 0 \pmod p$, then $\mu \not\equiv 0 \pmod p$, and the subscript $i + \lambda_1 p + q$ becomes $i + \lambda_2 p - \mu$, which can be put in the form $\lambda p + j$, where $1 \leqslant j \leqslant p$, $j \neq i$, and $\lambda > N$. Hence, using (3.18) and (3.20), we find that

$$(3.21) \qquad |u(i + \lambda_1 p) - u(i + \lambda_1 p + q)| > \epsilon \qquad \text{for all } q \not\equiv 0 \pmod p.$$

If $q \equiv 0 \pmod p$, then q is of the form $(\lambda_2 - \lambda_1)p - \mu p$, where $1 \leqslant \mu \leqslant m$ for a certain $m \leqslant L - 1$. The subscript $i + \lambda_1 p + q$ now takes the form $i + (\lambda_2 - \mu)p$. From the last two inequalities, together with $\lambda_2 > \lambda_1 + L$, we see that $\lambda_2 - \mu > N$ for all μ in question. Consequently, using (3.19), we obtain

$$(3.22) \qquad |u(i + \lambda_2 p) - u(i + \lambda_2 p - \mu p)| \leqslant \mu\epsilon/2L^2 < \epsilon/2L \qquad (1 \leqslant \mu \leqslant m).$$

Finally, from (3.20) and (3.22), we find that

$$(3.23) \qquad \begin{aligned} &|u(i + \lambda_1 p) - u(i + \lambda_1 p + q)| \\ &\geqslant |u(i + \lambda_1 p) - u(i + \lambda_2 p)| - |u(i + \lambda_2 p) - u(i + \lambda_1 p + q)| \\ &> b - a - \epsilon/L > \epsilon \end{aligned}$$

for all $q \equiv 0 \pmod p$. If we set $k = i + \lambda_1 p$, then $k > N > K$, and it follows from (3.21) and (3.23) that $|u_k - u_{k+q}| > \epsilon$ for a certain $k > K$ and *all* q in the interval $(h, h+L)$. This contradiction shows that $z = \{u_k\} \notin$ (ap); and it follows at once that $x = z - y \notin$ (ap) since $y \in \overline{|Z_p|} \subset$ (ap). This completes the proof.

In view of (2.4), (3.6), and (3.15), the cardinal number and the density of the set x' of limit points of $x = \{s_k\}$ appear to play certain roles which we now investigate more fully. For example, because of (3.6), we see that if $x \in Z$ and x' is finite, then $x \in Z_F$. The result (2.4) is of the same type. We now establish the following fact.

(3.24) *If* $x \in$ (ap) *and* x' *is finite, then* $x \in Z_F$.

Proof. Let $c_1, c_2, \ldots, c_r$ be the distinct limit points of x, and choose the positive number ϵ smaller than $\min_{i \neq j} |c_i - c_j|/3$. With this ϵ we return to the definition of (ap) in §1 and choose any admissible q so that we have $|s_k - s_{k+q}| < \epsilon$ for all $k > K$. Then for all $k > K_1 \geqslant K$ every s_k will fall in one of the intervals $(c_i - \epsilon, c_i + \epsilon)$. Let $k_j = K_1 + j$ $(1 \leqslant j \leqslant q)$ and notice that if s_{k_j} lies in the interval $(c_{i_j} - \epsilon, c_{i_j} + \epsilon)$, then $s(k_j + \lambda q)$ will lie in the same interval for all $\lambda \geqslant 0$. Since $s(k_j + \lambda q) \to c_{i_j}$ as $\lambda \to \infty$, it is apparent that $\{s_k\} \in \overline{|Z_q|} \subset Z_F$, and that $q \geqslant r$.

The following corollary is immediate.

(3.25) *If* $x \in$ (usp) *and* x' *is finite, then* $x \in Z_F$.

The sequence $y_0 = \{t_k^0\}$ definied in the proof of (3.4) is such that $y_0 \in$ (usp) $\subset$ (ap), and its set of limit points y_0' is the denumerably infinite set (0, 1, 1/2, 1/4, . . .). We have seen in (3.6) that elements of this type do not occur in Z. On the other hand, the question arises whether there exist elements $x \in$ (usp) for which card(x') is more than denumerable infinite. The next examples show not only that such x exist, but that the density of x' can, in a sense, be arbitrary.

(3.26) *There exist elements* x_1, x_0, x_θ *in* (usp), *and hence in* (ap), *such that* (i) x_1' *is the closed interval* $[0, 1]$; (ii) x_0' *is the Cantor ternary set;* (iii) x_θ' *is a non-dense perfect set on* $[0, 1]$ *of measure* θ $(0 < \theta < 1)$.

Proof. (i) Let $s_k \equiv 0.a_0a_1a_2 \ldots$ (scale 2) if $k \geqslant 1$ has the form

$$k = a_0 + 2a_1 + 2^2a_2 + \ldots \qquad (a_i = 0 \text{ or } 1).$$

If $x_1 \equiv \{s_k\}$, then obviously $x_1' = [0, 1]$. Let $\epsilon > 0$ be given and fix q so that $2^{-q} < \epsilon$. In the characterization of (usp) in (1.6) choose $K = 1$ and $p = 2^q$. If $\lambda \geqslant 0$ is given as

$$\lambda = b_0 + 2b_1 + 2^2b_2 + \ldots \qquad (b_1 = 0 \text{ or } 1),$$

then $k + \lambda p$ may be written as $a_0 + 2a_1 + \ldots + 2^{q-1}a_{q-1} + 2^qc_q + \ldots$. Then

$$s_{k+\lambda p} = 0.a_0a_1 \ldots a_{q-1}c_q \ldots \quad \text{(scale 2)},$$

and one sees that $|s_k - s_{k+\lambda p}| \leqslant 2^{-q} < \epsilon$ for all $k > K$ and all $\lambda \geqslant 0$. Therefore $x_1 \in$ (usp).

(ii) Let $k = a_0 + 2a_1 + 2^2a_2 + \ldots$ as in (i), and set $t_k \equiv 0.d_0d_1d_2 \ldots$ (scale 3; $d_i = 2a_i$). The sequence $\{t_k\}$ is precisely the set of all right end points of the open intervals deleted in the geometric construction of C, the Cantor ternary set. Consequently, if $x_0 \equiv \{t_k\}$, it is clear that $x_0' = C$.

Before proceeding it will be helpful to make the following obvious remarks. The stage-q in the construction of C yields 2^{q-1} new points t_k for the values of k, $2^{q-1} \leqslant k \leqslant 2^q - 1$, obtained by setting $a_{q-1} = 1$; $a_0, a_1, \ldots, a_{q-2} = 0$ or 1; and $a_i = 0$ otherwise. Also at stage-q there remain 2^q closed intervals, each of length 3^{-q}, in which the next stage of the construction occurs.

We now specify $\epsilon > 0$ and fix q so that $\max(2\cdot 3^{1-q}, 2\cdot I_{q-1}) < \epsilon$, where I_{q-1} will be clarified in (iii) and is introduced here for expedience. Let $p = 2^{q-1}$ and consider for the moment only those values of $k = a_0 + 2a_1 + \ldots + 2^{q-2}a_{q-2} + 2^{q-1}$ corresponding to the right end points t_k obtained at stage-q. With λ as in (i), and $p = 2^{q-1}$, we find that

$$k + \lambda p = a_0 + 2a_1 + \ldots + 2^{q-2}a_{q-2} + 2^{q-1}b_0' + \ldots .$$

Consequently, $t_k = 0.d_0d_1 \ldots d_{q-2}\, 200 \ldots$, $t_{k+\lambda p} = 0.d_0d_1 \ldots d_{q-2}d'_{q-1} \ldots$ (scale 3), so that

$$\text{(3.27)} \qquad |t_k - t_{k+\lambda p}| < 3^{1-q}, \qquad k = p, p+1, \ldots, 2p-1; \text{ all } \lambda \geqslant 0.$$

Now choose $K = 2^{q-1}$ and let $k_i = p + i$ for $i = 0, 1, \ldots, p - 1$. Then every $j > K$ is of the form $k_i + \lambda_i p$ for some i and some λ_i. Hence, using (3.27),

$$|t_j - t_{j+\lambda p}| \leqslant |t(k_i + \lambda_i p) - t(k_i)| + |t(k_i) - t(k_i + \lambda p + \lambda_i p)| < 2\cdot 3^{1-q} < \epsilon$$

for all $j > K$ and all $\lambda \geqslant 0$. Thus $\{t_k\} \in$ (usp).

(iii) Let $0 < \theta < 1$ and modify the construction of C by removing at stage-1 a centred open interval of length $(1 - \theta)/3$; by removing at stage-2 a pair of centred open intervals, each of length $(1 - \theta)/3^2$; etc. (By centring we ensure the equality of the lengths, I_q, of the residual closed intervals at stage-q—merely a matter of convenience.) The resulting set C_θ is, of course, a non-dense perfect set of measure θ. By the familiar pairing of the removed intervals in the construction of two non-dense perfect sets, we can enumerate the right end points of C_θ in such a way that a right end point $u_k \in C_\theta$ is paired with its corresponding point $t_k \in C$. Then, evidently, $x_\theta \equiv \{u_k\}$ is such that $x_\theta' = C_\theta$. Moreover, it is clear from (3.27) that if $t_k \in C$ is any right end point at stage-q, then all of the points $t_{k+\lambda p}$ ($\lambda = 0, 1, 2, \ldots$) lie in a certain interval of length 3^{1-q} remaining at stage-$(q - 1)$. By the similarity mapping of $\{u_k\}$ onto $\{t_k\}$ it follows that all of the points $u_{k+\lambda p}$ ($\lambda = 0, 1, 2, \ldots$) lie in the corresponding interval of length I_{q-1} $[=o(1)]$ remaining at stage-$(q - 1)$ in the construction of C_θ. Therefore, recalling the choice of q in (ii), we have

$$|u_k - u_{k+\lambda p}| < I_{q-1} < \epsilon/2,$$

and the details proceed as in (ii). Thus, $x_\theta \in$ (usp).

As concluding remarks we note that, by (3.15), the elements x_1, x_0, x_θ are not in Z, although, by (3.10) and (3.12), they are in $\bar{Z}$. We observe also that, by (3.6), elements with the density types of x_0 and x_θ do not occur in Z.

4. Concerning convergence fields. Is the set (ac) the bounded convergence field of a regular matrix method of summability? This question was raised by Lorentz, who proved that the answer is negative **(6**, §7**)**. We now consider the corresponding question with respect to the sets (usp) and $\bar{Z}$. For (usp) the answer is immediately negative since (usp) is separable by (3.4) and (3.10), and the result of Agnew **(1**, p. 99**)** cited earlier can be applied. With regard to $\bar{Z}$ we have the following propositions.

(4.1) *The set $\bar{Z}$ is not the bounded convergence field of any regular matrix method with non-negative terms.*

(4.2) *The set $\bar{Z}$ is not the bounded convergence field of any regular Nörlund method.*

Proof of (4.1). Let $A = (a_{nk})$ be any regular matrix of non-negative terms whose convergence field $\mathfrak{F}(A)$ includes $\bar{Z}$. For each $p \geqslant 1$ there are p periodic sequences of 0's and 1's, say X_i^p, containing for each i a single "1" per period in the ith position ($i = 1, 2, \ldots, p$). Since each $X_i^p \in Z_p \subset \bar{Z} \subset \mathfrak{F}(A)$, it follows that each X_i^p is summable-A. Consequently, the conditions

$$(4.3) \qquad \lim_n \sum_{\lambda=0}^{\infty} a_{n,i+\lambda p} \equiv A_i^p \text{ exists}, \qquad i = 1, 2, \ldots, p; p = 1, 2, 3, \ldots,$$

are necessary in order that $\mathfrak{F}(A) \supset \bar{Z}$. It is now only a matter of checking details to verify that each method (Z, p) is *perfect*, and therefore consistent with every regular matrix method not weaker than (Z, p) (**2**, pp. 90–95). Since each X_i^p is summable-(Z, p) to the value $1/p$, we conclude that each limit A_i^p in (4.3) must be equal to $1/p$. Let $\epsilon > 0$ be given and fix p' so that $2/p' < \epsilon$. Then an integer $N(\epsilon)$ exists such that

$$\sum_{\lambda=0}^{\infty} a_{n,i+\lambda p'} < 2/p' < \epsilon \qquad \text{for all } n > N, \text{and all } i = 1, 2, \ldots, p'.$$

In view of the assumption $a_{nk} \geqslant 0$, we conclude that $a_{nk} < \epsilon$ for all $n > N$ and all $k = 1, 2, 3, \ldots$. This condition implies that the method A possesses summability functions (**6**, §6), from which it follows that A is effective for sequences of 0's and 1's that are not ultimately periodic. But (2.4) shows that no such sequences belong to $\bar{Z}$, and this completes the proof.

Proof of (4.2). Let $N = (q_{n-k}/Q_n)$ be a regular Nörlund matrix such that $\mathfrak{F}(N) \supset \bar{Z}$. Then $Q_n \equiv q_0 + q_1 + \ldots + q_n \neq 0$ and the following necessary and sufficient conditions for regularity are satisfied:

$$(4.4) \qquad \lim_n q_{n-k}/Q_n = 0 \qquad (k = 0, 1, 2, \ldots),$$

$$(4.5) \qquad \sum_{k=0}^{n} |q_n| = O(|Q_n|) \qquad (n \to \infty).$$

All methods N may be separated into two disjoint classes according as (a) $\sum |q_k| < \infty$; or (b) $\sum |q_k| = \infty$, and by (4.5) the condition (b) is equivalent to (b') $|Q_n| \to \infty$. To dispose of the case (b') it is sufficient to observe that the condition $|Q_n| \to \infty$ entails the conditions (4.4), *uniformly* in k (**7**, pp. 37–38). This, in turn, endows the method N with summability functions, and the proof is completed as in the preceding theorem.

In case (a) we set $Q \equiv \sum q_k$ and notice that (4.5) implies $Q \neq 0$. Then the necessary conditions (4.3), with $A_i^p = 1/p$, may be written in the form

$$(4.6) \qquad \lim_n \sum_{\lambda=0}^{\lambda_{ni}} q_{i+\lambda p} = Q/p \qquad (i = 0, 1, \ldots, p-1; p = 1, 2, 3, \ldots),$$

where λ_{ni} is a certain index $\leqslant n$. We complete the argument by showing that (4.6) implies that all q_k are equal, and hence equal to zero, and this contradicts $q_0 \neq 0$. Hence, if possible, let m be a positive integer such that $q_m \neq q_0$, and fix $j > m$ so that $\sum_{k>j} |q_k| < |q_m - q_0|/2$. If we now fix $p > j$, then

$$\left| \sum_{\lambda=0}^{\lambda_{n0}} q_{\lambda p} - \sum_{\lambda=0}^{\lambda_{nm}} q_{m+\lambda p} \right| \geqslant |q_0 - q_m| - \sum_{k>j} |q_k| > |q_m - q_0|/2 > 0,$$

for all n sufficiently large. This involves a contradiction to (4.6) for $i = 0$ and $i = m$, and hence the integer m does not exist.

As final remarks we conjecture (i) that (4.1) is true without the restriction $a_{nk} \geqslant 0$; and (ii) that the set (ap) is not the bounded convergence field of any regular matrix method.

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