L. Accardi and Y. G. Lu
Nagoya Math. J.
Vol. 126 (1992), 25–87

THE LOW DENSITY LIMIT IN FINITE TEMPERATURE CASE

L. ACCARDI AND Y. G. LU*

§ 1. Introduction

The low density limit in the Boson Fock case has been investigated in [1] where also the physical meaning and their motivations have been explained (cf. also [0]). From these papers one knows how the number processes can be obtained from a quantum Hamiltonian model via a certain limit procedure.

But the physically more meaningful case is the one in which the initial state of the reservoir is a finite temperature rather than Fock state. The present paper is devoted to do this case.

In the present paper, our physical model and essential assumptions are same as ones of [1] (except the Fock state). We consider a "System + Reservoir" model described by a system Hilbert space H_0 and a (one particle) reservoir Hilbert space H_1; the system Hamiltonian H_S; the (one particle) reservoir Hamiltonian Δ, and the associated one particle reservoir evolution

$$S_t^0 := \exp(-it\Delta) \tag{1.0}$$

which is a one parameter unitary group on $B(H_1)$.

Let be given, on the CCR C^*-algebra over H_1 (cf. [11]), a gauge invariant quasi-free state $\varphi^{(z)}$ with fugacity z^2, i.e. for each $f \in H_1$,

$$\varphi^{(z)}(W(f)) = \exp\left(-\frac{1}{2}\langle f, (1 + z^2 e^{-\frac{1}{2}H\beta})(1 - z^2 e^{-\frac{1}{2}H\beta})^{-1} f\rangle\right) \tag{1.1}$$

where H is a self-adjoint operator which, will be supposed in this paper, commutes with Δ and $W(f)$ is the Weyl operator with test function f. Moreover up to GNS-construction one can write the left hand side of (1.1) as

$$\langle \Phi_z, W(f)\ \Phi_z \rangle$$

Received March 29, 1991.

*On leave of absence from Beijing Normal University

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

where the Weyl operators act on a Hilbert space $\mathscr{H}_z$ and Φ_z is a cyclic vector for the Weyl algebra. For any Hilbert space $\mathscr{H}$, we shall denote $\Gamma(\mathscr{H})$ the Fock space over $\mathscr{H}$.

As in [1], we consider the situation in which the interaction between the System and Reservoir is of the form

$$V := i(D \otimes A^+(g_0)A(g_1) - D^+ \otimes A(g_1)^+A(g_0)) \tag{1.2}$$

and the time evolution in the interaction picture is defined by

$$U_t := e^{itH_{\mathrm{fr}}}.\ e^{-itH_{\mathrm{tot}}} \tag{1.3}$$

where

$$H_{\mathrm{fr}} := H_0 \otimes 1 + 1 \otimes d\Gamma(-\Delta) \tag{1.4a}$$

is called the free Hamiltonian and

$$H_{\mathrm{tot}} := H_{\mathrm{fr}} + V \tag{1.4b}$$

the total Hamiltonian.

A simple computation shows that the time evolution $U(t)$ is the solution of the following ordinary differential equation

$$\frac{d}{dt}U_t = iV(t) \cdot U_t \tag{1.5}$$

and where

$$V(t) := e^{itH_{\mathrm{fr}}}\, V\, e^{-itH_{\mathrm{fr}}} \tag{1.6}$$

Now, as in [1], let us suppose that

1) g_0 and g_1 have disjoint spectrum sets, i.e.

$$\langle g_0, S_t^0 \quad g_1 \rangle = 0, \qquad \forall t \in \mathbf{R} \tag{1.7}$$

2) the rotating wave approximation:

$$e^{itH_0}\, D\, e^{-itH_0} = e^{-it(\omega_0 - \omega_1)}\, D, \qquad \text{for some } \omega_0 \neq \omega_1. \tag{1.8}$$

These two conditions have rather good physical explanations and under them, the same arguments as in [1] (or [0]) imply that (1.6) can be rewritten as

$$V(t) = i(D \otimes A^+(S_t g_0)A(S_t g_0) - D^+ \otimes A(S_t g_1)^+A(S_t g_0)) \tag{1.9}$$

where S_t is a unitary group satisfying

$$S_t g_0 = S_t^0\, e^{-it\omega_0} g_0 \qquad S_t g_1 = S_t^0\, e^{it\omega_1} g_1. \tag{1.10}$$

In this paper we shall directly start our research from (1.9) and in certain sense **forget** the motivation of this assumption, i.e. starting from (1.9), we investigate, purely mathematically, the low density limit. Of course one of physically meaningful case of our study is the one with the conditions (1.7) and (1.8). Moreover, we shall suppose that the unitary group $\{S_t\}_{t\in\mathbf{R}}$ commutes with Δ; there exists a non-zero subset K of H_1 such that $g_0, g_1 \in K$,

(1.11) $$\int_{\mathbf{R}} |\langle f, S_t g\rangle| \, dt < \infty, \ \forall f, g \in K$$

and for each $f, g \in K$, the series

$$\sum_{n=1}^{\infty} z^n \cdot \int_{\mathbf{R}} |\langle f, S_t \exp(-\beta n H) g\rangle| \, dt$$

has a positive convergence radius.

With the condition (1.11), one can define a Hilbert space $\{K, (\cdot|\cdot)\}$ in same way as [1], where, for each $f, g \in K$,

(1.12) $$(f|g) := \int_{\mathbf{R}} \langle f, S_t g\rangle \, dt.$$

Inspired by [1], in this paper we shall investigate the limit

(1.13) $$\lim_{z\to 0} \langle u \otimes W\left(z\int_{S/z^2}^{T/z^2} S_u f du\right) \Phi_z, \ U_{t/z^{2v}} \otimes W\left(z\int_{S'/z^2}^{T'/z^2} S_u f' du\right) \Phi_z \rangle.$$

It will be proved that the limit (1.13) exists and is equal to

(1.14) $$\langle u \otimes W(\chi_{[S,T]} \otimes f)\Psi, \ U(t)v \otimes W(\chi_{[S',T']} \otimes f')\Psi\rangle$$

where the scalar product in (1.14) is meant in a Fock quantum Brownian motion in the sense of [1] (cf. also [0]) and $U(t)$ is a Markovian cocycle satisfying a quantum stochastical differential equation (q.s.d.e.). More precisely, the main result in the present paper is the following:

THEOREM (1.1). *For each $f, f', g_0, g_1 \in K$, $u, v \in H_0$, $D \in B(H_0)$, $S, T, S', T' \in \mathbf{R}$, $t \geq 0$, if t, g_0, g_1 satisfy the condition*

(1.15)
$$t < \frac{1}{16\|D\|}; \ \|g\|_S^2 := \left(\max_{g,g' \in \{g_0, g_1, e^{-\frac{1}{2}\beta H} g_0, e^{-\frac{1}{2}\beta H} g_1\}} \int_{-\infty}^{\infty} |\langle g, S_t g'\rangle| \, dt\right)^2 < \frac{1}{16\|D\|}$$

then the low density limit (1.13) *exists and is equal to* (1.14) *where,* $U(t)$ *is the solution of* q.s.d.e.

(1.16) $$U(t) = 1 + \sum_{\varepsilon\in\{0,1\}} \int_0^t \Big[D_1(\varepsilon) \otimes dN_s(g_\varepsilon, g_{1-\varepsilon}) + D_2(\varepsilon) \otimes dN_s(g_\varepsilon, g_\varepsilon) + (D_3(\varepsilon) + D_\varepsilon \langle g_{1-\varepsilon}, e^{-\frac{1}{2}\beta H} g_\varepsilon\rangle) \otimes 1 ds\Big] U(s)$$

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

on $H_0 \otimes \Gamma\Big(L^2(\mathbf{R}) \otimes (K, (\cdot|\cdot))\Big)$ *and*

(i) Ψ *is the vacuum of* $\Gamma\Big(L^2(\mathbf{R}) \otimes (K, (\cdot|\cdot))\Big)$;

(ii) $N_s(g, g') := N(\chi_{[0,s]} \otimes |g\rangle\langle g'|)$;

(iii) *For any operator* T *on* $L^2(\mathbf{R}) \otimes K$, $N(T)$ *denotes the number operator generated by* T *characterized by the condition*

$$e^{it\,N(T)} W(\xi)\Psi = W(e^{itT}\xi)\Psi, \ \forall\ \xi \in L^2(\mathbf{R}) \otimes K$$

for T *self-adjoint, and extended by complex linearity to arbitrary* T.

(iv) $D_1(\varepsilon)$ $D_2(\varepsilon)$ *and* $D_3(\varepsilon)$ *are given by* (6.80), (6.81) *and* (6.65) *respectively.*

Remark. Since in the Hilbert space $\Gamma(L^2(\mathbf{R}) \otimes K)$, the set $\{W(\chi \otimes f)\Psi;\ \chi \in L^2(\mathbf{R}), f \in K\}$ is a total subset, we know that the linear functions

$$\langle u \otimes W(\chi_{[S,T]} \otimes f)\Psi, v \otimes W(\chi_{[S',T']} \otimes f')\Psi\rangle$$

can separate (bounded) operators. Therefore the Theorem (1.1) gives a stochastic process as the low density limit of the time evolution of the original Hamiltonian model.

§2. Preliminaries

In this section, some preparations will be done for the following sections.

LEMMA (2.1). *For each* $f, f' \in K$, $S, S', T, T' \in \mathbf{R}$,

$$\lim_{z \to 0} \langle z \int_{S/z^2}^{T/z^2} S_u f du,\ z \int_{S'/z^2}^{T'/z^2} S_u f' du \rangle \tag{2.1}$$

$$= \langle \chi_{[S,T]}, \chi_{[S',T']} \rangle_{L^2(\mathbf{R})} \cdot \int_{\mathbf{R}} \langle f, S_t f' \rangle\, dt.$$

One can find the proof in [1].

LEMMA (2.2). *For each* $n \in \mathbf{N}$, $\{f_k\}_{k=1}^n \subset K$, $\{S_k, T_k\}_{k=1}^n \subset \mathbf{R}$, $\{x_k\}_{k=1}^n \subset \mathbf{R}$,

$$\lim_{z \to 0} \langle \Phi_z, W\Big(x_1 z \int_{S_1/z^2}^{T_1/z^2} S_u f_1 du\Big) \cdots W\Big(x_n z \int_{S_n/z^2}^{T_n/z^2} S_u f_n du\Big) \Phi_z \rangle$$

$$= \langle \Psi, W\Big(x_1 \chi_{[S_1,T_1]} \otimes f_1\Big) \cdots W\Big(x_n \chi_{[S_n,T_n]} \otimes f_n\Big) \Psi \rangle \tag{2.2}$$

and the convergence is uniform for $\{x_k, S_k, T_k\}_{k=1}^n$ *in a bounded set of* $\mathbf{R}$, *where,* Ψ *is the vaccum of* $\Gamma\Big(L^2(\mathbf{R}) \otimes (K, (\cdot|\cdot))\Big)$.

One can find the proof in [1].

Denote H_1^ι the conjugate Hilbert space of H_1, i.e.

$$\iota : H_1 \longrightarrow H_1,\ \iota(\lambda f) := \bar{\lambda}\iota(f) \tag{2.3}$$

$$\langle \iota(f), \iota(g) \rangle_\iota := \langle g, f \rangle \tag{2.4}$$

then, H_1^ι is a Hilbert space. It is well known that up to a unitary isomorphism

$$\Gamma(H_1) = \Gamma(H_1) \otimes \Gamma(H_1^\iota) \tag{2.5}$$

$$W(f) = W(Q_+ f) \otimes W(Q_- f) \tag{2.6}$$

for each $f \in H_1$ and

$$\Phi_z = \Phi \otimes \Phi^\iota \tag{2.7}$$

where Φ, Φ^ι are the vaccum vectors in $\Gamma(H_1)$ and $\Gamma(H^\iota{}_1)$ respectively and

$$Q_+ := \sqrt{\frac{Q+1}{2}} = \sqrt{\frac{1}{2}\cdot\frac{2}{1 - z^2\exp(-\frac{1}{2}H\beta)}} = \frac{1}{\sqrt{1 - z^2\exp(-\frac{1}{2}H\beta)}} \tag{2.8}$$

and

$$Q_- := \iota\sqrt{\frac{Q-1}{2}} = \iota\sqrt{\frac{1}{2}\cdot\frac{2\exp(-\frac{1}{2}H\beta)}{1 - z^2\exp(-\frac{1}{2}H\beta)}} \tag{2.9}$$

$$= z\iota\sqrt{\frac{\exp(-\frac{1}{2}H\beta)}{1 - z^2\exp(-\frac{1}{2}H\beta)}} =: zQ^-.$$

Moreover, we have

$$A(f) = A(Q_+ f) \otimes 1 + 1 \otimes A^+(Q_- f). \tag{2.10}$$

So up to a unitary isomorphism:

$$A^+(f)A(g) = \Big(A^+(Q_+ f) \otimes 1 + 1 \otimes A(Q_- f)\Big)\cdot \tag{2.11}$$

$$\cdot\Big(A(Q_+ g) \otimes 1 + 1 \otimes A^+(Q_- g)\Big)$$

$$= A^+(Q_+ f)A(Q_+ g) \otimes 1 + z\Big(A^+(Q_+ f) \otimes A^+(Q^- g)$$

$$+ A(Q_+ g) \otimes A(Q^- f)\Big) + z^2 1 \otimes A(Q^- f)A^+(Q^- g).$$

By the CCR, the last term of the right hand side of (2.11) is equal to

$$z^2\, 1 \otimes \Big[A^+(Q^- g)A(Q^- f) + \langle Q^- f, Q^- g \rangle_\iota\Big]$$

therefore the right hand side of (2.11) can be rewritten as

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

$$\text{(2.11a)}\quad A^+(Q_+f)A(Q_+g)\otimes 1 + z\Big(A^+(Q_+f)\otimes A^+(Q^-g) + A(Q_+g)\otimes A(Q_-f)\Big) + + z^2\Big(1\otimes A^+(Q^-g)A(Q^-f) + \langle Q^-f, Q^-g\rangle_\iota\Big)$$
$$=: A_0(f, g) + z\Big(A_{-1}(f, g) + A_1(f, g)\Big) + z^2A_2(f, g) + z^2A_{-2}(f, g).$$

Our starting point of this paper is, as in [1], the iterative solution of (1.5):

$$\text{(2.12)}\qquad U_t = \sum_{n=0}^{\infty}(-i)^n\int_0^t dt_1\cdots\int_0^{t_{n-1}} dt_n V(t_1)\cdots V(t_n).$$

Now, we want to show what is the difference between the Fock and non-Fock cases.

In (2.11a), the term $A_0(f, g)$ is similar to that in the Fock case and the term $z^2A_{\pm 2}(f, g)$ will, roughly speaking, not play any important role. In the term $z\Big(A_{-1}(f, g) + A_1(f, g)\Big)$, if there exists some $A(Q^-g)$ acting on $W\Big(z\int_{S/z^2}^{T/z^2} Q_-S_u f du\Big)$ or $W\Big(z\int_{S'/z^2}^{T'/z^2} Q_-S_u f' du\Big)$, since $Q_- = zQ^-$ *and* Q^- tends to $\iota\circ\exp(-\frac{1}{4}\beta H)$ as $z\to 0$, we shall get so "many" z that the term will go to zero. But the new situation is not the same as that in the Fock case: there it is not possible that all the operators are "used" to produce scalar products, but here is possible. For example, in the case of $n = 2$, we can get the following term

$$\text{(2.13)}\qquad A(S_{t_1}Q_+g_0)\otimes A(Q_-S_{t_1}g_1)\cdot A^+(S_{t_2}Q_+g_0)\otimes A^+(Q_-S_{t_2}g_1).$$

From it one can get the following term.

$$\text{(2.14)}\qquad \langle Q_+g_0, S_{t_2-t_2}Q_+g_0\rangle\cdot\langle Q_-g_1, Q_-S_{t_2-t_2}g_1\rangle_\iota.$$

So in the low density limit, the following term will be obtained

$$\text{(2.15)}\quad \lim_{z\to 0}\int_0^{t/z^2} dt_1\int_0^{t_1} dt_2\langle Q_+g_0, S_{t_2-t_1}Q_+g_0\rangle\cdot\langle Q_-g_1, Q_-S_{t_2-t_1}g_1\rangle_\iota$$
$$\lim_{z\to 0}\int_0^{t/z^2} dt_1\int_0^{t_1} dt_2 z^2\langle Q_+g_0, S_{t_2-t_1}Q_+g_0\rangle\cdot\langle Q^-g_1, Q^-S_{t_2-t_1}g_1\rangle_\iota$$
$$\lim_{z\to 0}\int_0^t dt_1\int_{-t_1/z^2}^0 dt_2\langle Q_+g_0, S_{t_2}Q_+g_0\rangle\cdot\langle Q^-g_1, Q^-S_{t_2}g_1\rangle_\iota$$
$$=\int_0^t dt_1\int_{-\infty}^0 du\langle g_0, S_ug_0\rangle\cdot\overline{\langle e^{-\frac{1}{4}\beta H}g_1, S_ue^{-\frac{1}{4}\beta H}g_1\rangle}.$$

In the case of $n\geq 2$, one gets similar terms.

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

§ 3. The collective terms and negligible terms

The section is devoted to distinguish the collective terms from the negligible terms. First of all by (1.20) and (1.7), for each $n \in \mathbf{N}$, $\varepsilon \in \{0,1\}^n$ up to a unitary isomorphism,

$$
(3.1) \qquad A^+(S_{t_1} g_{\varepsilon(1)}) A(S_{t_1} g_{1-\varepsilon(1)}) \cdots A^+(S_{t_n} g_{\varepsilon(n)}) A(S_{t_n} g_{1-\varepsilon(n)})
$$

$$
= \Big(A^+(S_{t_1} Q_+ g_{\varepsilon(1)}) A(S_{t_1} Q_+ g_{1-\varepsilon(1)}) \otimes 1 + z\Big(A^+(S_{t_1} Q_+ g_{\varepsilon(1)}) \otimes A^+(Q^- S_{t_1} g_{1-\varepsilon(1)})
$$

$$
+ A(S_{t_1} Q_+ g_{1-\varepsilon(1)}) \otimes A(Q^- S_{t_1} g_{\varepsilon(1)}) \Big)
$$

$$
+ z^2 1 \otimes \Big(A^+(S_{t_1} Q^- g_{1-\varepsilon(1)}) A(Q^- S_{t_1} g_{\varepsilon(1)}) + \langle Q^- g_{\varepsilon(1)}, Q^- g_{1-\varepsilon(1)} \rangle_\iota \Big) \Big) \cdot \cdots \cdot
$$

$$
\Big(A^+(S_{t_n} Q_+ g_{\varepsilon(n)}) A(S_{t_n} Q_+ g_{1-\varepsilon(n)}) \otimes 1 + z \Big(A^+(S_{t_n} Q_+ g_{\varepsilon(n)}) \otimes A^+(Q^- S_{t_n} g_{1-\varepsilon(n)} \Big)
$$

$$
+ A(S_{t_n} Q_+ g_{1-\varepsilon(n)}) \otimes A(Q^- S_{t_n} g_{\varepsilon(n)}) \Big)
$$

$$
+ z^2 1 \otimes (A^+(Q^- S_{t_n} g_{1-\varepsilon(n)}) A(Q^- S_{t_n} g_{\varepsilon(n)}) + \langle Q^- g_{\varepsilon(n)}, Q^- g_{1-\varepsilon(n)} \rangle_\iota) \Big).
$$

Expanding the right hand side of (3.1) and using the notations in the right hand side of (1.20a), one can write (3.1) to

$$
(3.2) \qquad \sum_{\sigma \in \{0, \pm 1, \pm 2\}^n} A_{\sigma(1)}(S_{t_1} g_{\varepsilon(1)}, S_{t_1} g_{1-\varepsilon(1)}) \cdots A_{\sigma(n)}(S_{t_n} g_{\varepsilon(1)}, S_{t_n} g_{1-\varepsilon(n)}) z^{\sum_{k=1}^n |\sigma(k)|}.
$$

In order to write (3.2) to a more clear form, let us label A_1 by $1 \le i_1 < \cdots < i_k \le n$; A_{-1} by $1 \le i'_1 < \cdots < i'_{k'} \le n$; A_2 by $1 \le i''_1 < \cdots < i''_{k''} \le n$ and A_{-2} by $1 \le i'''_1 < \cdots < i'''_{k'''} \le n$, then the terms A_0 are naturally labeled by the indices set $\{1,\dots, n\} \setminus (i_h\}_{h=1}^k \cup (i'_h\}_{h=1}^{k'} \cup (i''_h\}_{h=1}^{k''} \cup (i'''_h\}_{h=1}^{k'''})$. Thus (3.2) is equal to the following expression

$$
(3.3) \qquad \sum_{k=0}^{n} \sum_{1 \le i_1 < \cdots < i_k \le n} \sum_{k'=0}^{n-k} \sum_{\substack{1 \le i'_1 < \cdots < i'_{k'} \le n \\ \{i_h\}_{h=1}^k \cap \{i'_h\}_{h=1}^{k'} = \emptyset}} \sum_{k''=0}^{n-k-k'} \sum_{\substack{1 \le i''_1 < \cdots < i''_{k''} \le n \\ \{i''_h\}_{h=1}^{k''} \cap (\{i_h\}_{h=1}^k \cup \{i'_h\}_{h=1}^{k'}) = \emptyset}}
$$

$$
\sum_{k'''=0}^{n-k-k'-k''} \sum_{\substack{1 \le i'''_1 < \cdots < i'''_{k'''} \le n \\ \{i'''_h\}_{h=1}^{k'''} \cap (\{i_h\}_{h=1}^k \cup \{i'_h\}_{h=1}^{k'} \cup \{i''_h\}_{h=1}^{k''}) = \emptyset}}
$$

$$
z^{k+k'+2(k''+k''')} A_0(S_{t_1} g_{\varepsilon(1)}, S_{t_1} g_{1-\varepsilon(1)}) \cdots A_1(S_{t_{i_1}} g_{\varepsilon(i_1)}, S_{t_{i_1}} g_{1-\varepsilon(i_1)}) \cdots
$$

$$
A_{-1}(S_{t_{i'_1}} g_{\varepsilon(i'_1)}, S_{t_{i'_1}} g_{1-\varepsilon(i'_1)}) \cdots A_2(S_{t_{i''_1}} g_{\varepsilon(i''_1)}, S_{t_{i''_1}} g_{1-\varepsilon(i''_1)}) \cdots
$$

$$A_1(S_{t_{i_k}} g_{\varepsilon(i_k)}, S_{t_{i_k}} g_{1-\varepsilon(i_k)}) \cdots A_{-1}(S_{t_{i'_{k'}}} g_{\varepsilon(i'_{k'})}, S_{t_{i'_{k'}}} g_{1-\varepsilon(i'_{k'})}) \cdots$$

$$A_2(S_{t_{i''_{k''}}} g_{\varepsilon(i''_{k''})}, S_{t_{i''_{k''}}} g_{1-\varepsilon(i''_{k''})}) \cdots A_0(S_{t_n} g_{\varepsilon(n)} S_{t_n} g_{1-\varepsilon(n)})$$

$$\prod_{h=1}^{k'''} \langle Q^- g_{\varepsilon(i'''_h)}, Q^- g_{1-\varepsilon(i'''_h)} \rangle_\iota$$

$$= \sum_{k=0}^{n} \sum_{1\le i_1<\cdots<i_k\le n} \sum_{k'=0}^{n-k} \sum_{\substack{1\le i'_1<\cdots<i'_{k'}\le n \\ \{i_h\}_{h=1}^k \cap \{i'_h\}_{h=1}^{k'}=\emptyset}} \sum_{k''=0}^{n-k-k'} \sum_{\substack{1\le i''_1<\cdots<i''_{k''}\le n \\ \{i''_h\}_{h=1}^{k''}\cap(\{i_h\}_{h=1}^k \cup \{i'_h\}_{h=1}^{k'})=\emptyset}}$$

$$\sum_{k'''=0}^{n-k-k'-k''} \sum_{\substack{1\le i'''_1<\cdots<i'''_{k'''}\le n \\ \{i'''_h\}_{h=1}^{k'''}\cap(\{i_h\}_{h=1}^k \cup \{i'_h\}_{h=1}^{k'} \cup \{i''_h\}_{h=1}^{k''})=\emptyset}}$$

$$z^{k+k'+2(k''+k''')} A^+(S_{t_1} Q_+ g_{\varepsilon(1)}) A(S_{t_1} Q_+ g_{1-\varepsilon(1)}) \cdots A(S_{t_{i_1}} Q_+ g_{1-\varepsilon(i_1)})$$

$$\cdots A^+(S_{t_{i'_1}} Q_+ g_{\varepsilon(i'_1)}) \cdots A(S_{t_{i_k}} Q_+ g_{1-\varepsilon(i_k)})$$

$$\cdots A^+(S_{t_{i'_{k'}}} Q_+ g_{\varepsilon(i'_{k'})}) \cdots A^+(S_{t_n} Q_+ g_{\varepsilon(n)}) \mathrm{A}(S_{t_n} Q_+ g_{1-\varepsilon(n)})$$

$$\otimes A(Q^- S_{t_{i_1}} g_{\varepsilon(i_1)}) \cdots A^+(Q^- S_{t_{i'_1}} g_{1-\varepsilon(i'_1)}) \cdots$$

$$A(Q^- S_{t_{i''_1}} g_{1-\varepsilon(i''_1)}) A^+(Q^- S_{t_{i''_1}} g_{\varepsilon(i''_1)}) \cdots$$

$$A(Q^- S_{t_{i_k}} g_{\varepsilon(i_k)}) \cdots A+(Q^- S_{t_{i'_{k'}}} g_{1-\varepsilon(i'_{k'})}) \cdots$$

$$A(Q^- S_{t_{i''_{k''}}} g_{1-\varepsilon(i''_{k''})}) \cdots A^+(Q^- S_{t_{i''_{k''}}} g_{\varepsilon(i''_{k''})})$$

$$\prod_{h=1}^{k'''} \langle Q^- g_{\varepsilon(i'''_h)}, Q^- g_{1-\varepsilon(i'''_h)} \rangle_\iota$$

$$=: \mathrm{I}_n^\varepsilon + \mathrm{II}_n^\varepsilon$$

where, I_n^ε denotes those terms satisfying $k''=0$ and $\mathrm{II}_n^\varepsilon$ denotes those terms satisfying $k''\ge 1$.

Now, we write I_n^ε and $\mathrm{II}_n^\varepsilon$ in normally ordered form. Then, one has

LEMMA (3.1). *For each* $n\in \boldsymbol{N}$, $\varepsilon\in\{0,1\}^n$,

$$\mathrm{I}_n^\varepsilon = \sum_{k=0}^{n} \sum_{1\le i_1<\cdots<i_k\le n} \sum_{k'=0}^{n-k} \sum_{\substack{1\le i'_1<\cdots<i'_{k'}\le n \\ \{i_h\}_{h=1}^k \cap \{i'_h\}_{h=1}^{k'}=\emptyset}} \sum_{k'''=0}^{n-k-k'} \sum_{\substack{1\le i'''_1<\cdots<i'''_{k'''}\le n \\ \{i'''_h\}_{h=1}^{k'''}\cap(\{i_h\}_{h=1}^k \cup \{i'_h\}_{h=1}^{k'})=\emptyset}} \tag{3.4}$$

$$\sum_{m=0}^{n-k-k'-k'''} \sum_{\substack{1\le q_1<\cdots<q_m\le n \\ \{q_h\}_{h=1}^m \subset\{1,\dots,n\}\setminus(\{i_h\}_{h=1}^k \cup \{i'_h\}_{h=1}^{k'} \cup \{i'''_h\}_{h=1}^{k'''})}} \sum_{m'=0}^{k'\wedge(n-k-k'''-m)} \sum_{\substack{1\le q'_1<\cdots<q'_{m'}\le i_{k'} \\ \{q'_h\}_{k=1}^{m'}\subset\{i'_h\}_{h=1}^{k'}}}$$

$$\sum_{(p_1,\dots,p_m,p'_1,\dots,p'_{m'})} z^{k+k'}$$

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

$$\prod_{h=1}^{m} \langle S_{t_{p_h}} Q_+ g_{1-\varepsilon(p_h)},\ S_{t_{q_h}} Q_+ g_{\varepsilon(q_h)} \rangle \cdot \prod_{h=1}^{m'} \langle S_{t_{p'_h}} Q_+ g_{1-\varepsilon(p'_h)},\ S_{t_{q'_h}} Q_+ g_{\varepsilon(q'_h)} \rangle$$

$$\prod_{\alpha \in \{1,\cdots,n\} \setminus (\{i_h\}_{h=1}^{k} \cup \{i'_h\}_{h=1}^{k'} \cup \{i'''_h\}_{h=1}^{k'''} \cup \{q_h\}_{h=1}^{m})} A^+(S_{t_\alpha} Q_+ g_{\varepsilon(\alpha)})$$

$$\prod_{\alpha \in \{i'_h\}_{h=1}^{k'} \setminus \{q'_h\}_{h=1}^{m'}} A^+(S_{t_\alpha} Q_+ g_{\varepsilon(\alpha)})$$

$$\prod_{\alpha \in \{1,\cdots,n\} \setminus (\{i'_h\}_{h=1}^{k'} \cup \{i'''_h\}_{h=1}^{k'''} \cup \{p_h\}_{h=1}^{m} \cup \{p'_h\}_{h=1}^{m'})} A(S_{t_\alpha} Q_+ g_{1-\varepsilon(\alpha)})$$

$$\otimes \sum_{m''=0}^{k' \wedge k} \sum_{\substack{1 \le q''_1 < \cdots < q''_{m''} \le i'_{k'} \\ \{q''_h\}_{h=1}^{m''} \subset \{i'_h\}_{h=1}^{k'}}} \sum_{(p''_1, \cdots, p''_{m''})}$$

$$\prod_{h=1}^{m} \langle Q_- S_{t_{p''_h}} g_{\varepsilon(p''_h)},\ Q_- S_{t_{q''_h}} g_{1-\varepsilon(q''_h)} \rangle_\iota$$

$$\prod_{\alpha \in \{i'_h\}_{h=1}^{k'} \setminus \{q''_h\}_{h=1}^{m''}} A^+(Q_- S_{t_\alpha} g_{1-\varepsilon(\alpha)}) \cdot \prod_{\alpha \in \{i_h\}_{h=1}^{k} \setminus \{p''_h\}_{h=1}^{m''}} A(Q_- S_{t_\alpha} g_{\varepsilon(\alpha)})$$

$$z^{2k'''} \prod_{h=1}^{k'''} \langle Q_- g_{1-\varepsilon(i'''_h)},\ Q_- g_{\varepsilon(i'''_h)} \rangle_\iota$$

where, $\sum_{(p_1,\cdots,p_m,p'_1,\cdots,p'_{m'})}$ *means the sum for all* $1 \le p_1, \cdots, p_m,\ p'_1, \cdots p'_{m'} \le n$ *which satisfy*

(i) $\{p_h\}_{h=1}^{m} \cup \{p'_h\}_{h=1}^{m'} \subset \{1,\dots,n\} \setminus (\{i'_h\}_{h=1}^{k'} \cap \{i'''_{h=1}\}_{h=1}^{k'''})$;

(ii) $| \{p_h\}_{h=1}^{m} \cup \{p'_h\}_{h=1}^{m'} | = m + m'$;

(iii) $p_h < q_h$ for all $h = 1,\dots,m$ and $p'_h < q'_h$ for all $h = 1,\dots, m'$;

and $\sum_{(p''_1,\cdots,p''_{m})}$ means the sum for all $1 \le p''_1, \cdots, p''_m \le n$ satisfying

(i) $\{p''_h\}_{h=1}^{m''} \subset \{i_h\}_{h=1}^{k}$;

(ii) $| \{p''_h\}_{h=1}^{m''} | = m''$;

(iii) $p''_h < q''_h$ for all $h = 1,\dots, m''$.

Proof. Notice that the indices $\{i'''_h\}_{h=1}^{k'''}$ label only scalar product terms which can be thought as "normally ordered", therefore, we can assume, without loss of generality, that $k''' = 0$.

By (3.3) and the definition of I_n^ε $(k'' = 0)$, one has

$$\text{(3.5)} \qquad \mathrm{I}_n^\varepsilon = \sum_{k=0}^{n} \sum_{1 \le i_1 < \cdots < i_k \le n} \sum_{k'=0}^{n-k} \sum_{\substack{1 \le i'_1 < \cdots < i'_{k'} \le n \\ \{i_h\}_{h=1}^{k} \cap \{i'_h\}_{h=1}^{k'} = \emptyset}} z^{k+k'}$$

$$\sum_{m=0}^{n-k-k'} C_n(1, m)(A^+ \cdots)_{(n-k-k'-m)} \cdot$$

$$\sum_{m'=0}^{k'} C_n(2, m')(A^+ \cdots)_{(k'-m')} \cdot (A \cdots)_{(n-k-k'+k-m-m')}$$

$$\otimes \sum_{m'''} C_n(3, m'')(A^+\cdots)_{(k'-m'')}\cdot(A\cdots)_{(k-m'')}$$

where, $C_n(1, m)$, $C_n(2, m')$ and $C_n(3, m'')$ are the products of some scalar products; $(A^+\cdots)_{(n-k-k'-m)}$ is the product of the creators which is a $n-k-k'-m$'s elements subset of

$$\left\{A^+(S_{t_\alpha}Q^+g_{\varepsilon(\alpha)});\ \alpha\in\{1,\cdots,n\}\setminus(\{i_h\}_{h=1}^k\cup\{i'_h\}_{h=1}^{k'})\right\} \tag{3.6}$$

whose complement is used to produce the scalar products; $(A^+\cdots)_{(k'-m')}$ is the product of the creators which is a $k'-m$'s elements subset of

$$\left\{A^+(S_{t_{i'_h}}Q^+g_{\varepsilon(i'_h)})\right\}_{h=1}^{k'} \tag{3.7}$$

whose complement is used to produce the scalar products. So, $m\le n-k-k'$ and $m'\le k'$. For each fixed m, one can use the set $\{q_h\}_{h=1}^m\subset\{1,\ldots,n\}\setminus(\{i_h\}_{h=1}^k\cup\{i'_h\}_{h=1}^{k'})$ to label the set of creators

$$\left\{A^+(S_{t_\alpha}Q^+g_{\varepsilon(\alpha)});\ \alpha\in\{1,\ldots,n\}\setminus(\{i_h\}_{h=1}^k\cup\{i'_h\}_{h=1}^{k'})\right\}\setminus(A^+\cdots)_{(n-k-k'-m)}. \tag{3.8}$$

Without loss of generality, we can suppose that

$$1\le q_1<\cdots<q_m\le n.$$

For each fixed m', one can use the set $\{q'_h\}_{h=1}^{m'}\subset\{i'_h\}_{h=1}^{k'}$ to label the set of creators

$$\left\{A^+(S_{t_{i'_h}}Q^+g_{\varepsilon(i'_h)})\right\}_{h=1}^{k'}\setminus(A^+\cdots)_{(k'-m')}. \tag{3.9}$$

Without loss of generality, we can suppose that

$$1\le q'_1<\cdots<q'_{m'}\le n.$$

For each fixed m, m', $1\le q_1<\cdots<q_m\le n$, $1\le q'_1<\cdots q'_{m'}<n$, there is a $m+m''$s elements subset of annihilators

$$\left\{A(S_{t_\alpha}Q^+g_{\varepsilon(\alpha)});\ \alpha\in\{1,\ldots,n\}\setminus\{i_h\}_{h=1}^k\cup\{i'_h\}_{h=1}^{k'})\right\} \tag{3.10}$$

which is used to produce scalar products with

$$\left\{A^+(S_{t_\alpha}Q^+g_{\varepsilon(\alpha)});\ \alpha\in\{1,\ldots,n\}\setminus\{i_h\}_{h=1}^k\cup\{i'_h\}_{h=1}^{k'})\right\}\setminus\left\{A^+(S_{t_{i'_h}}Q^+g_{\varepsilon(i'_h)})\right\}_{h=1}^{k'}. \tag{3.11}$$

If we label the subset by $\{p_h\}_{h=1}^m$ and $\{p'_h\}_{h=1}^{m'}$ respectively, then, $1\le p_1,\ldots,p_m$, $p'_1,\ldots,p_{m'}\le n$; $\{p_h\}_{h=1}^m\cup\{p'_h\}_{h=1}^m\subset\{1,\ldots,n\}\setminus\{i'_h\}_{h=1}^{k'}$ and $|\{p_h\}_{h=1}^m\cup\{p'_h\}_{h=1}^{m'}|=m+m'$. Moreover, $m+m'\le n-k'$, i.e. $m'\le n-k'-m$. So, we obtain the following

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

(3.12)
$$\mathrm{I}_n^{\varepsilon}=\sum_{k=0}^{n}\sum_{1\le i_1<\cdots<i_k\le n}\sum_{k'=0}^{n-k}\sum_{\substack{1\le i'_1<\cdots<i'_{k'}\le n\\ \{i_h\}_{h=1}^{k}\cap\{i'_h\}_{h=1}^{k'}=\emptyset}}$$
$$\sum_{m=0}^{n-k-k'}\sum_{\substack{1\le q_1<\cdots<q_m\le n\\ \{q_h\}_{h=1}^{m}\subset\{1,\cdots,n\}\setminus(i_h)_{h=1}^{k}\cup\{i'_h\}_{h=1}^{k'}}}\sum_{m'=0}^{k'\wedge(n-k'-m)}\sum_{\substack{1\le q'_1<\cdots<q'_{m'}\le i'_{k'}\\ \{q'_h\}_{h=1}^{m'}\subset\{i'_h\}_{h=1}^{k'}}}$$
$$\sum_{(p_1,\cdots,p_m,p'_1,\cdots,p'_{m'})} z^{k+k'}$$
$$\prod_{h=1}^{m}\langle S_{t_{p_h}}Q_+g_{1-\varepsilon(p_h)},\ S_{t_{q_h}}Q_+g_{\varepsilon(q_h)}\rangle\cdot\prod_{h=1}^{m'}\langle S_{t_{p'_h}}Q_+g_{1-\varepsilon(p'_h)},\ S_{t_{q'_h}}Q_+g_{\varepsilon(q'_h)}\rangle$$
$$\prod_{\alpha\in\{1,\cdots,n\}\setminus(\{i_h\}_{h=1}^{k}\cup\{i'_h\}_{h=1}^{k'}\cup\{q_h\}_{h=1}^{m})}A^+(S_{t_\alpha}Q_+g_{\varepsilon(\alpha)})$$
$$\prod_{\alpha\in\{i'_h\}_{h=1}^{k'}\setminus\{q'_h\}_{h=1}^{m'}}A^+(S_{t_\alpha}Q_+g_{\varepsilon(\alpha)})$$
$$\prod_{\alpha\in\{1,\cdots,n\}\setminus(\{i'_h\}_{h=1}^{k'}\cup(p_h)_{h=1}^{m}\cup\{p'_h\}_{h=1}^{m'})}A(S_{t_\alpha}Q_+g_{1-\varepsilon(\alpha)})$$
$$\otimes\sum_{m''}C_n(3,m'')(A^+\cdots)_{(k'-m'')}\cdot(A\cdots)_{(k-m'')}.$$

Similarly, we can deal with

(3.13)
$$\sum_{m''}C_n(3,m'')(A^+\cdots)_{(k'-m'')}\cdot(A\cdots)_{(k-m'')}$$

and get that (3.12) is equal to

(3.13a)
$$\sum_{m''=0}^{k'\wedge k}\sum_{\substack{1\le q''_1<\cdots<q''_{m''}\le i'_{k'}\\ \{q''_h\}_{h=1}^{m''}\subset\{i'_h\}_{h=1}^{k'}}}\sum_{p''_1,\cdots,p''_{m''}}$$
$$\prod_{h=1}^{m}\langle Q_-S_{t_{p''_h}}g_{\varepsilon(p''_h)},\ Q_-S_{t_{q''_h}}g_{\varepsilon(q''_h)}\rangle_\iota$$
$$\prod_{\alpha\in\{i'_h\}_{h=1}^{k'}\setminus\{q''_h\}_{h=1}^{m''}}A^+(Q_-S_{t_\alpha}g_{1-\varepsilon(\alpha)})\cdot\prod_{\alpha\in\{i_h\}_{h=1}^{k}\setminus\{p''_h\}_{h=1}^{m''}}A^+(Q_+S_{t_\alpha}g_{\varepsilon(\alpha)})\cdot$$

Combining together (3.12) and (3.13a), one finishes the proof.

As an appication of this Lemma, we can easily obtain the following

COROLLARY (3.2). For each $n\in\mathbf{N}$, $\varepsilon\in\{0,1\}^n$, $f,f'\in K$, $T,S,T',S'\in\mathbf{R}$,

(3.4)
$$\langle W\left(z\int_{S/z^2}^{T/z^2}Q_+S_ufdu\right)\Phi_F\otimes W\left(z^2\int_{S/z^2}^{T/z^2}Q_-S_ufdu\right)\Phi_F^{\iota},$$

$$\int_0^{t/z^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \mathrm{I}_n^{\varepsilon} \cdot$$

$$\cdot W\left(z \int_{S'/z^2}^{T'/z^2} Q_+ S_u f' du\right) \Phi_F \otimes W\left(z^2 \int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\right) \Phi_F^{\zeta} \rangle$$

$$= \sum_{k=0}^{n} \sum_{1 \le i_1 < \cdots < i_k \le n} \sum_{k'=0}^{n-k} \sum_{\substack{1 \le i_1' < \cdots < i_{k'}' \le n \\ \{i_h\}_{h=1}^{k} \cap \{i_h'\}_{h=1}^{k'} = \emptyset}} \sum_{k'''=0}^{n-k-k'} \sum_{\substack{1 \le i_1''' < \cdots < i_{k'''}''' \le n \\ \{i_h'''\}_{h=1}^{k'''} \cap (\{i_h\}_{h=1}^{k} \cup \{i_h'\}_{h=1}^{k'}) = \emptyset}}$$

$$\sum_{m=0}^{n-k-k'-k'''} \sum_{\substack{1 \le q_1 < \cdots < q_m \le n \\ \{q_h\}_{h=1}^{m} \subset \{1, \cdots, n\} \setminus (\{i_h\}_{h=1}^{k} \cup \{i_h'\}_{h=1}^{k'} \cup \{i_h'''\}_{h=1}^{k'''})}} \sum_{m'=0}^{k' \wedge (n-k'-k'''-m)} \sum_{\substack{1 \le q_1' < \cdots < q_{m'}' \le i_{k'}' \\ \{q_h'\}_{h=1}^{m'} \subset \{i_h'\}_{h=1}^{k'}}}$$

$$\sum_{(p_1, \cdots, p_m, p_1', \cdots, p_{m'}')} z^{k+k'} \int_0^{t/z^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n$$

$$\prod_{h=1}^{m} \langle S_{t_{p_h}} Q_+ g_{1-\varepsilon(p_h)},\ S_{t_{p_h}} Q_+ g_{\varepsilon(q_h)} \rangle \cdot \prod_{h=1}^{m'} \langle S_{t_{p_h'}} Q_+ g_{1-\varepsilon(p_h')},\ S_{t_{p_h'}} Q_+ g_{\varepsilon(q_h')} \rangle$$

$$z^{n-k-k'-m-k'''} \prod_{\alpha \in \{1, \cdots, n\} \setminus (\{i_h\}_{h=1}^{k} \cup \{i_h'\}_{h=1}^{k'} \cup \{i_h'''\}_{h=1}^{k'''} \cup \{q_h\}_{h=1}^{m})} \int_{S/z^2}^{T/z^2} du \langle S_u Q_+ f,\ S_{t_\alpha} Q_+ g_{\varepsilon(\alpha)} \rangle$$

$$z^{k'-m'} \prod_{\alpha \in \{i_h'\}_{h=1}^{k'} \setminus \{q_h'\}_{h=1}^{m'}} \int_{S/z^2}^{T/z^2} du \langle S_u Q_+ f,\ S_{t_\alpha} Q_+ g_{\varepsilon(\alpha)} \rangle$$

$$z^{n-k'-m-m'-k'''}$$

$$\prod_{\alpha \in \{1, \cdots, n\} \setminus (\{i_h'\}_{h=1}^{k'} \cup \{i_h'''\}_{h=1}^{k'''} \cup \{p_h\}_{h=1}^{m} \cup \{p_h'\}_{h=1}^{m'})} \int_{S/z^2}^{T/z^2} du \langle S_{t_\alpha} Q_+ g_{1-\varepsilon(\alpha)},\ S_u Q_+ f' \rangle \cdot$$

$$\sum_{m''=0}^{k' \wedge k} \sum_{\substack{1 \le q_1'' < \cdots < q_{m''}'' \le i_{k'}' \\ \{q_h''\}_{h=1}^{m''} \subset \{i_h'\}_{h=1}^{k'}}} \sum_{(p_1'', \cdots, p_{m''}'')} \prod_{h=1}^{m} \langle Q^- S_{t_{p_h''}} g_{\varepsilon(p_h'')},\ Q^- S_{t_{q_h''}} g_{1-\varepsilon(q_h'')} \rangle_{\zeta}$$

$$z^{2(k'-m'')} \prod_{\alpha \in \{i_h'\}_{h=1}^{k'} \setminus \{q_h''\}_{h=1}^{m''}} \int_{S/z^2}^{T/z^2} du \langle Q^- S_u f,\ Q^- S_{t_\alpha} g_{1-\varepsilon(\alpha)} \rangle_{\zeta} \cdot$$

$$\cdot z^{2(k'-m'')} \prod_{\alpha \in \{i_h\}_{h=1}^{k} \setminus \{p_h''\}_{h=1}^{m''}} \int_{S'/z^2}^{T'/z^2} du \langle Q^- S_{t_\alpha} g_{\varepsilon(\alpha)},\ Q^- S_u f \rangle_{\zeta}$$

$$z^{2k''} \prod_{h=1}^{k'''} \langle Q^- g_{\varepsilon(i_h''')},\ Q^- g_{1-\varepsilon(i_h''')} \rangle_{\zeta}$$

$$\langle W\left(z \int_{S/z^2}^{T/z^2} Q_+ S_u f du\right) \Phi_F \otimes W\left(z^2 \int_{S/z^2}^{T/z^2} Q^- S_u f du\right) \Phi_F^{\zeta},$$

$$W\left(z \int_{S'/z^2}^{T'/z^2} Q_+ S_u f' du\right) \Phi_F \otimes W\left(z^2 \int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\right) \Phi_F^{\zeta} \rangle.$$

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

COROLLARY (3.3). *For each* $n \in \mathbf{N}$, $\varepsilon \in \{0, 1\}^n$, $f, f' \in K$, $T, S, T', S' \in \mathbf{R}$,

$$|\langle W\left(z\int_{S/z^2}^{T/z^2} Q_+ S_u f du\right) \Phi_F \otimes W\left(z^2 \int_{S/z^2}^{T/z^2} Q^- S_u f du\right) \Phi_F^\iota, \tag{3.15}$$

$$\int_0^{t/z^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \mathrm{I}_n^\varepsilon \cdot$$

$$\cdot W\left(z\int_{S'/z^2}^{T'/z^2} Q_+ S_u f' du\right) \Phi_F \otimes W\left(z^2 \int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\right) \Phi_F^\iota \rangle | \leq$$

$$\leq \sum_{k=0}^{n} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{k'=0}^{n-k} \sum_{\substack{1 \leq i'_1 < \cdots < i'_{k'} \leq n \\ \{i_h\}_{h=1}^{k} \cap \{i'_h\}_{h=1}^{k'} = \emptyset}} \sum_{k'''=0}^{n-k-k'} \sum_{\substack{1 \leq '''_1 < \cdots < i'''_{k'''} \leq n \\ \{i_{h'''}\}_{h=1}^{k'''} \cap \{i_h\}_{h=n}^{k}\} \cup \{i'_h\}_{h=1}^{k'}) = \emptyset}}$$

$$\sum_{m=0}^{n-k-k'-k'''} \sum_{\substack{1 \leq q_1 < \cdots < q_m \leq n \\ \{q_h\}_{h=1}^{m} \subset \{1, \cdots, n\} \setminus (\{i_h\}_{h=1}^{k}\} \cup \{i'_h\}_{h=1}^{k'} \cup \{i'''_h\}_{h=1}^{k'''})}} \sum_{m'=0}^{k' \wedge n-k-k'-k'''-m)}$$

$$\sum_{\substack{1 \leq q'_1 < \cdots < q'_{m'} \leq i'_{k'} \\ \{q'_h\}_{h=1}^{m'} \subset (\{i'_h\} \subset \{i'_h\}_{h=1}^{k'}}} \sum_{(p_1, \cdots, p_m, p'_1, \cdots, p'_{m'})} \sum_{m''=0}^{k' \wedge k} \sum_{\substack{1 \leq q''_1 < \cdots < q''_{m''} \leq i'_{k'} \\ \{q''_h\}_{h=1}^{m''} \subset \{i'_h\}_{h=1}^{k'}}} \sum_{(p'_1, \cdots, p''_{m''})}$$

$$z^{2(n+k+k'-m-m'-2m'')} \cdot C_1^{2(n-m-m'-m'')} \int_0^{t/z^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n$$

$$\prod_{h=1}^{m} | \langle S_{t_{p_h}} Q_+ g_{1-\varepsilon(p_h)}, S_{t_{q_h}} Q_+ g_{\varepsilon(q_h)} \rangle | \cdot \prod_{h=1}^{\bar{m}} | \langle S_{t_{\alpha_h}} Q_+ g_{1-\varepsilon(\alpha_h)}, S_{t_{\beta_h}} Q_+ g_{\varepsilon(\beta_h)} \rangle |$$

$$\prod_{h=1}^{k'''} | \langle Q^- g_{\varepsilon(i'''_h)}, Q^- g_{1-\varepsilon(i'''_h)} \rangle_\iota | \cdot \prod_{h=1}^{m} | < Q^- S_{t_{p''_h}} g_{\varepsilon(p''_h)}, Q^- S_{t_{q''_h}} g_{1-\varepsilon(q''_h)} \rangle_\iota |$$

$$| \langle W\left(z\int_{S/z^2}^{T/z^2} Q_+ S_u f du\right) \Phi_F \otimes W\left(z^2\int_{S/z^2}^{T/z^2} Q^- S_u f du\right) \Phi_F^\iota,$$

$$W\left(z\int_{S'/z^2}^{T'/z^2} Q_+ S_u f' du\right) \Phi_F \otimes W\left(z^2\int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\right) \Phi_F^\iota \rangle |$$

where, here and in the following,

$$C_1 := \max_{F=f,f';G=g_0,g_1;P=Q_+,\iota\circ Q^-,I} \int_{-\infty}^{\infty} | \langle PG, S_t PF \rangle | dt \tag{3.16}$$

and

$$\bar{m} := | \{q'_h\}_{h=1}^{m'} \cup \{q'_h\}_{h=1}^{m'} | \tag{3.17}$$

$$\{\beta_h\}_{h=1}^{\bar{m}} = \{q'_h\}_{h=1}^{m''} \cup (\{q''_h\}_{h=1}^{m''} \setminus \{q'_h\}_{h=1}^{m'}). \tag{3.18}$$

Moreover, $\{\alpha_h\}_{h=1}^{\bar{m}}$ is chosen in the following way:

$$\alpha_h = \begin{cases} p'_{h0} & \text{if } \beta_h = q'_{h0} ; \\ p''_{h0} & \text{if } \beta_h = q''_{h0} \end{cases} \tag{3.19}$$

With arguments similar to those of the proof of Lemma (3.1), one can get the following

LEMMA (3.4). *For each $n \in \mathbf{R}$, $\varepsilon \in \{0,1\}^n$,*

$$\mathrm{II}_n^{\varepsilon} = \sum_{k=0}^{n} \sum_{1\le i_1<\cdots<i_k\le n} \sum_{k'=0}^{n-k} \sum_{\substack{1\le i'_1<\dots<i'_{k'}\le n \\ \{i_h\}_{h=1}^{k}\cap\{i'_h\}_{h=1}^{k'}=\emptyset}} \sum_{k''=1}^{n-k-k'} \sum_{\substack{1\le i''_1<\cdots<i''_{k''}\le n \\ \{i''_h\}_{h=1}^{k''}\cap(\{i_h\}_{h=1}^{k}\cup\{i'_h\}_{h=1}^{k'})=\emptyset}} \tag{3.20}$$

$$\sum_{k'''=0}^{n-k-k'} \sum_{\substack{1\le i'''_1<\cdots<i'''_{k'''}\le n \\ \{i'''_h\}_{h=1}^{k'''}\}\cap(\{i_h\}_{h=1}^{k}\cup\{i'_h\}_{h=1}^{k'}\cup\{i''_h\}_{h=1}^{k''})=\emptyset}}$$

$$\sum_{m=0}^{n-k-k'-k'''} \sum_{\substack{1\le q_1<\cdots<q_m\le n \\ \{q_h\}_{h=1}^{m}\subset\{1,\cdots,n\}\setminus(\{i_h\}_{h=1}^{k}\cup\{i'_h\}_{h=1}^{k'}\cup\{i''_h\}_{h=1}^{k''}\cup\{i'''_h\}_{h=1}^{k'''})}} \sum_{m'=0}^{k'\wedge(n-k'-k''-k'''-m)}$$

$$\sum_{\substack{1\le q'_1<\cdots<q'_{m'}\le i'_{k'} \\ \{q'_h\}_{h=1}^{m'}\subset\{i'_h\}_{h=1}^{k'}}} \sideset{}{'}\sum_{(p_1,\cdots,p_m,p'_1,\cdots,p'_{m'})} z^{k+k'+2k''}$$

$$\prod_{h=1}^{m} \langle S_{t_{p_h}} Q_+ g_{1-\varepsilon(p_h)}, S_{t_{q_h}} Q_+ g_{\varepsilon(q_h)} \rangle \cdot \prod_{h=1}^{m'} \langle S_{t_{p'_h}} Q_+ g_{1-\varepsilon(p'_h)}, S_{t_{q'_h}} Q_+ g_{\varepsilon(q'_h)} \rangle$$

$$\prod_{\alpha\in\{1,\cdots,n\}\setminus(\{i_h\}_{h=1}^{k}\cup\{i''_h\}_{h=1}^{k''}\cup\{i'''_h\}_{h=1}^{k'''}\cup\{q_h\}_{h=1}^{m}\cup\{q'_h\}_{h=1}^{m'})} A^+(S_{t_\alpha} Q_+ g_{\varepsilon(\alpha)})$$

$$\prod_{\alpha\in\{1,\cdots,n\}\setminus(\{i_h\}_{h=1}^{k}\cup\{i''_h\}_{h=1}^{k''}\cup\{i'''_h\}_{h=1}^{k'''}\cup\{p_h\}_{h=1}^{m}\cup\{p'_h\}_{h=1}^{m'})} A(S_{t_\alpha} Q_+ g_{1-\varepsilon(\alpha)})$$

$$\otimes \sum_{d'=0}^{k'\wedge(k+k'')} \sum_{\substack{1\le \vec{q}_1<\cdots<\vec{q}_{d'}\le i'_{k'} \\ \{\vec{q}_h\}_{h=1}^{d'}\subset\{i'_h\}_{h=1}^{k'}}} \sum_{d''=0}^{k'\wedge(k+k''-d')} \sum_{\substack{1\le \vec{q}''_1<\cdots<\vec{q}''_{d''}\le i''_{k''} \\ \{q''_h\}_{h=1}^{d''}\subset\{i''_h\}_{h=1}^{k''}}}$$

$$\sideset{}{'}\sum_{(\vec{p}_1,\cdots,p'_{d'},p''_1,\cdots,p''_{d''})} \prod_{h=1}^{d''} \langle Q_- S_{t_{p'_h}} g_{\varepsilon(\vec{p}_h)}, Q_- S_{t_{q'_h}} g_{1-\varepsilon(\vec{q}_h)} \rangle_\iota$$

$$\prod_{h=1}^{d''} \langle Q_- S_{t_{p''_h}} g_{\varepsilon(\vec{p}''_h)}, Q_- S_{t_{q''_h}} g_{1-\varepsilon(\vec{q}''_h)} \rangle_\iota$$

$$z^{2k'''} \prod_{h=1}^{k'''} \langle Q_- g_{\varepsilon(i''_h)}, Q_- g_{1-\varepsilon(i''_h)} \rangle_\iota$$

$$\prod_{\alpha\in(\{i'_h\}_{h=1}^{k'}\cup\{i''_h\}_{h=1}^{k''})\setminus(\{\vec{q}''_h\}_{h=1}^{d''}\cup\{\vec{q}_h\}_{h=1}^{d'})} A^+(Q_- S_{t_\alpha} g_{1-\varepsilon(\alpha)})$$

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

$$\prod_{\alpha\in(\{i'_h\}_{h=1}^{k'}\cup\{i''_h\}_{h=1}^{k''})\setminus(\{\bar{q}''_h\}_{h=1}^{d''}\cup\{\bar{q}_h\}_{h=1}^{d'})} A^+(Q^- S_{t_\alpha} g_{1-\varepsilon(\alpha)})$$

where, $\sum'_{(p_1,\dots,p_m,p'_1,\dots,p'_{m'})}$ *means the sum for all* $1\le p_1,\cdots,p_m,\ p'_1,\cdots,p'_{m'}\le n$ *which satisfying*

(i) $\{p_h\}_{h=1}^m\cup\{p'_h\}_{h=1}^{m'}\subset\{1,\dots,n\}\setminus(\{i'_h\}_{h=1}^{k'}\cup\{i''_h\}_{h=1}^{k''}\cup\{i'''_h\}_{h=1}^{k'''})$;

(ii) $|\{p_h\}_{h=1}^m\cup\{p'_h\}_{h=1}^{m'}|=m+m'$;

(iii) $p_h<q_h$ for all $h=1,\dots,m$ *and* $p'_h<q'_h$ for all $h=1,\dots,m'$;

and $\sum'_{(\bar{p}_1,\dots,\bar{p}_{d'},\ \bar{p}'_1,\dots,\bar{p}'_{d''})}$, *means the sum for all* $1\le\bar{p}_1,\dots,\bar{p}_{d'},\ \bar{p}'_1,\dots,\bar{p}'_{d''}\le n$ *satisfying*

(i) $\{\bar{p}_h\}_{h=1}^{d'}\cup\{\bar{p}'_h\}_{h=1}^{d''}\subset\{i_h\}\}_{h=1}^{k}\cup\{i''_h\}_{h=1}^{k''}$;

(ii) $|\{\bar{p}_h\}_{h=1}^{d'}\cup\{\bar{p}'_h\}_{h=1}^{d''}|=d'+d''$;

(iii) $\bar{p}_h<\bar{q}_h$ for all $h=1,\dots,d'$ and $\bar{p}'_h<\bar{p}''_h$ for all $h=1,\dots,d''$.

COROLLARY (3.5). *For each* $n\in\mathbf{N}$, $\varepsilon\in\{0,1\}^n$, $f, f'\in K$, $T, S, T', S'\in\mathbf{R}$,

$$\begin{aligned}
(3.21)\qquad &\Big|\Big\langle W\Big(z\int_{S/z^2}^{T/z^2}Q_+S_u f du\Big)\Phi_F\otimes W\Big(z^2\int_{S/z^2}^{T/z^2}Q^-S_u f du\Big)\Phi_F^c,\\
&\int_0^{t/z^2}dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n \mathrm{II}_n^{\varepsilon}\cdot\\
&\cdot W\Big(z\int_{S'/z^2}^{T'/z^2}Q_+S_u f' du\Big)\Phi_F\otimes W\Big(z^2\int_{S'/z^2}^{T'/z^2}Q^-S_u f' du\Big)\Phi_F^c\Big\rangle\Big|\\
&\le\sum_{k=0}^{n}\sum_{1\le i_1<\cdots<i_k\le n}\sum_{k'=0}^{n-k}\sum_{\substack{1\le i'_1<\cdots<i'_{k'}\le n\\ \{i_h\}_{h=1}^k\cap\{i'_h\}_{h=1}^{k'}=\emptyset}}\sum_{k''=1}^{n-k-k'}\sum_{\substack{1\le i''_1<\cdots<i''_{k''}\le n\\ \{i''_h\}_{h=1}^{k''}\cap(\{i_h\}_{h=1}^k\cup\{i'_h\}_{h=1}^{k'})=\emptyset}}\\
&\sum_{k'''=0}^{n-k-k'}\sum_{\substack{1\le i'''_1<\cdots<i'''_{k'''}\le n\\ \{i'''_h\}_{h=1}^{k'''}\cap(\{i_h\}_{h=1}^k\cup\{i'_h\}_{h=1}^{k'}\cup\{i''_h\}_{h=1}^{k''})=\emptyset}}\\
&\sum_{m=0}^{n-k-k'-k'''}\sum_{\substack{1\le q_1<\cdots<q_m\le n\\ \{q_h\}_{h=1}^m\subset\{1,\cdots,n\}\setminus(\{i_h\}_{h=1}^k\cup\{i'_h\}_{h=1}^{k'}\cup\{i''_h\}_{h=1}^{k''}\cup\{i'''_h\}_{h=1}^{k'''})}}\sum_{m'=0}^{k'\wedge(n-k'-k''-k'''-m)}\\
&\sum_{\substack{1\le q'_1<\cdots<q'_{m'}<i'_{k'}\\ \{q'_h\}_{h=1}^{m'}\subset\{i'_h\}_{h=1}^{k'}}}\sum'_{p_1,\cdots,p_m,p'_1,\cdots,p'_{m'})}\int_0^{t/z^2}dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n\\
&\sum_{d'=0}^{k'\wedge(k+k'')}\sum_{\substack{1\le\bar{q}_1<\cdots<\bar{q}_{d'}\le i_{k'}\\ \{\bar{q}_h\}_{h=1}^{d'}\subset\{i'_h\}_{h=1}^{k'}}}\sum_{d''=0}^{k''\wedge(k+k''-d'')}\sum_{\substack{1\le\bar{q}''_1<\cdots<\bar{q}''_{d''}\le i''_{k''}\\ \{\bar{q}''_h\}_{h=1}^{d''}\subset\{i''_h\}_{h=1}^{k''}}}\sum'_{(\bar{p}_1,\cdots,\bar{p}_{d'},\bar{p}'_1,\cdots,\bar{p}'_{d''})}\\
&\prod_{h=1}^{\bar{d}+d''+m+m'}|\langle S_{t_{\alpha_h}}Q_+g_{1-\varepsilon(\alpha_h)},\ S_{t_{\beta_h}}Q_+g_{\varepsilon(\beta_h)}\rangle|
\end{aligned}$$

$$C_1^{2(n-m-m'-d'-d'')}\cdot z^{2(n-m-m'+2k''+k+k'-2d'-2d'')}$$

$$\prod_{h=1}^{k'''}|\langle Q^-g_{1-\varepsilon(i_h'')},Q^-g_{\varepsilon(i_h'')}\rangle|\cdot\prod_{h=1}^{d'}|\langle Q^-S_{t_{p_h'}}g_{\varepsilon(\bar p_h')},Q^-S_{t_{p_h'}}g_{1-\varepsilon(\bar p_h')}\rangle_\iota|$$

$$\langle W\left(z\int_{S/z^2}^{T/z^2}Q_+S_ufdu\right)\Phi_F\otimes W\left(z^2\int_{S/z^2}^{T/z^2}Q^-S_ufdu\right)\Phi_F^\iota,$$

$$W\left(z\int_{S'/z^2}^{T'/z^2}Q_+S_uf'du\right)\Phi_F\otimes W\left(z\int_{S'/z^2}^{T'/z^2}Q^-S_uf'du\right)\Phi_F^\iota\rangle|$$

where,

$$\bar d:=\left|\{\bar q_h'\}_{h=1}^{d'}\backslash\{q_h'\}_{h=1}^{m'}\right| \tag{3.22}$$

$$\{\beta\}_{h=1}^{\bar d+d''+m+m'}=\{q_h\}_{h=1}^{m}\cup\{q_h'\}_{h=1}^{m'}\cup\{\bar p_h'\}_{h=1}^{d''}\cup\{\bar\beta_h\}_{h=1}^{\bar d} \tag{3.23}$$

with

$$\{\bar\beta_h\}_{h=1}^{\bar d}:=\{\bar q_h'\}_{h=1}^{d'}\backslash\{q_h'\}_{h=1}^{m'}. \tag{3.24}$$

§4. The limit of the collective and the Negligible terms

In the section, we shall deal with the limits

$$\lim_{z\to0}\langle W\left(z\int_{S/z^2}^{T/z^2}Q_+S_ufdu\right)\Phi_F\otimes W\left(z^2\int_{S/z^2}^{T/z^2}Q^-S_ufdu\right)\Phi_F^\iota,$$
$$\int_0^{t/z^2}dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n\mathrm{I}_n^\varepsilon\cdot$$
$$\cdot W\left(z\int_{S'/z^2}^{T'/z^2}Q_+S_uf'du\right)\Phi_F\otimes W\left(z^2\int_{S'/z^2}^{T'/z^2}Q^-S_uf'du\right)\Phi_F^\iota\rangle \tag{4.1}$$

and

$$\lim_{z\to0}\langle W\left(z\int_{S/z^2}^{T/z^2}Q_+S_ufdu\right)\Phi_F\otimes W\left(z^2\int_{S/z^2}^{T/z^2}Q^-S_ufdu\right)\Phi_F^\iota,$$
$$\int_0^{t/z^2}dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n\mathrm{II}_n^\varepsilon\cdot$$
$$\cdot W\left(z\int_{S'/z^2}^{T'/z^2}Q_+S_uf'du\right)\Phi_F\otimes W\left(z^2\int_{S'/z^2}^{T'/z^2}Q^-S_uf'du\right)\Phi_F^\iota\rangle \tag{4.2}$$

for fixed $n\in\mathbf{N}$.

First of all, we have the following

LEMMA (4.1). *For each* $n\in\mathbf{N}$, $0\le m\le n$, $2\le q_1<\cdots<q_m\le n$, $1\le p_1,\cdots,p_m\le n-1$, $p_h<q_h$, $h=1,\dots,m$, $\{f_h\}_{h=1}^m\subset L^1(\mathbf{R})$,

$$z^{-2m}\int_0^t dt_1\cdots\int_0^{t_{n-1}}dt_n\prod_{h=1}^m\left|f_h\left(\frac{t_{q_h}-t_{p_h}}{z^2}\right)\right|\le\frac{t^{n-m}}{(n-m)!}\prod_{h=1}^m\int_{-\infty}^\infty\left|f_h(t)\right|dt \tag{4.3}$$

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

as $z \to 0$.

The proof is similar to that of the lemma (3.3) of [1].

LEMMA (4.2). *For each* $n \in \mathbf{N}$, $0 \leq m \leq n$, f, $g \in K$, $2 \leq q_1 < \cdots < q_m \leq n$, $1 \leq p_1, \ldots, p_m \leq n-1$, $p_h < q_h$, $h = 1, \ldots, m$,

$$z^{-2m} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \prod_{h=1}^{m} \left| \langle Q_+ f, S_{\frac{t_{q_h}-t_{p_h}}{z^2}} Q_+ g \rangle \right| = O(1) \tag{4.4}$$

as $z \to 0$.

Remark. Lemma (4.2) is not a direct corollary of Lemma (4.1) because in (4.4), the operator Q_+ depends on z.

Proof. By the definition, one has

$$Q_+^2 = (1 + z^2 e^{-\frac{1}{2}\beta H})^{-1} = \sum_{p=0}^{\infty} z^{2p} e^{-\frac{1}{2}p\beta H}.$$

So,

$$z^{-2m} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \prod_{h=1}^{m} \left| \langle Q_+ f, S_{\frac{t_{q_h}-t_{p_h}}{z^2}} Q_+ g \rangle \right| \tag{4.5}$$

$$\leq \sum_{p=0}^{\infty} z^{2p} \cdot z^{-2m} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \prod_{h=1}^{m} \left| \langle f, S_{\frac{t_{q_h}-t_{p_h}}{z^2}} e^{-\frac{1}{2}p\beta H} g \rangle \right|.$$

Applying Lemma (4.1) to (4.5), one finds a majorization of the left hand side of (4.5) by

$$\sum_{p=0}^{\infty} z^{2p} \cdot \frac{t^{n-m}}{(n-m)!} \left(\int_{-\infty}^{\infty} | \langle f, S_t e^{-\frac{1}{2}p\beta H} g \rangle | \, dt \right)^m. \tag{4.5a}$$

Combining together this and the fact that if the series

$$\sum_{n=1}^{\infty} z^n a_n, \; a_n \geq 0, \; n = 1, 2, \ldots$$

has a positive convergence radius, then for any $m \in \mathbf{N}$, the series

$$\sum_{n=1}^{\infty} z^n (a_n)^m, \; a_n \geq 0, \; n = 1, 2, \ldots$$

also has a positive convergence radius, the proof is ended.

THEOREM (4.3). *For each* $n \in \mathbf{N}$,

$$\lim_{z \to 0} \langle W\left(z \int_{S/z^2}^{T/z^2} Q_+ S_u f du \right) \Phi_F \otimes W\left(z^2 \int_{S/z^2}^{T/z^2} Q^- S_u f du \right) \Phi_F', \tag{4.6}$$

$$\int_0^{t/z^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \mathrm{II}_n^{\varepsilon} \cdot$$

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

$$\cdot W\left(z\int_{S'/z^2}^{T'/z^2} Q_+ S_u f' du\right) \Phi_F \otimes W\left(z\int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\right) \Phi_F' \rangle = 0.$$

Proof. In (3.21), one has

$$z^{2(n+k+k'+2k''-2d'-2d''-m-m')}\int_0^{t/z^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \tag{4.7}$$

$$\prod_{h=1}^{\bar{d}+d''+m+m'} | \langle S_{t_{\alpha_h}} Q_+ g_{\varepsilon(\alpha_h)}, S_{t_{\beta_h}} Q_+ g_{1-\varepsilon(\beta_h)} \rangle |$$

$$\leq z^{-2(m+m'+\bar{d}+d'')} \cdot z^{2(k+k'+2k''-\bar{d}-d')} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n$$

$$\prod_{h=1}^{\bar{d}+d''+m+m'} | \langle Q_+ g_{\varepsilon(\alpha_h)}, S_{\frac{t_{\beta_h}-t_{\alpha_h}}{z^2}} Q_+ g_{1-\varepsilon(\beta_h)} \rangle |.$$

By definition, $1 \leq \beta_1 < \cdots < \beta_{\bar{d}+d''+m+m'} \leq n$, so applying Lemma (4.2) to (4.7), one gets

$$z^{-2(m+m'+\bar{d}+d'')} \cdot z^{2(k+k'+2k''+\bar{d}-d')} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \tag{4.8}$$

$$\prod_{h=1}^{\bar{d}+d''+m+m'} | \langle Q_+ g_{\varepsilon(\alpha_h)}, S_{\frac{t_{\beta_h}-t_{\alpha_h}}{z^2}} Q_+ g_{1-\varepsilon(\beta_h)} \rangle |$$

$$\leq z^{2(k'+k''+\bar{d}-d')} \cdot O(1).$$

Notice that $d' \leq k'$ and in $\mathrm{II}_n^\varepsilon$, $k'' \geq 1$, so, the right hand side of (4.8) goes to zero as $z \to 0$.

In the following we shall consider the limit of the right hand side of (3.14). The first step is to show that we need only to consider the situation in which there are the same number of A_{-1} and A_1, i.e. $k = k'$.

THEOREM (4.4). *In the right hand side of* (3.15), *if* $k \neq k'$, *then*

$$\lim_{z\to 0} z^{2(n+k+k'-m'-m-2m'')} \cdot \int_0^{t/z^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n$$

$$\prod_{h=1}^{m} | \langle S_{t_{p_h}} Q_+ g_{1-\varepsilon(p_h)}, S_{t_{q_h}} Q_+ g_{\varepsilon(q_h)} \rangle | \cdot \prod_{h=1}^{\bar{m}} | \langle S_{t_{p_h}} Q_+ g_{1-\varepsilon(\alpha_h)}, S_{t_{\beta_h}} Q_+ g_{\varepsilon(\beta_h)} \rangle | = 0.$$

Proof. With the change of variables

$$z^2 t_j \hookrightarrow t_j, \; j = 1,\ldots, n \tag{4.10a}$$

one has

$$z^{2(n+k+k'-m'-m-2m'')} \cdot \int_0^{t/z^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \tag{4.10}$$

$$\prod_{h=1}^{m} | \langle S_{t_{p_h}} Q + g_{1-\varepsilon(p_h)}, S_{t_{q_h}} Q + g_{\varepsilon(q_h)} \rangle | \cdot \prod_{h=1}^{\bar{m}} | \langle S_{t_{\alpha_h}} Q + g_{1-\varepsilon(\alpha_h)}, S_{t_{\beta_h}} Q + g_{\varepsilon(\beta_h)} \rangle |$$
$$= z^{-2(\bar{m}+m)} \cdot z^{2(k+k'-m'+\bar{m}-2m'')} \cdot \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n$$
$$\prod_{h=1}^{m} | \langle Q + g_{1-\varepsilon(p_h)}, S_{\frac{t_{q_h}-t_{p_h}}{z^2}} Q + g_{\varepsilon(q_h)} \rangle | \cdot \prod_{h=1}^{\bar{m}} | \langle Q + g_{1-\varepsilon(\alpha_h)}, S_{\frac{t_{\beta_h}-t_{\alpha_h}}{z^2}} Q + g_{\varepsilon(\beta_h)} \rangle |.$$

By the defininion one has $\{\beta_h\}_{h=1}^{\bar{m}} \cap \{q_h\}_{h=1}^{m} = \emptyset$, therefore

$$(4.11) \quad z^{-2(\bar{m}+m)} \cdot \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \prod_{h=1}^{m} | \langle Q + g_{1-\varepsilon(p_h)}, S_{\frac{t_{q_h}-t_{p_h}}{z^2}} Q + g_{\varepsilon(q_h)} \rangle |$$
$$\prod_{h=1}^{\bar{m}} | \langle Q + g_{1-\varepsilon(\alpha_h)}, S_{\frac{t_{\beta_h}-t_{\alpha_h}}{z^2}} Q + g_{\varepsilon(\beta_h)} \rangle | = O(1).$$

By the definition (3.17), we obtain that

$$(4.12) \qquad \bar{m} \geq m'$$

so,

$$(4.13) \qquad k + k' + \bar{m} - m' - 2m'' \geq k + k' - 2m''.$$

If $k \neq k'$, then $2m'' \leq 2(k \wedge k') < k + k'$ and this implies that the right hand side of (4.10) is less or equal to

$$(4.13a) \qquad z^{k+k'-2m''} \cdot O(1) \leq z \cdot O(1)$$

and goes to 0 as $z \to 0$.

Our second step is to show that with the condition $k = k'$, one needs only to consider the situation in which the projection on the conjugate Fock space of all $A_{\pm 1}$ are used to produce scalar products in the normally ordered form, i.e. in the notations of Corollary (3.2), $m'' = k \ (= k')$.

COROLLARY (4.5). *If in the right hand side of* (3.15), $k = k'$ *and* $m'' < k$, *then* (4.9) *is true also.*

Proof. In the present case, by repeating the proof of Theorem (4.4), one finds a majorization of the quantity in the left hand side of (4.9) by

$$(4.13b) \qquad z^{k+k'-2m'} \cdot O(1) = z^2 \cdot O(1)$$

this ends the proof.

Theorem (4.4) and Corollary (4.5) show that in the following we need only to consider the limit of

$$(4.14)\qquad \mathrm{I}_n^{\varepsilon}(1, z, t) := \sum_{k=0}^{[n/2]} \sum_{1\le i_1<\cdots<i_k\le n} \sum_{\substack{1\le i_1'<\cdots<i_k'\le n \\ \{i_h\}_{h=1}^k\cap\{i_h'\}_{h=1}^k=\emptyset;\, i_h<i_h',\, h=1,\dots,k}}$$

$$\sum_{k'''=0}^{n-2k} \sum_{\substack{1\le i_1'''<\cdots<i_{k'''}'''\le n \\ \{i_h'''\}_{h=1}^{k'''}\cap(\{i_h\}_{h=1}^k\cup\{i_h'\}_{h=1}^k)=\emptyset}}$$

$$\sum_{m=0}^{n-2k-k'''} \sum_{\substack{1\le q_1<\cdots<q_m\le n \\ \{q_h\}_{h=1}^m\subset\{1,\dots,n\}\setminus(\{i_h,i_h'\}_{h=1}^k\cup\{i_h'''\}_{h=1}^{k'''})}} \sum_{m'=0}^{k\wedge(n-k-m-k''')} \sum_{\substack{1\le q_1'<\cdots<q_{m'}'\le i_k' \\ \{q_h'\}_{h=1}^{m'}\subset\{i_h'\}_{h=1}^k}}$$

$$\sum_{(p_1,\dots,p_m,p_1',\dots,p_{m'}')}$$

$$z^{2(n-m'-m)}\cdot\int_0^{t/z^2}dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n$$

$$\prod_{h=1}^m \langle S_{t_{p_h}}Q_+g_{1-\varepsilon(p_h)},\, S_{t_{q_h}}Q_+g_{\varepsilon(q_h)}\rangle\cdot\prod_{h+1}^{m'}\langle S_{t_{p_h'}}Q_+g_{1-\varepsilon(p_h')},\, S_{t_{q_h'}}Q_+g_{\varepsilon(q_h')}\rangle$$

$$\prod_{\alpha\in\{1,\dots,n\}\setminus(\{i_h,i_h'\}_{h=1}^k\cup\{q_h\}_{h=1}^m)} \int_{S/z^2}^{T/z^2} du\,\langle S_uQ_+f,\, S_{t_\alpha}Q_+g_{\varepsilon(\alpha)}\rangle$$

$$\prod_{\alpha\in\{i_h'\}_{h=1}^k\setminus\{q_h'\}_{h=1}^{m'}} \int_{S/z^2}^{T/z^2} du\,\langle S_uQ_+f,\, S_{t_\alpha}Q_+g_{\varepsilon(\alpha)}\rangle$$

$$\prod_{\alpha\in\{1,\dots,n\}\setminus(\{i_h'\}_{h=1}^k\cup\{p_h\}_{h=1}^m\cup\{p_h'\}_{h=1}^{m'})} \int_{S/z^2}^{T/z^2} du\,\langle S_{t_\alpha}Q_+g_{1-\varepsilon(\alpha)},\, S_uQ_+f'\rangle$$

$$\sum_{\sigma\in S_k'}\prod_{h+1}^{k}\langle Q_-S_{t_{i_{\sigma_h}}}g_{\varepsilon(i_{\sigma_h})},\, Q_-S_{t_{i_h'}}g_{1-\varepsilon(i_h')}\rangle_\iota\cdot\prod_{h=1}^{k'''}\langle Q_-g_{\varepsilon(i_h''')},\, Q_-g_{1-\varepsilon(i_h''')}\rangle_\iota$$

$$\langle W\Big(z\int_{S/z^2}^{T/z^2}Q_+S_ufdu\Big)\Phi_F\otimes W\Big(z^2\int_{S/z^2}^{T/z^2}Q_-S_ufdu\Big)\Phi_F',$$

$$W\Big(z\int_{S'/z^2}^{T'/z^2}Q_+S_uf'du\Big)\Phi_F\otimes W\Big(z^2\int_{S'/z^2}^{T'/z^2}Q_-S_uf'du\Big)\Phi_F'\rangle$$

where, for each $\{i_h, i_h'\}_{h=1}^k$, $i_h<i_h'$, $h=1,\dots,k$,

$$(4.15)\qquad S_k' := \Big\{\sigma\in S_k : i_{\sigma(h)}<i_h',\ h=1,\dots,k\Big\}.$$

Remark. In the sum $\sum_{\substack{1\le i_1'<\cdots<i_k'\le n \\ \{i_h\}_{h=1}^k\cap\{i_h'\}_{h=1}^k=\emptyset;\, i_h<i_h',\, h=1,\dots,k}}$, the conditions $i_h<i_h'$, $h=1,\dots,k$ dues to on the conjugate space, all operators are used to produce scalar products $(m''=k)$, so all creators should be to the right hand side of annihila-

tors.

Stating from (4.14), our third step is to show that those terms in the right hand side of (4.14) in which $m'' < k$ tend to zero, i.e. one needs only to consider the terms in which the projection on the Fock space (not only the conjugate one) of all A_{-1} and A_1 are used to produce scalar products in the normally ordered form.

THEOREM (4.6). *In the right hand side of* (4.14), *if* $m'' < k$, *then the quantity*

$$z^{2(n-m-m')} \cdot \int_0^{t/z^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \tag{4.16}$$

$$\prod_{h=1}^{m} | \langle S_{t_{p_h}} Q_+ g_{1-\varepsilon(p_h)}, S_{t_{p_h}} Q_+ g_{\varepsilon(q_h)} \rangle | \cdot \prod_{h=1}^{m'} | \langle S_{t_{p'_h}} Q_+ g_{1-\varepsilon(p'_h)}, S_{t_{q'_h}} Q_+ g_{\varepsilon(q'_h)} \rangle |$$

$$\prod_{h=1}^{k} | \langle Q^- S_{t_{i_{oh}}} g_{\varepsilon(i_{o_h})}, Q^- S_{t_{i'_h}} g_{1-\varepsilon(i'_h)} \rangle |$$

tends to zero as $z \to 0$.

Proof. With the change of variables (4.10a) and since the product

$$\prod_{h=1}^{m'} | \langle S_{t_{p'_h}} Q_+ g_{1-\varepsilon(p'_h)}, S_{t_{q'_h}} Q_+ g_{\varepsilon(q'_h)} \rangle | \tag{4.17}$$

is less than or equal to

$$(2 \| g_0 \| \cdot \| g_1 \|)^{m'}$$

we know that the left side of (4.16) is equal to

$$z^{2(k-m')} \cdot z^{-2(k+m)} \cdot \int_0^{t} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \prod_{h=1}^{m} | \langle Q_+ g_{1-\varepsilon(p_h)}, S_{\frac{t_{q_h}-t_{p_h}}{z^2}} Q_+ g_{\varepsilon(q_h)} \rangle |$$

$$\prod_{h=1}^{k} | \langle \iota Q^- g_{\varepsilon(i_{o_h})}, S_{\frac{t_{i_h}-t_{i_{oh}}}{2}} \iota Q^- g_{1-\varepsilon(i'_h)} \rangle | \cdot O(1).$$

Notice that the sets of indices $\{q_h\}_{h=1}^{m}$ *and* $\{i_h\}_{h=1}^{k}$ are disjoint and by (4.2), the quantity (4.18) becomes to

$$z^{2(k-m')} . O(1)$$

and it tends to zero if $m' < k$.

Remark. The proof of Theorem (4.6) can not give the conclusion that $m = k$ because $\{q'_h\}_{h=1}^{m'} \subset \{i'_h\}_{h=1}^{k}$ are NOT disjoint.

Theorem (4.6) shows that $\{q'_h\}_{h=1}^{m'} = \{i_h\}_{h=1}^{k}$ therefore

$$\lim_{z \to 0} \mathrm{I}_n^{\varepsilon}(1, z, t) = \lim_{z \to 0} \mathrm{I}_n^{\varepsilon}(2, z, t) \tag{4.19}$$

$$:= \lim_{z\to 0} \sum_{k=0}^{[n/2]} \sum_{1\le i_1<\cdots<i_k\le n} \sum_{\substack{1\le i_1'<\cdots<i_{k'}'\le n \\ \{i_h\}_{h=1}^k\cap\{i_h'\}_{h=1}^k=\emptyset;\, i_h<i_h',\, h=1,\cdots,k}}$$

$$\sum_{k'''=0}^{n-2k} \sum_{\substack{1\le i_1'''<\cdots<i_{k'''}'''\le n \\ \{i_h'''\}_{h=1}^{k'''}\cap(\{i_h\}_{h=1}^k\cup\{i_h'\}_{h=1}^k)=\emptyset}}$$

$$\sum_{m=0}^{n-2k-k'''} \sum_{\substack{1\le q_1<\cdots<q_m\le n \\ \{q_h\}_{h=1}^m\subset\{1,\cdots,n\}\setminus(\{i_h,i_h'\}_{h=1}^k\cup\{i_h'''\}_{h=1}^{k'''})}} \sum_{(p_1,\cdots,p_m,p_1',\cdots,p_k',\{i_h'\}_{h=1}^k)}$$

$$z^{2(n-k-m)}\cdot\int_0^{t/z^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n$$

$$\prod_{h=1}^m \langle S_{t_{p_h}} Q_+ g_{1-\varepsilon(p_h)},\ S_{t_{q_h}} Q_+ g_{\varepsilon(q_h)}\rangle \cdot \prod_{h=1}^k \langle S_{t_{p_h'}} Q_+ g_{1-\varepsilon(p_h')},\ S_{t_{i_h'}} Q_+ g_{\varepsilon(i_h')}\rangle$$

$$\prod_{\alpha\in\{1,\cdots,n\}\setminus(\{i_h,i_h'\}_{h=1}^k\cup\{i_h'''\}_{h=1}^{k'''}\cup\{q_h\}_{h=1}^m)} \int_{S/z^2}^{T/z^2} du\,\langle S_u Q_+ f,\ S_{t_\alpha} Q_+ g_{\varepsilon(\alpha)}\rangle$$

$$\prod_{\alpha\in\{i_h'\}_{h=1}^k\setminus\{q_h'\}_{h=1}^{m'}} \int_{S/z^2}^{T/z^2} du\,\langle S_u Q_+ f,\ S_{t_\alpha} Q_+ g_{\varepsilon(\alpha)}\rangle$$

$$\prod_{\alpha\in\{1,\cdots,n\}\setminus(\{i_h'\}_{h=1}^k\cup\{i_h'''\}_{h=1}^{k'''}\cup\{p_h\}_{h=1}^m\{p_h'\}_{h=1}^{m'})} \int_{S/z^2}^{T/z^2} du\,\langle S_{t_\alpha} Q_+ g_{1-\varepsilon(\alpha)},\ S_u Q_+ f'\rangle\cdot$$

$$\sum_{\sigma\in S_k'} \prod_{h=1}^k \langle Q_- S_{t_{i_{\sigma h}}} g_{\varepsilon(i_{\sigma h})},\ Q_- S_{t_{i_h'}} g_{1-\varepsilon(i_h')}\rangle_\iota$$

$$\prod_{h=1}^{k'''} \langle Q_- g_{\varepsilon(i_h''')},\ Q_- g_{1-\varepsilon(i_h''')}\rangle_\iota$$

$$\langle W\Big(z\int_{S/z^2}^{T/z^2} Q_+ S_u f du\Big)\Phi_F \otimes W\Big(z^2\int_{S/z^2}^{T/z^2} Q_- S_u f du\Big)\Phi_F^\iota,$$

$$W\Big(z\int_{S'/z^2}^{T'/z^2} Q_+ S_u f' du\Big)\Phi_F \otimes W\Big(z^2\int_{S'/z^2}^{T'/z^2} Q_- S_u f' du\Big)\Phi_F^\iota\rangle$$

where, $\sum_{(p_1,\dots,p_m,p_1',\dots,p_k',\{i_h'\}_{h=1}^k)}$ means the sum for all $1\le p_1,\cdots,p_m,p_1',\cdots,p_k'\le n$ which satisfies

(i) $\{p_h\}_{h=1}^m\cup\{p_h'\}_{h=1}^k\subset\{1,\dots,n\}\setminus(\{i_h'\}_{h=1}^k\cup\{i_h'''\}_{h=1}^{k'''})$;

(ii) $|\{p_h\}_{h=1}^m\cup\{p_h'\}_{h=1}^k| = m+k$;

(iii) $p_h<q_h$ for all $h=1,\dots,m$ and $p_h'<i_h'$ for all $h=1,\dots,k$.

Our fourth step is to show that we need only to investigate the situation in which

(4.20a) $$p_h = q_h - 1,\ h=1,\dots,m$$

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

and

(4.20b) $$p'_h = i'_h - 1,\ h = 1,\ldots,k.$$

THEOREM (4.7). *For each* m, k, *those terms of right hand side of* (4.19) *in which either* (4.20a) *or* (4.20b) *does not hold will tend to zero as* $z \to 0$.

Proof. We should prove that if there exists a $h = 1,\ldots,m$, such that $q_h - p_h \geq 2$ or a $h = 1,\ldots,k$, such that $i'_h - p'_h \geq 2$, then,

(4.21) $$z^{2(n-k-m)} \cdot \int_0^{t/z^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n$$

$$\prod_{h=1}^{m} |\langle S_{t_{p_h}} Q_+ g_{1-\varepsilon(p_h)},\ S_{t_{q_h}} Q_+ g_{\varepsilon(q_h)} \rangle| \cdot \prod_{h=1}^{k} |\langle S_{t_{p'_h}} Q_+ g_{1-\varepsilon(p'_h)},\ S_{t_{i'_h}} Q_+ g_{\varepsilon(i'_h)} \rangle| \longrightarrow 0.$$

The fact $\{q_h\}_{h=1}^m \cap \{i_h\}_{h=1}^k = \emptyset$ implies that the function in (4.21) is made of type II terms in the sense of [1] and therefore the same arguments as those of the proof of Lemma (3.3) imply the Theorem.

Theorem (4.7) shows that

(4.22) $$\lim_{z\to 0} \mathrm{I}_n^{\varepsilon}(1, z, t) = \lim_{z\to 0} \mathrm{I}_n^{\varepsilon}(2, z, t) = \lim_{z\to 0} \mathrm{I}_n^{\varepsilon}(3, z, t)$$

$$:= \lim_{z\to 0} \sum_{k=0}^{[n/2]} \sum_{1\leq i_1<\cdots<i_k\leq n} \sum_{\substack{1\leq i'_1<\cdots<i'_k\leq n \\ \{i_h\}_{h=1}^k \cap \{i'_h\}_{h=1}^k = \emptyset;\ \{i'_h-1\}_{h=1}^k \cap \{i'_h\}_{h=1}^k = \emptyset;\ i_h < i'_h,\ h=1,\cdots,k}}$$

$$\sum_{k'''=0}^{n-2k} \sum_{\substack{1\leq i'''_1<\cdots<i'''_{k'''}\leq n \\ \{i'''_h\}_{h=1}^{k'''} \cap \{i_h\}_{h=1}^k \cup \{i'_h\}_{h=1}^k) = \emptyset}}$$

$$\sum_{m=0}^{n-2k-k'''} \sum_{\substack{1\leq q_1<\cdots<q_m\leq n \\ \{q_h\}_{h=1}^m \subset \{1,\cdots,n\} \backslash (\{i_h, i'_h\}_{h=1}^k \cup \{i'''_h\}_{h=1}^{k'''});\ \{q_h-1\}_{h=1}^m \bigcap (\{i'_h, i'_h-1\}_{h=1}^k \cup {}_{h=1}^{k'''}) = \emptyset}}$$

$$z^{2(n-k-m)} \cdot \int_0^{t/z^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n$$

$$\sum_{\sigma\in S'_k} \prod_{h=1}^{k} \langle Q^- S_{t_{i_{\sigma h}}} g_{\varepsilon(i_{\sigma_h})},\ Q^- S_{t_{i_h}} g_{1-\varepsilon(i_h)} \rangle_\iota$$

$$\prod_{h=1}^{m} \langle S_{t_{q_h-1}} Q_+ g_{1-\varepsilon(q_h-1)},\ S_{t_{q_h}} Q_+ g_{\varepsilon(q_h)} \rangle \cdot | \prod_{h=1}^{k} \langle S_{t_{i'_h-1}} Q_+ g_{1-\varepsilon(i'_h-1)},\ S_{t_{i'_h}} Q_+ g_{\varepsilon(i'_h)} \rangle$$

$$\prod_{\alpha\in\{1,\cdots,n\} \backslash (\{i_h, i'_h\}_{h=1}^k \cup \{i'''_h\}_{h=1}^{k'''} \cup \{q_h\}_{h=1}^m)} \int_{S/z^2}^{T/z^2} du \langle S_u Q_+ f,\ S_{t_\alpha} Q_+ g_{\varepsilon(\alpha)} \rangle$$

$$\prod_{\alpha\in\{1,\dots,n\}\setminus(\{i'_h\}_{h=1}^{k}\cup\{i''_h\}_{h=1}^{k'''}\cup\{q_h-1\}_{h=1}^{m}\cup\{i'_h-1\}_{h=1}^{k})}\int_{S'/z^2}^{T'/z^2}du\,\langle S_uQ_+g_{1-\varepsilon(\alpha)},\ S_uQ_+f'\rangle\cdot$$

$$\prod_{h=1}^{k'''}\langle Q^-g_{\varepsilon(i''_h)},\ Q^-g_{1-\varepsilon(i''_h)}\rangle_\iota$$

$$\langle W\left(z\int_{S/z^2}^{T/z^2}Q_+S_ufdu\right)\Phi_F\otimes W\left(z^2\int_{S/z^2}^{T/z^2}Q^-S_ufdu\right)\Phi_F^\iota,$$

$$W\left(z\int_{S'/z^2}^{T'/z^2}Q_+S_uf'du\right)\Phi_F\otimes W\left(z^2\int_{S'/z^2}^{T'/z^2}Q^-S_uf'du\right)\Phi_F^\iota\rangle$$

where $\{i'_h-1\}_{h=1}^{k}\cap\{i'_h\}_{h=1}^{h}=\emptyset$ dues to $\{p'_h\}_{h=1}^{k}\subset\{1,\dots,n\}\setminus\{i'_h\}_{h=1}^{k}$; $\{q_h-1\}_{h=1}^{m}\cap\{i'_h-1,\ i'_h\}_{h=1}^{k}=\emptyset$ dues to $\{p_h\}_{h=1}^{m}\subset\{1,\dots,n\}\setminus\{i'_h\}_{h=1}^{k}$ and $|\{p_h\}_{h=1}^{m}\cup\{p'_h\}_{h=1}^{k}|=m+k$.

Remark. In (4.21), we consider the integral of the function

$$\prod_{h=1}^{m}|\langle S_{t_{p_h}}Q_+g_{1-\varepsilon(p_h)},\ S_{t_{q_h}}Q_+g_{\varepsilon(q_h)}\rangle|\cdot\prod_{h=1}^{k}|\langle S_{t_{p'_h}}Q_+g_{1-\varepsilon(p'_h)},\ S_{t_{i'_h}}Q_+g_{\varepsilon(i'_h)}\rangle|$$

but not of the function

(4.24)
$$\prod_{h=1}^{m}|\langle S_{t_{p_h}}Q_+g_{1-\varepsilon(p_h)},\ S_{t_{p_h}}Q_+g_{\varepsilon(q_h)}\rangle|\cdot\prod_{h=1}^{k}|\langle Q^-S_{t_{i_{\sigma(h)}}}g_{\varepsilon(i_{\sigma(h)})},\ Q^-S_{t_{i'_h}}g_{1-\varepsilon(i'_h)}\rangle_\iota|$$

this is due to the fact that if in (4.24) $p_h=q_h-1$ for any $h=1,\dots,m$ but there exists a $h\in\{1,\dots,k\}$ such that $i_{\sigma(h)}<i'_h-1$, the limit

$$\lim_{z\to0}z^{2(n-k-m)}\int_0^{t/z^2}dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n \tag{4.25}$$

$$\prod_{h=1}^{m}|\langle S_{t_{p_h}}Q_+g_{1-\varepsilon(p_h)},\ S_{t_{p_h}}Q_+g_{\varepsilon(q_h)}\rangle\cdot|\prod_{h=1}^{k}|\langle S_{t_{p'_h}}Q_+g_{1-\varepsilon(p'_h)},\ S_{t_{i'_h}}Q_+g_{\varepsilon(p'_h)}\rangle|$$

$$\prod_{h=1}^{k}|\langle Q^-S_{t_{i_{\sigma(h)}}}g_{\varepsilon(i_{\sigma(h)})},\ Q^-S_{t_{i'_h}}g_{1-\varepsilon(i'_h)}\rangle_\iota|$$

may be not zero. For example, $n=4$, $k=1$, $m=2$, $i_1=1$, $i'_1=4$, $q_1=2$, $q_1=3$, then, (4.25) becomes

$$\lim_{z\to0}z^{2(4-2-1)}\cdot\int_0^{t/z^2}dt_1\int_0^{t_1}dt_2\int_0^{t_2}dt_3\int_0^{t_3}dt_4 \tag{4.26}$$

$$|\langle S_{t_1}Q_+g_{1-\varepsilon(1)},S_{t_2}Q_+g_{\varepsilon(2)}\rangle|\cdot|\langle S_{t_2}Q_+g_{1-\varepsilon(2)},S_{t_3}Q_+g_{\varepsilon(3)}\rangle|\cdot$$

$$|\langle S_{t_3}Q_+g_{1-\varepsilon(3)},S_{t_4}Q_+g_{\varepsilon(4)}\rangle|\cdot|\langle Q^-S_{t_1}g_{\varepsilon(1)},Q^-S_{t_4}g_{1-\varepsilon(4)}\rangle_\iota|$$

$$= \lim_{z\to 0} \int_0^t dt_1 \int_{-t_1/z^2}^0 dt_2 \,|\, \langle Q_+ g_{1-\varepsilon(1)}, S_{t_2} Q_+ g_{\varepsilon(2)} \rangle \,|$$

$$\int_{-t_1/z^2 - t_2}^0 dt_3 |\, \langle Q_+ g_{1-\varepsilon(2)}, S_{t_3} Q_+ g_{\varepsilon(3)} \rangle \,|$$

$$\int_{-t_1/z^2 - t_2 - t_3}^0 dt_4 \,|\, \langle Q_+ g_{1-\varepsilon(3)}, S_{t_4} Q_+ g_{\varepsilon(4)} \rangle \,| \cdot |\, \langle Q^- g_{\varepsilon(1)}, Q^- S_{t_4+t_3+t_2} g_{1-\varepsilon(4)} \rangle_\iota \,|.$$

We shall prove that this kind of limit exists and is, usually, not equal to zero.

Our fifth step is to prove that as $z \to 0$ only those terms can remain, in which the set of indices $\{i_h, i'_h\}_{h=1}^k$ satisfies

$$i_1 < i'_1 < \cdots < i_k < i'_k. \tag{4.27}$$

THEOREM (4.8). *In the right hand side of* (4.22), *those terms in which* (4.27) *does not hold will tend to zero as* $z \to 0$.

Before proving the theorem we first prove a lemma:

LEMMA (4.9). *Suppose that* $f \in L^1(\mathbf{R}_+)$ *and is bounded and* $m \geq 3$, *then*

(4.28)

$$z^{-2(m'+1)} \int_0^t dt_1 \cdots \int_0^{t_{m-1}} dt_m f\left(\frac{t_m - t_1}{z^2}\right) f\left(\frac{t_m - t_{m-1}}{z^2}\right) \cdot \prod_{h=1}^{m'} f\left(\frac{t_{q_h} - t_{q_h-1}}{z^2}\right) \to 0$$

for $m' < m-2$, *where*, $2 \leq q_1 < \cdots < q_{m'} \leq m-1$.

Proof. If $q_1 > 2$ or $q_{m'} < m-1$, by the boundedness of f, the left hand side of (4.28) is less or equal to

$$C \cdot z^{-2(m'+1)} \int_0^t dt_1 \cdots \int_0^{t_{m-1}} dt_m f\left(\frac{t_m - t_1}{z^2}\right) \cdot \prod_{h=1}^{m'} f\left(\frac{t_{q_h} - t_{q_h-1}}{z^2}\right) \tag{4.29}$$

where C is a constant. It is clear that (4.29) is a type II term in the sense of [1], therefore by the same arguments as those of the proof of Lemma (3.3) in [1] (4.28) follows.

Now, suppose that

$$q_1 = 2 \quad q_{m'} = m - 1.$$

Introduce the *connected decomposition*

$$\{q_h\}_{h=1}^{m'} = \{q_h\}_{h=1}^{x_1} \cup \{q_h\}_{h=x_1+1}^{x_2} \cup \cdots \cup \{q_h\}_{h=x_{y-1}+1}^{m'}$$

characterized by the following properties:

(i) $q_1 = q_2 - 1 = \cdots = q_{x_1} - (x_1 - 1), \cdots, q_{x_{y-1}} = q_{x_{y-1}+1} - 1 = \cdots = q_{m'} - (m' - x_{y-1} - 1)$;

(ii) $q_{x_1} + 1 < q_{x_1+1}, \cdots, q_{x_{y-1}} + 1 < q_{x_{y-1}+1}$.

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

Then, similarly to the proof of Lemma (3.3) of [1], one has

(4.30) $$z^{-2(m'+1)}\int_0^t dt_1\cdots\int_0^{t_{m-1}} dt_m f\left(\frac{t_m-t_1}{z^2}\right) f\left(\frac{t_m-t_{m-1}}{z^2}\right)\cdot\prod_{h=1}^{m'} f\left(\frac{t_{q_h}-t_{q_h-1}}{z^2}\right)$$

$$\le z^{-2}\int_0^t dt_1\cdots\int_0^{t'_{q_1-2}} dt'_{q_1-1}\int_{-t_{q_1-1}/z^2}^0 f(t'_{q_1}dt'_{q_1}\cdots$$

$$\int_{-t'_{q_1-1}/z^2}^0 f(t_{q_{x_1}})dt'_{q_{x_1}}\int_0^{t'_{q_1-1}} dt'_{q_{x_1}+1}$$

$$\cdots\int_0^{t'_{q_{x_1+1}-2}} dt'_{q_{x_1+1}-1}\int_{-t'_{q_{x_1+1}-1}/z^2}^0 f(t'_{q_{x1}+1})dt'_{q_{x1}+1}\cdots\int_{-t'_{q_{x_1+1}-1}/z^2}^0 f(t'_{q_{x_2}})dt'_{q_{x_2}}\cdots$$

$$\int_0^{t'_{q_{x_y-1}-2}} dt'_{q_{x_y-1}-1}\int_{-t'_{q_{x_y+1}-1}/z^2}^0 f(t'_{q_{x_y-1}})dt'_{q_{x_y-1}}\cdots$$

$$\int_{-t'_{q_{x_y-1}-1}/z^2}^0 f(t'_{q_{m'}})dt'_{q_{m'}}\int_0^{t'_{q_{x_y-1}-1}+z^2(t'_{q_{x_y-1}}+\cdots+t'_{q_{m'}})} dt_m f\left(\frac{t_m-t_1}{z^2}\right) f\left(\frac{t_m-t_{m-1}}{z^2}\right)$$

where, since $q_{m'}=m-1$, then,

$$t_{m-1}:=t'_{q_{x_y-1}-1}+z^2(t'_{q_{x_y-1}}+\cdots+t'_{q_{m'}}).$$

With the change of variable

(4.31) $$(t_m-t_1)/z^2=s_m$$

the right hand side of (4.30) is majorized by

$$C_n\cdot\int_0^t dt_1\cdots\int_0^{t'_{q_1-2}} dt'_{q_1-1}\int_0^{t'_{q_1-1}} dt'_{q_{x_1}+1}\cdots$$

(4.32) $$\cdots\int_0^{t'_{q_{x_y-1}-2}} dt'_{q_{x_y-1}-1}\cdots\int_{-t_1/z^2}^{-(t_1-t'_{q_{x_y-1}-1})/z^2+t'_{q_{x_y-1}}+\cdots+t'_{q_{m'}}} ds_m f(s_m)$$

and this vanishes as $z\to 0$.

Proof of Theorem (4.8). It is enough to show that if either there exists a $h=1,\dots,k-1$, such that

(4.33) $$\left|\{i'_h\}_{h=1}^k\cap\{i_h+1,\cdots,i_{h+1}-1\}\right|\ne 1$$

or there exists a $h=1,\dots,k-1$, such,that

(4.34) $$\left|\{i_h\}_{h=1}^k\cap\{i'_h+1,\cdots,i'_{h+1}-1\}\right|\ne 1$$

then,

(4.35)

$$\lim_{z\to 0} z^{2(n-k-m)}\cdot\int_0^{t/z^2} dt_1\int_0^{t_1} dt_2\cdots\int_0^{t_{n-1}} dt_n\prod_{h=1}^m\left|\langle S_{t_{q_h-1}}Q+g_{1-\varepsilon(q_h-1)},S_{t_{q_h}}Q+g_{\varepsilon(q_h)}\rangle\right|$$

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

$$\prod_{h=1}^{k} |\langle S_{t_{i'_h-1}} Q_+ g_{1-\varepsilon(i'_h-1)}, S_{t_{i'_h}} Q_+ g_{\varepsilon(i'_h)} \rangle| \prod_{h=1}^{k} |\langle Q^- S_{t_{i_h}} g_{\varepsilon(i_h)}, Q^- S_{t_{i'_h}} g_{1-\varepsilon(i'_h)} \rangle| = 0.$$

We shall prove (4.35) separately distinguishing 4 cases.

(i) There exists a $h = 1,\ldots,k-1$, such that

$$\left|\{i'_h\}_{h=1}^{k} \cap \{i_h+1,\cdots,i_{h+1}-1\}\right| = 0. \tag{4.36}$$

In this case, denoting by h_0 the minimum h satisfying (4.36), since $i_h < i'_h$ and $i'_h \notin \{i_h\}_{h=1}^{k}$ for any h, we have

$$i'_{h_0-1} < i_{h_0} < i_{h_0+1} < i'_{h_0} \tag{4.37}$$

and therefore

$$\begin{aligned} & z^{2(n-k-m)} \cdot \int_0^{t/z^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \\ & \prod_{h=1}^{m} |\langle S_{t_{q_h-1}} Q_+ g_{1-\varepsilon(q_h-1)}, S_{t_{q_h}} Q_+ g_{\varepsilon(q_h)} \rangle\rangle| \cdot \\ & \prod_{h=1}^{k} |\langle S_{t_{i'_h-1}} Q_+ g_{1-\varepsilon(i'_h-1)}, S_{t_{i'_h}} Q_+ g_{\varepsilon(i'_h)} \rangle| \prod_{h=1}^{k} |\langle Q^- S_{t_{i_h}} g_{\varepsilon(i_h)}, Q^- S_{t_{i'_h}} g_{1-\varepsilon(i'_h)} \rangle_\iota| \\ & \leq C_n \cdot z^{2(n-k-m)} \cdot \int_0^{t/z^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \\ & \prod_{h=1}^{m} |\langle S_{t_{q_h-1}} Q_+ g_{1-\varepsilon(q_h-1)}, S_{t_{q_h}} Q_+ g_{\varepsilon(q_h)} \rangle| \cdot \\ & \prod_{h\in\{1,\cdots,\hat{h}_0,\cdots,k\}} |\langle S_{t_{i'_h-1}} Q_+ g_{1-\varepsilon(i'_h-1)}, S_{t_{i'_h}} Q_+ g_{\varepsilon(i'_h)} \rangle| \\ & |\langle Q^- S_{t_{i_{h_0}}} g_{\varepsilon(i_{h_0})}, Q^- S_{t_{i'_{h_0}}} g_{1-\varepsilon(i'_{h_0})} \rangle_\iota| =: J_n(1,t) \end{aligned} \tag{4.38}$$

where, C_n is a constant. Now, we shall show that $J_n(1,t)$ goes to zero as $z \to 0$. In the following, we denote $\{i'_h\}_{h=1}^{k} \cup \{q_h\}_{h=1}^{m}$ as $\{\bar{q}_h\}_{h=1}^{m+k}$. By (4.36), one knows that

$$\{i'_h, i'_h - 1\}_{h\in\{1,\cdots,\hat{h}_0,\cdots,k\}} \cap \{i_{h_0}+1,\cdots,i_{h_0+1}-1\} = \emptyset. \tag{4.39}$$

Moreover, with the change of variable (4.10a) and enlarging t_{h_0-1} to t, one finds that

$$\begin{aligned} j_n(1,t) = {} & C_n \cdot z^{-2(k+m)} \cdot \int_0^{t} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \\ & \prod_{h=1}^{m} |\langle Q_+ g_{1-\varepsilon(q_h-1)}, S_{\frac{t_{q_h}-t_{q_h-1}}{z^2}} Q_+ g_{\varepsilon(q_h)} \rangle| \cdot \\ & \prod_{h\in\{1,\cdots,\hat{h}_0,\cdots,k\}} |\langle Q_+ g_{1-\varepsilon(i'_h-1)}, S_{\frac{t_{i'_h}-t_{i'_h-1}}{z^2}} Q_+ g_{\varepsilon(i'_h)} \rangle| \end{aligned} \tag{4.40}$$

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

$$|\langle Q^- g_{\varepsilon(i_{h_0})}, Q^- S_{\frac{t_{i'_{h_0}}-t_{i_{h_0}}}{z^2}} g_{1-\varepsilon(i'_{h_0})}\rangle_t|$$

$$\leq C_n \cdot z^{-2(k+m)} \cdot \int_0^t dt_1 \cdots \int_0^{t_{h_0-2}} dt_{h_0-1} \cdot \int_0^t dt_{h_0} \cdots \int_0^{t_{n-1}} dt_n$$

$$\prod_{h=1}^m |\langle Q_+ g_{1-\varepsilon(q_h-1)}, S_{\frac{t_{q_h}-t_{q_h-1}}{z^2}} Q_+ g_{\varepsilon(q_h)}\rangle| \cdot$$

$$\prod_{h=1,\cdots,\hat{h}_0,\cdots,k} |\langle Q_+ g_{1-\varepsilon(i'_h-1)}, S_{\frac{t_{i'_h}-t_{i'_h}-1}{z^2}} Q_+ g_{\varepsilon(i'_h)}\rangle|$$

$$|\langle Q^- g_{\varepsilon(i_{h_0})}, Q^- S_{\frac{t_{i'_{h_0}}-t_{i_{h_0}}}{z^2}} g_{1-\varepsilon(i'_{h_0})}\rangle_t|.$$

Putting

$$p := |\{h \in \{1,\dots,m\}: i_{h_0} < q_h < i'_{h_0}\}| \tag{4.41}$$

and

$$\{\beta_h\}_{h=1}^p := \{q_h : i_{h_0} < q_h < i'_{h_0},\ h = 1,\dots,m\}. \tag{4.42}$$

Then,

$$J_n(1, t) \leq C'_n \cdot z^{-2(p+1)} \cdot \int_0^t dt_{i_{h_0}} \cdots \int_0^{t_{i'_{h_0}-1}} dt_{i'_{h_0}} \tag{4.43}$$

$$\prod_{h=1}^p |\langle Q_+ g_{1-\varepsilon(\beta_h-1)}, S_{\frac{t_{\beta_h}-t_{\beta_h-1}}{z^2}} Q_+ g_{\varepsilon(\beta_h)}\rangle| \cdot$$

$$|\langle Q_+ g_{1-\varepsilon(i'_{h_0}-1)}, S_{\frac{t_{i'_{h_0}}-t_{i'_{h_0}-1}}{z^2}} Q_+ g_{\varepsilon(i'_{h_0})}\rangle| \cdot |\langle Q^- g_{\varepsilon(i_{h_0})}, Q^- S_{\frac{t_{i'_{h_0}}-t_{i_{h_0}}}{z^2}} g_{1-\varepsilon(i'_{h_0})}\rangle_t|.$$

Applying Lemma (4.8) to the right hand side of (4.43), we know that term goes to zero as $z \to 0$.

(ii) There exists a $h = 1,\dots,k-1$, such that

$$\left|\{i'_h\}_{h=1}^k \cap \{i_h + 1,\cdots,i_{h+1} - 1\}\right| = d \geq 2. \tag{4.44}$$

In this case, denoting by h_0 the minimum h satisfying (4.44), then since $i_h < i'_h$ for any $h = 1,\dots,k$, we know that not only $i'_{h_0} \in \{i_{h_0} + 1,\cdots,i_{h_0+1} - 1\}$ but also $i'_{h_0-1} \in \{i_{h_0} + 1,\cdots,i_{h_0+1} - 1\}$, i.e.

$$i_{h_0-1} < i_{h_0} < i'_{h_0-1} < i'_{h_0}. \tag{4.45}$$

Considering the piece $i'_{h_0-1},\cdots,i'_{h_0}$, one gets (4.43) again. This implies (4.35).

(iii) There exists a $h = 1,\dots,k-1$, such that

$$\left|\{i_h\}_{h=1}^k \cap \{i'_h + 1,\cdots,i'_{h+1} - 1\}\right| = 0. \tag{4.46}$$

In this case, $i_{h+1} < i'_h$, considering the piece $i_h,\cdots,i'_h$, one gets (4.43) again.

(iv) There exists a $h = 1,\dots,k-1$, such that

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

$$\left|\{i_h\}_{h=1}^{k} \cap \{i'_h + 1, \cdots, i'_{h+1} - 1\}\right| \geq 2. \tag{4.47}$$

In this case, since $i_h < i'_h$, $h = 1, \ldots, k$, we can write (4.40) as

$$\left|\{i_j\}_{j=h+1}^{k} \cap \{i'_h + 1, \cdots, i'_{h+1} - 1\}\right| \geq 2 \tag{4.48}$$

therefore there exist h_1, $h_2 \in \{h + 1, \ldots, k\}$ such that

$$i'_h < i_{h_1} < i_{h_2} < i'_{h+1} \tag{4.49}$$

considering the piece $i_{h_1}, \cdots, i'_{h_1+1}$, one gets (4.43) again.

From the above arguments our proof follows.

Summing up, we know that as $z \to 0$ the limit of $\mathrm{I}_n^{\varepsilon}$ is equal to the limit of the following quantity:

$$\begin{aligned}
\mathrm{I}_n^{\varepsilon}(4, z, t) := \sum_{k=0}^{[n/2]} \sum_{1 \leq i_1 < i'_1 < \cdots < i_k < i'_k \leq n} & \\
\sum_{k'''=0}^{n-2k} \sum_{\substack{1 \leq i'''_1 < \cdots < i'''_{k'''} \leq n \\ \{i'''_h\}_{h=1}^{k'''} \cap (\{i_h\}_{h=1}^{k} \cup \{i'_h\}_{h=1}^{k}) = \emptyset}} & \\
\sum_{m=0}^{n-2k-k'''} \sum_{\substack{1 \leq q_1 < \cdots < q_m \leq n \\ \{q_h\}_{h=1}^{m} \subset \{1, \cdots, n\} \setminus (\{i_h, i'_h\}_{h=1}^{k} \cup \{i'''_h\}_{h=1}^{k'''}); \{q_h - 1\}_{h=1}^{m} \cap (\{i_h, i'_h - 1\}_{h=1}^{k} \cup \{i'''_h\}_{h=1}^{k'''}) = \emptyset}} & \\
z^{2(n-k-m)} \cdot \int_0^{t/z^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n & \\
\prod_{h=1}^{k} \left| \langle Q^- S_{t_{i_h}} g_{\varepsilon(i_h)}, Q^- S_{t_{i'_h}} g_{1-\varepsilon(i'_h)} \rangle_\iota \right. & \\
\prod_{h=1}^{m} \left| \langle S_{t_{q_h - 1}} Q_+ g_{1-\varepsilon(q_h - 1)}, S_{t_{q_h}} Q_+ g_{\varepsilon(q_h)} \rangle \cdot \prod_{h=1}^{k} \langle S_{t_{i'_h - 1}} Q_+ g_{1-\varepsilon(i'_h - 1)}, S_{t_{i'_h}} Q_+ g_{\varepsilon(i'_h)} \rangle \right. & \\
\prod_{\alpha \in \{1, \cdots, n\} \setminus (\{i_h, i'_h\}_{h=1}^{k} \cup \{i'''_h\}_{h=1}^{k'''} \cup \{q_h\}_{h=1}^{m}} \int_{S/z^2}^{T/z^2} du \langle S_u Q_+ f, S_{t_\alpha} Q_+ g_{\varepsilon(\alpha)} \rangle & \\
\prod_{\alpha \in \{1, \cdots, n\} \setminus (\{i'_h\}_{h=1}^{k} \cup \{i'''_h\}_{h=1}^{k'''} \cup \{q_h - 1\}_{h=1}^{m} \cup \{i'_h - 1\}_{h=1}^{k}} \int_{S'/z^2}^{T'/2} du \langle S_{t_\alpha} Q_+ g_{1-\varepsilon(\alpha)}, S_u Q_+ f' \rangle \cdot & \\
\prod_{h=1}^{k'''} \langle Q^- g_{\varepsilon(i'''_h)}, Q^- g_{1-\varepsilon(i'''_h)} \rangle_\iota & \\
\langle W\left(z \int_{S/z^2}^{T/z^2} Q_+ S_u f du\right) \Phi_F \otimes W\left(z^2 \int_{S/z^2}^{T/z^2} Q^- S_u f du\right) \Phi_F^\iota, & \\
W\left(z \int_{S'/z^2}^{T'/z^2} Q_+ S_u f' du\right) \Phi_F \otimes W\left(z^2 \int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\right) \Phi_F^\iota \rangle. &
\end{aligned} \tag{4.50}$$

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

Now let us investigate the limit. In order to do this we shall first prove two Lemmata.

LEMMA (4.10). *For each $m \in \mathbf{N}$, $\{f_h\}_{h=1}^{m+1} \subset L^1(\mathbf{R})$ and bounded, the limit*

(4.51)

$$\lim_{z\to 0} z^{-2m} \int_0^t dt_1 \cdots \int_0^{t_m} dt_{m+1} f_1\left(\frac{t_{m+1}-t_1}{z^2}\right) f_{m+1}\left(\frac{t_{m+1}-t_m}{z^2}\right) \cdot \prod_{h=1}^{m-1} f_{h+1}\left(\frac{t_{h+1}-t_h}{z^2}\right)$$

exists.

Proof. With the change of variables

(4.52)
$$(t_{h+1}-t_h)/z^2 \hookrightarrow t_{h+1},\ h=1,\dots,m$$

we have that

(4.53)

$$z^{-2m} \int_0^t dt_1 \cdots \int_0^{t_m} dt_{m+1} f_1\left(\frac{t_{m+1}-t_1}{z^2}\right) f_{m+1}\left(\frac{t_{m+1}-t_m}{z^2}\right) \cdot \prod_{h-1}^{m-1} f_{h+1}\left(\frac{t_{h+1}-t_h}{z^2}\right)$$
$$= \int_0^t dt_1 \cdots \int_{-t_1/z^2}^0 dt_2 f_2(t_2) \int_{-t_1/z^2-t_2}^0 dt_3 f_3(t_3) \cdots$$
$$\int_{-t/z^2-t^2-\dots-t_m}^0 dt_{m+1} f_1(t_{m+1}+\dots+t_2) f_{m+1}(t_{m+1})$$
$$=: M(m,z,t).$$

Moreover,

(4.54)
$$\left| M(m,z_1,t) - M(m,z_2,t) \right|$$
$$\le \Bigg| \int_0^t dt_1 \cdots \int_{-t_1/z_1^2}^0 dt_2 f_2(t_2) \int_{-t_1/z_1^2-t_2}^0 dt_3 f_3(t_3) \cdots$$
$$\int_{-t_1/z_1^2-t_2-\dots-t_m}^0 dt_{m+1} f_1(t_{m+1}+\dots+t_2) f_{m+1}(t_{m+1})$$
$$- \int_0^t dt_1 \cdots \int_{-t_1/z_1^2}^0 dt_2 f_2(t_2) \int_{-t_1/z_1^2-t_2}^0 dt_3 f_3(t_3) \cdots$$
$$\int_{-t_1/z_2^2-t_2-\dots-t_m}^0 dt_{m+1} f_1(t_{m+1}+\dots+t_2) f_{m+1}(t_{m+1})$$
$$+ \int_0^t dt_1 \left| \int_{-t_1/z_2^2}^{-t_1/z_1^2} dt_2 |f_2(t_2)| \right| \cdot \left(\max_{1\le h\le m+1} \int_{-\infty}^0 |f_h|\, dt \right)^{m-1} \sup_{t\in\mathbf{R}} |f(t)|$$
$$\le \cdots \le m \cdot \int_0^t dt_1 \left| \int_{-t_1/z_2^2}^{-t_1/z_1^2} dt_2 |f_2(t_2)| \right| \cdot \left(\max_{1\le h\le m+1} \int_{-\infty}^0 |f_h|\, dt \right)^{m-1} \sup_{t\in\mathbf{R}} |f(t)|$$
$$\longrightarrow 0$$

as z_1, $z_2 \to 0$. Therefore by the Cauchy principle the proof is finished.

LEMMA (4.11). *With the above symbols and assumptions*

(4.55)
$$\lim_{z\to 0} M(m, z, t) = \int_0^t dt_1 \int_{-\infty}^0 dt_2 f_2(t_2) \cdots \int_{-\infty}^0 dt_m f_m(t_m)$$
$$\int_{-\infty}^0 dt_{m+1} f_{m+1}(t_{m+1}) f_1(t_{m+1} + \cdots + t_2).$$

Proof.

(4.56)
$$\left| M(m, z, t) - \int_0^t dt_1 \int_{-\infty}^0 dt_2 f_2(t_2) \cdots \int_{-\infty}^0 dt_{m+1} f_{m+1}(t_{m+1}) f_1(t_{m+1} + \cdots + t_2) \right|$$
$$\leq \left| \int_0^t dt_1 \cdots \int_{-T}^0 dt_2 f_2(t_2) \int_{-t_1/z_1^2 - t_2}^0 dt_3 f_3(t_3) \cdots \right.$$
$$\left. \int_{-t_1/z_1^2 - t_2 - \cdots - t_m}^0 dt_{m+1} f_1(t_{m+1} + \cdots + t_2) f_{m+1}(t_{m+1}) \right|$$
$$- \int_0^t dt_1 \int_{-\infty}^0 dt_2 f_2(t_2) \cdots \int_{-\infty}^0 dt_{m+1} f_{m+1}(t_{m+1}) f_1(t_{m+1} + \cdots + t_2)$$
$$+ C_m \left| \int_0^t dt_1 \int_{-T}^{-t_1/z^2} | f_2(t_2) | dt_2 \right|$$
$$\leq \left| \int_0^t dt_1 \int_{-T}^0 dt_2 f_2(t_2) \int_{-T}^0 dt_3 f_3(t_3) \int_{-t_1/z_1^2 - t_2 - t_3}^0 dt_4 f_4(t_4) \cdots \right.$$
$$\int_{-t_1/z_1^2 - t_2 - \cdots - t_m}^0 dt_{m+1} f_1(t_{m+1} + \cdots + t_2) f_{m+1}(t_{m+1})$$
$$\left. - \int_0^t dt_1 \int_{-\infty}^0 dt_2 f_2(t_2) \cdots \int_{-\infty}^0 dt_{m+1} f_{m+1}(t_{m+1}) f_1(t_{m+1} + \cdots + t_2) \right|$$
$$+ C_m \left| \int_0^t dt_1 \int_{-T}^0 | f_2(t_2) | dt_2 \int_{-T}^{-t_1/z^2 - t_2} | f_3(t_3) | dt_3 \right|$$
$$+ C_m \left| \int_0^t dt_1 \int_{-T}^{-t_1/z^2} | f_2(t_2) | dt_2 \right|$$
$$\leq \cdots \leq \cdots$$
$$\leq C_m \left| \int_0^t dt_1 \int_{-T}^{-t_1/z^2} | f_2(t_2) | dt_2 \right|$$
$$+ C_m \left| \int_0^t dt_1 \int_{-T}^0 | f_2(t_2) | dt_2 \int_{-T}^{-t_1/z^2 - t_2} | f_3(t_3) | dt_3 \right|$$
$$+ \cdots +$$
$$+ C_m \left| \int_0^t dt_1 \int_{-T}^0 | f_2(t_2) | dt_2 \int_{-T}^0 | f_3(t_3) | dt_3 \cdots \right.$$

$$\int_{-T}^{-t_1/z^2-t_2-\cdots-t_m} |f_{m+1}(t_{m+1})|\, dt_{m+1} \Big|$$

$$+\Big|\int_0^t dt_1 \int_{-T}^0 dt_2 f_2(t_2)\cdots\int_{-T}^0 dt_{m+1} f_{m+1}(t_{m+1}) f_1(t_{m+1}+\cdots+t_2) -$$

$$-\Big|\int_0^t dt_1 \int_{-\infty}^0 dt_2 f_2(t_2)\cdots\int_{-\infty}^0 dt_{m+1} f_{m+1}(t_{m+1}) f_1(t_{m+1}+\cdots+t_2)\Big|$$

where,

(4.57) $$C_m := \Big(\max_{h=1,\cdots,m+1}\int_{-\infty}^0 |f_h(t)dt| \bigvee \max_{h=1,\cdots,m+1}\sup_{t\in\mathbf{R}}|f_h(t)| \bigvee 1\Big)^m.$$

For each $\eta>0$, we take T such that

(4.58) $$\max_{h=1,\cdots,m}\int_{-\infty}^{-T} |f_h(t)|\,dt < \eta$$

then,

(4.59)

$$\lim_{z\to0}\Big|M(m,z,t) - \int_0^t dt_1\int_{-\infty}^0 dt_2 f_2(t_2)\cdots\int_{-\infty}^0 dt_{m+1} f_{m+1}(t_{m+1}) f_1(t_{m+1}+\cdots+t_2)\Big|$$

$$\leq \eta\cdot t(C_m + C_m^2 + \cdots + C_m^m)$$

this shows the validity of (4.55).

Using the two Lemmata, we can make our sixth step:

THEOREM (4.12).

$$\lim_{z\to0} \mathrm{I}_n^\varepsilon = \sum_{k=0}^{[n/2]} \sum_{1\leq i_1<i_1'<\cdots<i_k<i_k'\leq n} \sum_{k'''=0}^{n-2k} \sum_{\substack{1\leq i_1'''<\cdots<i_{k'''}'''\leq n \\ \{i_h'''\}_{h=1}^{k'''}\cap(\{i_h\}_{h=1}^k\cup\{i_h'\}_{h=1}^k)=\emptyset}}$$

$$\sum_{m=0}^{n-2k-k'''} \sum_{\substack{1\leq q_1<\cdots<q_m\leq n \\ \{q_h\}_{h=1}^m\subset\{1,\cdots,n\}\setminus(\{i_h,i_h'\}_{h=1}^k\cup\{i_h'''\}_{h=1}^{k'''});\{q_h-1\}_{h=1}^m\cap(\{i_h',i_h'-1\}_{h=1}^k\cup\{i_h'''\}_{h=1}^{k'''})=\emptyset}}$$

$$\int_{0\leq t_{n-1}\leq\cdots\leq\widehat{t_{i_k'}}\cdots\leq\widehat{t_{q_m}}\leq\cdots\leq\widehat{t_{q_1}}\leq\cdots\leq\widehat{t_{i_1'}}\leq\cdots\leq t_i\leq t} dt_1\ldots\widehat{dt_{i_1'}}\ldots\widehat{dt_{q_1}}\ldots\widehat{dt_{q_m}}\ldots\widehat{dt_{i_k'}}\cdots dt_n$$

(4.60) $$\prod_{h=1}^k M(\varepsilon, i_h, i_h', \{q_h\}_{h=1}^m)\cdot N(\varepsilon, \{i_{h+1}, i_h'\}_{h=0}^k, \{q_h\}_{h=1}^m)$$

$$\prod_{h=1}^{k'''}\langle g_{1-\varepsilon(i_h''')}, e^{-\frac{1}{2}\beta H} g_{\varepsilon(i_h''')}\rangle\cdot\langle W(\chi_{|S,T|}\otimes f)\Psi, W(\chi_{|S',T'|}\otimes f')\Psi\rangle$$

with

(4.61) $$M(\varepsilon, i_h, i_h', \{q_h\}_{h=1}^m) := \int_{-\infty}^0 dt_{i_h+1}\cdots\int_{-\infty}^0 dt_{i_h'}$$

$$\prod_{\alpha=0}^{i'_h - i_h - 1} \langle g_{1-\varepsilon(i_h+\alpha)}, S_{t_{i_h+\alpha+1}} g_{\varepsilon(i_h+\alpha+1)} \rangle \overline{\langle g_{\varepsilon(i_h)}, S_{t_{i_h}+\cdots+t_{i'_h+1}} e^{-\frac{1}{2}\beta H} g_{1-\varepsilon(i'_h)} \rangle}.$$

Moreover, denoting

$$\{\beta_h\}_{h=1}^{\bar{m}} := \{q_h : q_h \notin \bigcup_{h=1}^{k} \{i_n + 1, \cdots, i'_h - 1\}, h = 1, \ldots, m\} \tag{4.62}$$

$$\bar{m} = m - \sum_{h=1}^{k} (i'_h - i_h - 1) \tag{4.63}$$

then $\{\beta_h\}_{h=1}^{\bar{m}}$ *has the connected decomposition*

$$\{\beta_h\}_{h=1}^{\bar{m}} = \{\beta_h\}_{h=1}^{r_1} \cup \{\beta_h\}_{h=r_1+1}^{r_2} \cup \cdots \cup \{\beta_h\}_{h=r_{x-1}+1}^{r_x} \tag{4.64}$$

the quantity $N\{\varepsilon, \{i_{h+1}, i'_h\}_{h=0}^{k}, \{q_h\}_{h=1}^{m})$ *is defined by the following expression:*

$$\begin{aligned} &\prod_{a \in \{1,\cdots,n\} \setminus (\{i'_h, i_h\}_{h=1}^{k} \cup \{i'''_h\}_{h=1}^{k'''} \cup \{q_h\}_{h=1}^{m})} (f \mid g_{\varepsilon(\alpha)}) \cdot \chi_{|S,T|}(t_\alpha) \\ &\prod_{a \in (\{1,\cdots,n\} \setminus (\{i'_h, i'_h - 1\}_{h=1}^{k} \cup \{i'''_h\}_{h=1}^{k'''} \cup \{q_h - 1\}_{h=1}^{m})) \cap \{1,\cdots,\beta_1 - 2\}} (g_{1-\varepsilon(\alpha)} \mid f') \cdot \chi_{|S',T'|}(t_\alpha) \\ &\cdot \chi_{|S',T'|}(t_{\beta_1 - 1}) (g_{1-\varepsilon(\beta_{r_1})} \mid f') \\ &\prod_{a \in (\{1,\cdots,n\} \setminus (\{i'_h, i'_h - 1\}_{h=1}^{k} \cup \{i'''_h\}_{h=1}^{k'''} \cup \{q_h - 1\}_{h=1}^{m})) \cap \{\beta_{r_1} + 1, \cdots, \beta_{r_1 + 1} - 2\}} (g_{1-\varepsilon(\alpha)} \mid f') \cdot \chi_{|S',T'|}(t_\alpha) \\ &\cdots\cdots \\ &\cdot \chi_{[S',T')}(t_{\beta_{r_{x-1}+1} - 1}) (g_{1-\varepsilon(\beta_{r_x})} \mid f') \\ &\prod_{a \in (\{1,\cdots,n\} \setminus (\{i'_h, i'_h - 1\}_{h=1}^{k} \cup \{q_h - 1\}_{h=1}^{m})) \cap \{\beta_{r_{x+1}} + 1, \cdots, n\}} (g_{1-\varepsilon(\alpha)} \mid f') \cdot \chi_{|S',T'|}(t_\alpha). \end{aligned} \tag{4.65}$$

Proof. Using Lemma (4.11), the proof of the lemma (3.4) in [1] and dominated convergence, it is enough to show for each subset R_z of $\mathbf{R}$, and $\xi, \eta \in K$,

$$\lim_{z \to 0} \int_{Rz} \langle Q_+ \xi, S_t Q_+ \eta \rangle \, dt = \lim_{z \to 0} \int_{Rz} \langle \xi, S_t \eta \rangle \, dt. \tag{4.66}$$

In fact

$$\begin{aligned} &\left| \int_{Rz} \langle Q_+ \xi, S_t Q_+ \eta \rangle \, dt - \int_{Rz} \langle \xi, S_t \eta \rangle \, dt \right| \\ &\leq \int_{Rz} \left| \langle Q_+ \xi, S_t Q_+ \eta \rangle - \langle \xi, S_t \eta \rangle \right| dt \end{aligned} \tag{4.67}$$

and

$$Q_+^2 = (1 - z^2 e^{-\frac{1}{2}\beta H})^{-1} = \sum_{n=0}^{\infty} z^{2n} e^{-\frac{1}{2}\beta n H}. \tag{4.68}$$

So,

$$\int_{Rz} \left| \langle Q_+ \xi, S_t Q_+ \eta \rangle - \langle \xi, S_t \eta \rangle \right| dt \leq z^2 \sum_{n=1}^{\infty} z^{2(n-1)} \tag{4.69}$$

$$\times \int_{-\infty}^{\infty} \left| \langle \xi, S_t e^{-\frac{1}{2}\beta n H} \eta \rangle \right| dt = z^2 \cdot O(1) \longrightarrow 0$$

§5. The uniform estimate

Having investigated the low density limit term by term, the present section will be devoted to study the uniform estimate.

Applying (3.24), (3.25) to (3.22), it will be obtained that

$$
\begin{aligned}
(5.1)\qquad & |\langle W\Big(z\int_{S/z^2}^{T/z^2} Q_+ S_u f du\Big)\Phi_F \otimes W\Big(z^2\int_{S/z^2}^{T/z^2} Q^- S_u f du\Big)\Phi_F^{\zeta}, \\
& \int_0^{t/z^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n (\mathrm{I}_n^{\varepsilon} + \mathrm{II}_n^{\varepsilon}) \cdot \\
& \cdot W\Big(z\int_{S'/z^2}^{T'/z^2} Q_+ S_u f' du\Big)\Phi_F \otimes W\Big(z^2\int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\Big)\Phi_F^{\zeta} \rangle| \le \\
& \le \sum_{k=0}^{n} \sum_{1\le i_1<\cdots<i_k\le n} \sum_{k'=0}^{n-k} \sum_{\substack{1\le i'_1<\cdots<i'_{k'}\le n \\ \{i_h\}_{h=1}^{k}\cap\{i'_h\}_{h=1}^{k'}=\emptyset}} \sum_{k''=0}^{n-k-k'} \sum_{\substack{1\le i''_1<\cdots<i''_{k''}\le n \\ \{i''_h\}_{h=1}^{k''}\cap(\{i_h\}_{h=1}^{k}\cup\{i'_h\}_{h=1}^{k'})=\emptyset}} \\
& \sum_{k'''=0}^{n-k-k'-k''} \sum_{\substack{1\le i'''_1<\cdots<i'''_{k'''}\le n \\ \{i'''_h\}_{h=1}^{k'''}\cap(\{i_h\}_{h=1}^{k}\cup\{i'_h\}_{h=1}^{k'}\cup\{i''_h\}_{h=1}^{k''})=\emptyset}} \\
& \sum_{m=0}^{n-k-k'-k''-k'''} \sum_{m'=0}^{k'\wedge(n-k'-k''-k'''-m)} \sum_{d'=0}^{k'\wedge(k+k'')} \sum_{d''=0}^{k''\wedge(k-k''-d')} \sum_{d=0}^{d'} (d'-\bar{d})! \\
& \sum_{\substack{1\le\beta_1<\cdots<\beta_{\bar{d}+d''+m+m'}\le n \\ \{\beta_h\}_{h=1}^{d+d''+m+m'}\subset\{1,\cdots,n\}\setminus(\{i_h\}_{h=1}^{k}\cup\{i'''_h\}_{h=1}^{k'''})}} \sum_{\substack{1\le\alpha_1,\cdots,\alpha_{d+d''+m+m'}\le n;\,\alpha_h<\beta_h,\,h=1,\cdots,\bar{d}+d''+m+m', \\ \{\alpha_h\}_{h=1}^{d+d''+m+m'}\subset\{1,\cdots,n\}\setminus(\{i'_h\}_{h=1}^{k'}\cup\{i'''_h\}_{h=1}^{k'''})}} \\
& C_1^{2(n-m-m'-d'-d'')} \cdot z^{2(n-m-m'-d'-d''+k'+k'')} \cdot \int_0^{t/z^2} dt_1 \int_0^{t_1} dt_t \cdots \int_0^{t_{n-1}} dt_n \\
& \prod_{h=1}^{\bar{d}+d''+m+m'} |\langle PS_{t_{\alpha_h}} g_{1-\varepsilon(\alpha_h)}, PS_{t_{\beta_h}} g_{\varepsilon(\beta_h)} \rangle| \\
& |\langle W\Big(z\int_{S/z^2}^{T/z^2} Q_+ S_u f du\Big)\Phi_F \otimes W\Big(z^2\int_{S/z^2}^{T/z^2} Q^- S_u f du\Big)\Phi_F^{\zeta}, \\
& W\Big(z\int_{S'/z^2}^{T'/z^2} Q_+ S_u f' du\Big)\Phi_F \otimes W\Big(z^2\int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\Big)\Phi_F^{\zeta} \rangle|
\end{aligned}
$$

where, the factor $(d'-\bar{d})$ is due to the fact that all the elements in

$$\{p_h\}_{h=1}^{m} \cup \{p'_h\}_{h=1}^{m'} \cup \{\bar{p}'_h\}_{h=1}^{d'} \cup \{\bar{p}''_h\}_{h=1}^{d''} \setminus \{\alpha_h\}_{h=1}^{\bar{d}+b''+m+m'}$$

being chose freely and $P = Q_+$ or Q^-.

In the following, for each $n \in \mathbf{N}$ and $\varepsilon \in \{0, 1\}^n$, we denote the left hand side of (5.1) by $\Delta(n, \varepsilon, t)$ then, one has

THEOREM (5.1). *For each* $n \in \mathbf{N}$,

$$\Delta(n, \varepsilon, t) \leq n^5 \cdot 8^n \max_{0 \leq m \leq n} \left(\| g \|_s + o_z(1) \right)^{2m} t^{n-m} \tag{5.2}$$

where, $o_z(1)$ *tends to zero as* $z \to 0$ *and* $\| g \|_s^2$ *is defined in* (1.15).

Proof. Put

$$f(t) := \sum_{\varepsilon \in \{0,1\}} \max_{P = Q^+, \ell^{-1} Q^-} \left| \langle Pg_\varepsilon, S_t P g_\varepsilon \rangle \right| \tag{5.3}$$

then,

$$\int_{-\infty}^{\infty} f(t) dt = \sum_{\varepsilon \in \{0,1\}} \max_{P = Q^+, \ell^{-1} Q^-} \int_{-\infty}^{\infty} \left| \langle Pg_\varepsilon, S_t P g_\varepsilon \rangle \right| dt \tag{5.4}$$

this is controlled by

$$2 \cdot \| g \|_s^2 + o_z(1) \tag{5.5}$$

for z small enough. Moreover, f is a bounded symmetric function, therefore using Lemma (9.1) of [0] to the quantity

$$\sum_{\substack{1 \leq \alpha_1, \cdots, \alpha_{\bar{d}+d''+m+m'} \leq n; \alpha_h < \beta_h, h = 1, \cdots, \bar{d}+d'' m+m', \\ \{\alpha_h\}_{h=1}^{\bar{d}+d''+m+m'} \subset \{1, \cdots, n\} \setminus (\{i'_h\}_{h=1}^{k'} \cup \{i'''_h\}_{h=1}^{k'''})}} z^{2(n-m-m'-d'-d''+k'+k'')}$$

$$\int_0^{t/z^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \prod_{h=1}^{\bar{d}+d''+m+m'} \left| \langle S_{t_{\alpha_h}} P g_{1-\varepsilon(\alpha_h)}, S_{t_{\beta_h}} P g_{\varepsilon(\beta_h)} \rangle \right|$$

we find that

$$\Delta(n, \varepsilon, t) \leq \sum_{k=0}^{n} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{k'=0}^{n-k} \sum_{\substack{1 \leq i'_1 < \cdots < i'_{k'} \leq n \\ \{i_h\}_{h=1}^{k} \cap \{i'_h\}_{h=1}^{k'} = \emptyset}} \sum_{k''=0}^{n-k-k'} \sum_{\substack{1 \leq i''_1 < \cdots < i''_{k''} \leq n \\ \{i''_h\}_{h=1}^{k''} \cap (\{i_h\}_{h=1}^{k} \cup \{i'_h\}_{h=1}^{k'}) = \emptyset}} \tag{5.6}$$

$$\sum_{k'''=0}^{n-k-k'-k''} \sum_{\substack{1 \leq i'''_1 < \cdots < i'''_{k'''} \leq n \\ \{i'''_h\}_{h=1}^{k'''} \cap (\{i_h\}_{h=1}^{k} \cup \{i'_h\}_{h=1}^{k'} \cup \{i''_h\}_{h=1}^{k''}) = \emptyset}}$$

$$\sum_{m=0}^{n-k-k'-k''-k'''} \sum_{m'=0}^{k' \wedge (n-k'-k''-k'''-m)} \sum_{d'=0}^{k' \wedge (k+k'')} \sum_{d''=0}^{k'' \wedge (k+k''-d')} \sum_{\bar{d}=0}^{d'}$$

$$\sum_{\substack{1 \leq \beta_1 < \cdots < \beta_{\bar{d}+d''+m+m'} \leq n \\ \{\beta_h\}_{h=1}^{\bar{d}+d''+m+m'} \subset \{1, \cdots, n\} \setminus (\{i_h\}_{h=1}^{k} \cup \{i'''_h\}_{h=1}^{k'''})}} 2^{n-k-k'''} (d' - \bar{d})! \cdot c_1^{2(n-m-m'-d'-d'')} \cdot z^{2(k'+k''+\bar{d}-d')}$$

$$\frac{t^{n-\bar{d}-d''-m-m'}}{(n - \bar{d} - d'' - m - m')!} \cdot \left(\| g \|_s + o_z(1) \right)^{2(\bar{d}+d''+m+m')}$$

where,

$$c_1 := 1 + \max_{\varepsilon\in\{0,1\};Q'=-\frac{1}{2}\beta H,1;F=f,f'} \int_{-\infty}^{\infty} |\langle g_\varepsilon, S_t Q'F\rangle| \, dt.$$

Notice that the set

$$\{p_h\}_{h=1}^{m} \cup \{p'_h\}_{h=1}^{m'} \cup \{\bar{p}'_h\}_{h=1}^{d'} \cup \{\bar{p}''_h\}_{h=1}^{d''}$$

labels the annihilation operator which are used to produce scalar products, therefore we should have at least $m + m' + d' + d''$ creators, but each fixed time corresponds to two operators, this implies that

$$m + m' + d' + d'' + m + m' + d' + d'' \leq 2n \tag{5.8}$$

i.e.

$$m + m' + d' + d'' \leq n \tag{5.8a}$$

therefore,

$$\frac{(d'-d)!\cdot c_1^{2(n-m-m'-d'-d'')}\cdot z^{2(k'+k''+d-d')}}{(n-\bar{d}-d''-m-m')!}$$

$$\leq \frac{(d'-d)!\cdot c_1^{2(n-m-m'-d'-d'')}}{(n-d'-d''-m-m'+(d'-\bar{d}))!}$$

$$\leq \frac{c_1^{2(n-m-m'-d'-d'')}}{(n-d'-d''-m-m')!} = O(1). \tag{5.9}$$

Moreover, the sum

$$\sum_{m=0}^{n-k-k'-k''-k'''} \sum_{m'=0}^{k'\wedge(n-k'-k''-k'''-m)} \sum_{d'=0}^{k'\wedge(k+k'')} \sum_{d''=0}^{k''\wedge(k+k''-d')} \sum_{d=0}^{d'}$$

is less than n^5 and

$$\sum_{k=0}^{n} \sum_{1\leq i_1<\cdots<i_k\leq n} \sum_{k'=0}^{n-k} \sum_{\substack{1\leq i'_1<\cdots<i'_{k'}\leq n\\ \{i_h\}_{h=1}^{k}\cap\{i'_h\}_{h=1}^{k'}=\emptyset}} \sum_{k''=0}^{n-k-k'} \sum_{\substack{1\leq i''_1<\cdots<i''_{k''}\leq n\\ \{i''_h\}_{h=1}^{k''}\cap(\{i_h\}_{h=1}^{k}\cup\{i'_h\}_{h=1}^{k'})=\emptyset}} \tag{5.10}$$

$$\sum_{k'''=0}^{n-k-k'-k''} \sum_{\substack{1\leq i'''_1<\cdots<i'''_{k'''}\leq n\\ \{i'''_h\}_{h=1}^{k''}\cap(\{i_h\}_{h=1}^{k}\cup\{i'_h\}_{h=1}^{k'}\cup\{i''_h\}_{h=1}^{k''})=\emptyset}} 2^{n-k-k'''}$$

$$= \sum_{k=0}^{n} 2^{n-k}\binom{n}{k} \sum_{k'=0}^{n-k}\binom{n-k}{k'} \sum_{k''=0}^{n-k-k'}\binom{n-k-k'}{k''} \sum_{k'''=0}^{n-k-k'-k''}\binom{n-k-k'-k''}{k'''} 2^{-k'''}$$

$$= \sum_{k=0}^{n} 2^{n-k}\binom{n}{k} \sum_{k'=0}^{n-k}\binom{n-k}{k'} \sum_{k''=0}^{n-k-k'}\binom{n-k-k'}{k''} (3/2)^{n-k-k'-k''}$$

$$= \sum_{k=0}^{n} 2^{n-k} \binom{n}{k} \sum_{k'=0}^{n-k} \binom{n-k}{k'} (5/2)^{n-k-k'}$$

$$= \sum_{k=0}^{n} 2^{n-k} \binom{n}{k} (7/2)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} 7^{n-k} = 8^n$$

so the proof is finished.

THEOREM (5.2). *For each* $r \in \{0, \pm 1, \pm 2\}$, $\sigma \in \{0,1\}$,

$$(5.11) \qquad | \langle W\Big(z \int_{S/z^2}^{T/z^2} Q_+ S_u f du\Big)\Phi_F \otimes W\Big(z^2 \int_{S/z^2}^{T/z^2} Q^- S_u f du\Big)\Phi_F^{\zeta},$$

$$\frac{1}{z^2} \cdot z^{|r|} A_r(S_{t/z^2} g_\sigma, S_{t/z^2} g_{1-\sigma}) \int_0^{t/z^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n (\mathrm{I}_n^{\varepsilon} + \mathrm{II}_n^{\varepsilon}) \cdot$$

$$\cdot W\Big(z \int_{S'/z^2}^{T'/z^2} Q_+ S_u f' du\Big)\Phi_F \otimes W\Big(z^2 \int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\Big)\Phi_F^{\zeta} \rangle | \leq$$

$$\leq n^6 \cdot 8^n \max_{0 \leq m \leq n} \Big(\| g \|_s + o_z(1)\Big)^{2m} t^{n-m}.$$

Proof. The proof is almost the same as ones of Theorem (5.1) and the difference is only replacing

$$\frac{t^{n-\bar{d}-d''-m-m'}}{(n-\bar{d}-d''-m-m')!}$$

by

$$\frac{t^{n-\bar{d}-d''-m-m'-1}}{(n-\bar{d}-d''-m-m'-1)!}$$

§6. The low density limit and its properties

In the section we shall

i) prove that the limit (1.13) exists and formulate its explicit form;

ii) find a differential (or integro) equation satisfied by the limit (1.13).

THEOREM (6.1). *For each* $g_0, g_1, \in K$, $t \geq 0$, $D \in B(H_0)$ *satisfying* (1.15), *and each* $f, f' \in K$, $S, T, S', T'', \in \mathbf{R}$, $u, v \in H_0$, *the low density limit*

$$(6.1) \quad \lim_{z \to 0} \langle u \otimes W\Big(z \int_{S/z^2}^{T/z^2} S_u f du\Big) \Phi_z,\ U_{t/z^2} v \otimes W\Big(z \int_{S'/z^2}^{T'/z^2} S_u f' du\Big) \Phi_z \rangle$$

exists and is equal to

$$(6.2) \qquad \sum_{n=0}^{\infty} \sum_{\alpha \in \{0,1\}} \langle u, D_{\varepsilon(1)} \cdots D_{\varepsilon(n)} v \rangle$$

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

$$\sum_{k=0}^{[n/2]} \sum_{1\le i_1'<\cdots<i_k<i_k'\le n} \sum_{l=1}^{n-2k} \sum_{\substack{1\le j_1<\cdots<j_l\le n \\ \{j_h\}_{h=1}^{l}\subset\{1,\cdots,n\}\setminus(\{i_h,i_h'\}_{h=1}^{k}}}$$

$$\sum_{m=\Sigma_{h=1}^{k}(i_h'-i_h-1)}^{n-2k-l} \sum_{\substack{1\le q_1<\cdots<q_m\le n \\ \{q_h\}_{h=1}^{m}\subset\{1,\cdots,n\}\setminus(\{i_h,i_h'\}_{h=1}^{k}\cup\{j_h\}_{h=1}^{l};\{q_h-1\}_{h=1}^{m}\cap(\{i_h',i_h'-1\}_{h=1}^{k}=\emptyset}}$$

$$\int_{0\le t_{n-1}\le\cdots\le\widehat{t_{i_k'}}\cdots\le\widehat{t_{q_m}}\le\cdots\le\widehat{t_{q_1}}\le\cdots\le\widehat{t_{i_1'}}\le\cdots\le t_1\le t} dt_1\cdots\widehat{dt_{i_1'}}\cdots\widehat{dt_{q1}}\cdots\widehat{dt_{q_m}}\cdots \widehat{dt_{i_k'}}\cdots dt_n$$

$$\prod_{h=1}^{k} M(\varepsilon_1,i_h,i_h',\{q_h\}_{h=1}^{m})\cdot N(\varepsilon_1\{i_{h+1},i_h'\}_{h=0}^{k},\ \{q_h\}_{h=1}^{m})$$

$$\prod_{h=1}^{l}\langle g_{\varepsilon(j_h)},\ g_{1-\varepsilon(j_h)}\rangle\langle W(\chi_{|S,T|}\otimes f)\Psi,\ W(\chi_{|S',T'|}\otimes f')\Psi\rangle$$

where, $M(\varepsilon,\ i_h,\ i_h',\{q_h\}_{h=1}^{m})$, $h=1,\ldots,m$ *and* $N(\varepsilon,\{i_{h+1},i_h'\}_{h=0}^{k},\{q_h\}_{h=1}^{m})$ *are defined in* (4.61) *and* (4.65).

Proof.

$$(6.3)\qquad \left\langle u\otimes W\left(z\int_{S/z^2}^{T/z^2} S_u f du\right)\Phi_z,\ U_{t/z^2}v\otimes W\left(z\int_{S'/z^2}^{T'/z^2} S_u f' du\right)\Phi_z\right\rangle$$

$$=\sum_{n=0}^{\infty}\sum_{\varepsilon\in\{0,1\}}\langle u,\ D_{\varepsilon(1)}\cdots D_{\varepsilon(n)}v\rangle$$

$$\left\langle u\otimes W\left(z\int_{S/z^2}^{T/z^2} S_u f du\right)\Phi_z,\ \int_0^{t/z^2}dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n\right.$$

$$\left. A^+(S_{t_1}g_{\varepsilon(1)})A(S_{t_1}g_{1-\varepsilon(1)})\cdots A^+(S_{t_n}g_{\varepsilon(n)})A(S_{t_n}g_{1-\varepsilon(n)})v\otimes W\left(z\int_{S'/z^2}^{T'/z^2} S_u f' du\right)\Phi_z\right\rangle.$$

By the uniform estimate,

$$(6.4)\qquad \lim_{z\to0}\left\langle u\otimes W\left(z\int_{S/z^2}^{T/z^2} S_u f du\right)\Phi_z,\ U_{t/z^2}v\otimes W\left(z\int_{S'/z^2}^{T'/z^2} S_u f' du\right)\Phi_z\right\rangle$$

$$=\sum_{n=0}^{\infty}\sum_{\alpha\in\{0,1\}}\langle u,\ D_{\varepsilon(1)}\cdots D_{\varepsilon(n)}u\rangle$$

$$\lim_{z\to0}\left\langle u\otimes W\left(z\int_{S/z^2}^{T/z^2} S_u f du\right)\Phi_z,\ \int_0^{t'/z^2}dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n\right.$$

$$\left. A^+(S_{t_1}g_{\varepsilon(1)})A(S_{t_1}g_{1-\varepsilon(1)})\cdots A^+(S_{t_n}g_{\varepsilon(n)})A(S_{t_n}g_{1-\varepsilon(n)})v\otimes W\left(z\int_{S'/z^2}^{T'/z^2} S_u f' du\right)\Phi_z\right\rangle.$$

Applying Theorem (4.12) to (6.4), we finish the proof at once.

This theorem shows that for each $u,\ v\in H_0$, there exists a $G(t)\in H_0$, such that

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

(6.5)
$$\langle u, G(t) \rangle$$
$$= \lim_{z\to 0} \langle u \otimes W\left(z\int_{S/z^2}^{T/z^2} S_u f du\right) \Phi_z,\ U_{t/z^2} v \otimes W\left(z\int_{S'/z^2}^{T'/z^2} S_u f' du\right) \Phi_z \rangle.$$

For each $z > 0$, defining

(6.6)
$$\langle u, G_z(t) \rangle$$
$$:= \langle u \otimes W\left(z\int_{S/z^2}^{T/z^2} S_u f du\right) \Phi_z,\ U_{t/z^2} v \otimes W\left(z\int_{S'/z^2}^{T'/z^2} S_u f' du\right) \Phi_z \rangle$$

then

(6.7)
$$\langle u, G(0) \rangle = \lim_{z\to 0} \langle u, G_z(0) \rangle =$$
$$= \lim_{z\to 0} \langle u \otimes W\left(z\int_{S/z^2}^{T/z^2} S_u f du\right) \Phi_z,\ v \otimes W\left(z\int_{S'/z^2}^{T'/z^2} S_u f' du\right) \Phi_z \rangle$$
$$= \langle u, v \rangle \langle W\left(\chi_{|S,T|} \otimes f\right) \Psi,\ W\left(\chi_{|S',T'|} \otimes f'\right) \Psi \rangle.$$

So,

(6.8)
$$\langle u, G(t) \rangle = \lim_{z\to 0} \langle u, G_z(t) \rangle = \langle u, G(0) \rangle$$
$$+ \lim_{z\to 0} \int_0^t \frac{d}{ds} \langle u, G_z(s) \rangle\, ds.$$

Moreover, for each $z > 0$, $n \in \mathbf{N}$,

(6.9)
$$\Big| \frac{d}{ds} \langle u \otimes W\left(z\int_{S/z^2}^{T/z^2} S_u f du\right) \Phi_z,\ \int_0^{s/z^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n$$
$$V(t_1) \cdots V(t_n) v \otimes W\left(z\int_{S'/z^2}^{T'/z^2} S_u f' du\right) \Phi_z \rangle \Big| =$$
$$= \Big| \langle u \otimes W\left(z\int_{S/z^2}^{T/z^2} S_u f du\right) \Phi_z,\ \int_0^{s/z^2} dt_2 \int_0^{t_2} dt_2 \cdots \int_0^{t_{n-1}} dt_n$$
$$\frac{1}{z^2} V(s/z^2) V(t_2) \cdots V(t_n) v \otimes W\left(z\int_{S'/z^2}^{T'/z^2} S_u f' du\right) \Phi_z \rangle \Big|.$$

Hence from Theorem (5.2) follows that

(6.10)
$$\frac{d}{ds} \langle u, G_z(s) \rangle = \sum_{n=1}^{\infty} \langle u \otimes W\left(z\int_{S/z^2}^{T/z^2} S_u f du\right) \Phi_z,$$
$$\frac{d}{ds} \int_0^{s/z^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n (-i)^n V(t_1) \cdots V(t_n)\ v \otimes$$
$$W\left(z\int_{S'/z^2}^{T'/z^2} S_u f' du\right) \Phi_z \rangle$$

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

$$= \sum_{n-1}^{\infty} \frac{-i}{z^2} \langle u \otimes W\left(z\int_{S/z^2}^{T/z^2} S_u f du\right) \Phi_z,$$

$$V(s/z^2) \int_0^{s/z^2} dt_2 \int_0^{t_2} dt_3 \cdots \int_0^{t_{n-1}} dt_n (-i)^{n-1} V(t_2) \cdots V(t_n)$$

$$v \otimes W\left(z\int_{S'/z^2}^{T'/z^2} S_u f' du\right) \Phi_z \rangle$$

$$= \frac{-i}{z^2} \langle u \otimes W\left(z\int_{S/z^2}^{T/z^2} S_u f du\right) \Phi_z,\ V(s/z^2) U_{s/z^2} v \otimes W\left(z\int_{S'/z^2}^{T'/z^2} S_u f' du\right) \Phi_z \rangle.$$

Moreover, by expanding V to the sum of the products of creation and annihilation operators according to (1.15) and (1.10), we find that

(6.11)
$$\frac{d}{ds} \langle u,\ G_z(s) \rangle = \frac{1}{z^2} \langle u \otimes W\left(z\int_{S/z^2}^{T/z^2} S_u Q_+ f du\right) \Phi_F$$

$$\otimes W\left(z^2 \int_{S/z^2}^{T/z^2} Q^- S_u f du\right) \Phi_F^c \rangle$$

$$\sum_{\varepsilon \in \{0,1\}} D_\varepsilon \otimes \Big(A^+(S_{s/z^2} Q_+ g_\varepsilon) A(S_{s/z^2} Q_+ g_{1-\varepsilon}) \otimes 1$$

$$z \cdot A^+(S_{s/z^2} Q_+ g_\varepsilon) \otimes A^+(Q^- S_{s/z^2} g_{1-\varepsilon}) + z \cdot A(S_{s/z^2} Q_+ g_{1-\varepsilon}) \otimes A(Q^- S_{s/z^2} g_\varepsilon)$$

$$+ z^2 1 \otimes (A^+(Q^- S_{s/z^2} g_{1-\varepsilon})\, A(Q^- S_{s/z^2} g_\varepsilon) + \langle Q^- g_\varepsilon,\ Q^- g_{1-\varepsilon} \rangle_\iota)\Big) U_{s/z^2}$$

$$v \otimes W\left(z\int_{S'/z^2}^{T'/z^2} S_u Q_+ f' du\right) \Phi_F \otimes W\left(z^2 \int_{S'/z^2}^{T'/z^2} S_u Q^- f' du\right) \Phi_F^c \rangle$$

$$:= \frac{1}{z^2} \mathrm{I}_z + \frac{1}{z} \mathrm{II}_z + \frac{1}{z} \mathrm{III}_z + \mathrm{IV}_z.$$

Now our first step is to deal with $\frac{1}{z^2}\mathrm{I}_z$. We split $\frac{1}{z^2}\mathrm{I}_z$ into a sum of two terms, one of them corresponds to the case of the creator $A^+(S_{s/z^2}Q_+ g_\varepsilon)$ acting on the coherent vector $W(z\int_{S/z^2}^{T/z^2} S_u Q_+ f du)\Phi_F$ and the annihilation operator $A^+(S_{s/z^2}Q_+ g_{1-\varepsilon})$ acting on the coherent vector $W(z\int_{S'/z^2}^{T'/z^2} S_u Q_+ f' du)\Phi_F$; the other one corresponds to the case of the creator $A^+(S_{s/z^2}Q_+ g_\varepsilon)$ acting on the coherent vector $W(z\int_{S/z^2}^{T/z^2} S_u Q_+ f du)\Phi_F$ and the annihilation operator $A^+(S_{s/z^2}Q_+ g_{1-\varepsilon})$ commuting with the operator U_{s/z^2}, i.e.

(6.12)
$$\frac{1}{z^2}\mathrm{I}_z = \sum_{\varepsilon \in \{0,1\}} \langle D_\varepsilon^+ u \otimes W\left(z\int_{S/z^2}^{T/z^2} S_u Q_+ f du\right) \Phi_F \otimes W\left(z^2 \int_{S/z^2}^{T/z^2} Q^- S_u f du\right) \Phi_F^c$$

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

$$\int_{S/z^2}^{T/z^2} \langle S_u Q_+ f, S_{s/z^2} Q_+ g_\varepsilon \rangle du \cdot \int_{S'/z^2}^{T'/z^2} \langle S_{s/z^2} Q_+ g_{1-\varepsilon}, S_u Q_+ f' \rangle du \cdot U_{s/z^2}$$

$$v \otimes W\left(z \int_{S'/z^2}^{T'/z^2} S_u Q_+ f' du\right) \Phi_F \otimes W\left(z^2 \int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\right) \Phi_F' \rangle +$$

$$+ \sum_{\varepsilon \in \{0,1\}} \langle D_\varepsilon^+ u \otimes W\left(z \int_{S/z^2}^{T/z^2} S_u Q_+ f du\right) \Phi_F \otimes W\left(z^2 \int_{S/z^2}^{T/z^2} Q^- S_u f du\right) \Phi_F'$$

$$\frac{1}{z} \int_{S/z^2}^{T/z^2} \langle S_u Q_+ f, S_{s/z^2} Q_+ g_\varepsilon \rangle du \cdot \left[1 \otimes A(S_{s/z^2} Q_+ g_{1-\varepsilon}) \otimes, U_{s/z^2}\right]$$

$$v \otimes W\left(z \int_{S'/z^2}^{T'/z^2} S_u Q_+ f' du\right) \Phi_F \otimes W\left(z^2 \int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\right) \Phi_F' \rangle$$

$$:= \mathrm{I}_z(a) + \mathrm{I}_z(b).$$

It is obvious that

(6.13)

$$\lim_{z \to 0} \mathrm{I}_z(a) = \sum_{\varepsilon \in \{0,1\}} \chi_{[S,T]}(s) (f \mid g_\varepsilon) \cdot \chi_{[S',T']}(s) (g_{1-\varepsilon} \mid f') \langle D_\varepsilon^+ u, G(s) \rangle \quad \text{a.e.}$$

Moreover denoting for each $n \in \mathbf{N}$,

$$\mathrm{I}_z(n, a) := \sum_{\varepsilon \in \{0,1\}} \langle D_\varepsilon^+ u \otimes W\left(z \int_{S/z^2}^{T/z^2} S_u Q_+ f du\right) \Phi_F \otimes W$$

$$\times \left(z^2 \int_{S/z^2}^{T/z^2} Q^- S_u f du\right) \Phi_F',$$

$$\int_{S/z^2}^{T/z^2} \langle S_u Q_+ f, S_{s/z^2} Q_+ g_\varepsilon \rangle du \cdot \int_{S'/z^2}^{T'/z^2} \langle S_{s/z^2} Q_+ g_{1-\varepsilon}, S_u Q_+ f' \rangle du \cdot$$

$$\int_0^{s/z^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n V(t_1) \cdots V(t_n)$$

$$v \otimes W\left(z \int_{S'/z^2}^{T'/z^2} S_u Q_+ f' du\right) \Phi_F \otimes W\left(z^2 \int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\right) \Phi_F' \rangle$$

then the limit $\lim_{z \to 0} \mathrm{I}_z(n, a)$ exists.

In order to obtain the limit

(6.14) $$\lim_{z \to 0} \mathrm{I}_z(b)$$

we need the following

LEMMA (6.2). *The limit*

(6.15)

$$\lim_{z \to 0} \sum_{\varepsilon \in \{0,1\}} \langle D_\varepsilon^+ u \otimes W\left(z \int_{S/z^2}^{T/z^2} S_u Q_+ f du\right) \Phi_F \otimes W\left(z^2 \int_{S/z^2}^{T/z^2} Q^- S_u f du\right) \Phi_F',$$

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

$$\frac{1}{z}\left[1\otimes A(S_{s/z^2}Q_+g_{1-\varepsilon})\otimes 1,\ U_{s/z^2}\right]$$

$$v\otimes W\left(z\int_{S'/z^2}^{T'/z^2} S_uQ_+f'du\right)\Phi_F\otimes W\left(z^2\int_{S'/z^2}^{T'/z^2} Q^-S_uf'du\right)\Phi_F^{\zeta}\rangle$$

exists.

Proof. By the uniform estimate (Theorem (5.2)), it is enough to show that for each $n\in\mathbf{N}$,

(6.16)

$$\lim_{z\to 0}\sum_{\varepsilon\in\{0,1\}}\langle D_\varepsilon^+u\otimes W\left(z\int_{S/z^2}^{T/z^2} S_uQ_+fdu\right)\Phi_F\otimes W\left(z^2\int_{S/z^2}^{T/z^2} Q^-S_ufdu\right)\Phi_F^{\zeta},$$

$$\int_0^{s/z^2}dt_1\cdots\int_0^{t_{n-1}}dt_n\frac{1}{z}(-i)^n\left[1\otimes A(S_{s/z^2}Q_+g_{1-\varepsilon})\otimes 1,\ V(t_1)\cdots V(t_n)\right]$$

$$v\otimes W\left(z\int_{S'/z^2}^{T'/z^2} S_uQ_+f'du\right)\Phi_F\otimes W\left(z^2\int_{S'/z^2}^{T'/z^2} Q^-S_uf'du\right)\Phi_F^{\zeta}\rangle$$

exists.

Denoting the limit (6.16) by

(6.17) $$\lim_{z\to 0}J(n,z,s)$$

let us prove the lemma by induction.

First of all,

(6.18) $$J(1,z,s)=\sum_{\varepsilon,\varepsilon(1)\in\{0,1\}}$$

$$\langle D_\varepsilon^+u\otimes W\left(z\int_{S/z^2}^{T/z^2} S_uQ_+fdu\right)\Phi_F\otimes W\left(z^2\int_{S/z^2}^{T/z^2} Q^-S_ufdu\right)\Phi_F^{\zeta},$$

$$\int_0^{s/z^2}dt_1D_{\varepsilon(1)}\otimes\frac{1}{z}\left[A(S_{s/z^2}Q_+g_{1-\varepsilon})\otimes 1,\right.$$

$$\Big(A^+(S_{t_1}Q_+g_{\varepsilon(1)})A(S_{t_1}Q_+g_{1-\varepsilon(1)})\otimes 1+$$

$$z\Big(A^+(S_{t_1}Q_+g_{\varepsilon(1)})\otimes A^+(Q^-S_{t_1}g_{1-\varepsilon(1)})+A(S_{t_1}Q_+g_{1-\varepsilon(1)})\otimes A(Q^-S_{t_1}g_{\varepsilon(1)})\Big)+$$

$$\left.+z^21\otimes(\langle Q^-S_{t_1}g_{\varepsilon(1)},\ Q^-S_{t_1}g_{1-\varepsilon(1)}\rangle+A^+(Q^-S_{t_1}g_{1-\varepsilon(1)})A(Q^-S_{t_1}g_{\varepsilon(1)})\Big)\right]$$

$$v\otimes W\left(z\int_{S'/z^2}^{T'/z^2} S_uQ_+f'du\right)\Phi_F\otimes W\left(z^2\int_{S'/z^2}^{T'/z^2} Q^-S_uf'du\right)\Phi_F^{\zeta}\rangle$$

$$:=J_1(1,z,s)+J_2(1,z,s)+J_3(1,z,s)+J_4(1,z,s).$$

Notice that $J_3(1,z,s)=J_4(1,z,s)=0$, now let us investigate the first two terms.

(6.19)
$$J_1(1, z, s) = \sum_{\varepsilon,\varepsilon(1)\in\{0,1\}} \langle D^+_{\varepsilon(1)} D^+_{\varepsilon} u, v \rangle$$
$$\langle W\Big(z\int_{S/z^2}^{T/z^2} S_u Q_+ f du\Big)\Phi_F, \int_0^{s/z^2} dt_1 \frac{1}{z}\Big[A(S_{s/z^2}Q_+ g_{1-\varepsilon}), A^+(S_{t_1}Q_+ g_{\varepsilon(1)})\Big]$$
$$A(S_{t_1}Q_+ g_{1-\varepsilon(1)}) W\Big(z\int_{S'/z^2}^{T'/z^2} S_u Q_+ f' du\Big)\Phi_F \rangle$$
$$\langle W\Big(z\int_{S/z^2}^{T/z^2} Q^- S_u f du\Big)\Phi_F^{\zeta}, W\Big(z^2\int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\Big)\Phi_F^{\zeta} \rangle_{\zeta}$$
$$= \sum_{\varepsilon,\varepsilon(1)\in\{0,1\}} \langle D^+_{\varepsilon(1)} D^+_{\varepsilon} u, v \rangle$$
$$\int_0^{s/z^2} dt_1 \langle S_{s/z^2}Q_+ g_{1-\varepsilon}, S_{t_1}Q_+ g_{\varepsilon(1)} \rangle \int_{S'/z^2}^{T'/z^2} \langle S_{t_1}Q_+ g_{1-\varepsilon(1)}, S_u Q_+ f' \rangle du$$
$$\langle W\Big(z\int_{S/z^2}^{T/z^2} S_u Q_+ f du\Big)\Phi_F, W\Big(z\int_{S'/z^2}^{T'/z^2} S_u Q_+ f' du\Big)\Phi_F \rangle$$
$$\langle W\Big(z^2\int_{S/z^2}^{T/z^2} Q^- S_u f du\Big)\Phi_F^{\zeta}, W\Big(z^2\int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\Big)\Phi_F^{\zeta} \rangle_{\zeta}.$$

With the change of variable

(6.19a)
$$t_1 - s/z^2 = s_1$$

the right hand side of (6.19) becomes

(6.20)
$$\sum_{\varepsilon,\varepsilon(1)\in\{0,1\}} \langle D^+_{\varepsilon(1)} D^+_{\varepsilon} u, v \rangle$$
$$\int_{-s/z^2}^{0} ds_1 \langle Q_+ g_{1-\varepsilon}, S_{s_1} Q_+ g_{\varepsilon(1)} \rangle \int_{S'/z^2}^{T/z^2} \langle S_{s_1+s/z^2}Q_+ g_{1-\varepsilon(1)}, S_u Q_+ f' \rangle du$$
$$\langle W\Big(z\int_{S/z^2}^{T/z^2} S_u f du\Big)\Phi_z, W\Big(z\int_{S'/z^2}^{T'/z^2} S_u f' du\Big)\Phi_z \rangle.$$

Again with the change of variable

(6.21)
$$u - s_1 - s/z^2 = v$$

(6.20) becomes
$$\sum_{\varepsilon,\varepsilon(1)\in\{0,1\}} \langle D^+_{\varepsilon(1)} D^+_{\varepsilon} u, v \rangle$$

(6.22a)
$$\int_{-s/z^2}^{0} ds_1 \langle Q_+ g_{1-\varepsilon}, S_{s_1} Q_+ g_{\varepsilon(1)} \rangle \int_{(S'-s)/z^2-s_1}^{(T'-s)/z^2-s_1} \langle Q_+ g_{1-\varepsilon(1)}, S_u Q_+ f' \rangle dv$$
$$\langle W\Big(z\int_{S/z^2}^{T/z^2} S_u f du\Big)\Phi_z, W\Big(z\int_{S'/z^2}^{T'/z^2} S_u f' du\Big)\Phi_z \rangle$$

which tends to

(6.22) $$\sum_{\varepsilon,\varepsilon(1)\in\{0,1\}} \langle D^+_{\varepsilon(1)} D^+_\varepsilon u,\ v\rangle \int_{-\infty}^0 ds_1 \langle g_{1-\varepsilon},\ S_{s_1} g_{\varepsilon(1)}\rangle\ (g_{1-\varepsilon(1)} \mid f')\cdot \chi_{[S',T']}(s)$$
$$\langle W\left(\chi_{[S,T]}\otimes f\right)\Psi,\ W\left(\chi_{[S',T']}\otimes f'\right)\Psi\rangle.$$

Moreover,

(6.23) $$J_2(1,\ z,\ s) = \sum_{\varepsilon,\varepsilon(1)\in\{0,1\}}$$
$$\langle D^+_{\varepsilon(1)} D^+_\varepsilon u \otimes W\left(z\int_{S/z^2}^{T/z^2} S_u Q_+ f du\right)\Phi_F \otimes W\left(z^2\int_{S/z^2}^{T/z^2} Q^- S_u f du\right)\Phi'_F,$$
$$\int_0^{s/z^2} dt_1 1\otimes\left[A(S_{s/z^2}Q_+ g_{1-\varepsilon}),\ A^+(S_{t_1}Q_+ g_{\varepsilon(1)})\right]\otimes A^+(Q^- S_{t_1} g_{1-\varepsilon(1)})$$
$$v\otimes W\left(z\int_{S'/z^2}^{T'/z^2} S_u Q_+ f' du\right)\Phi_F \otimes W\left(z^2\int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\right)\Phi'_F\rangle$$
$$= \sum_{\varepsilon,\varepsilon(1)\in\{0,1\}} \langle D^+_{\varepsilon(1)} D^+_\varepsilon u,\ v\rangle$$
$$z^2\int_{S/z^2}^{T/z^2} \langle Q^- S_u f,\ Q^- S_{t_1} g_{1-\varepsilon(1)}\rangle_\iota\, du \int_0^{s/z^2} dt_1 \langle S_{s/z^2}Q_+ g_{1-\varepsilon},\ S_{t_1}Q_+ g_{\varepsilon(1)}\rangle$$
$$\langle W\left(z\int_{S/z^2}^{T/z^2} S_u f du\right)\Phi_Q,\ W\left(z\int_{S'/z^2}^{T'/z^2} S_u f' du\right)\Phi_Q\rangle.$$

With the same change of variables (6.19) and (6.21), one knows that the right hand side of (6.23) is equal to $z^2\cdot O(1)$ and tends to zero.

Assuming that the limit (6.16) exists for n, let see the situation in $n+1$. By the same arguments as in (6.17), one can write $J(n+1,\ z,\ s)$ as a sum of $J_1(n+1,\ z,\ s)$ and $J_2(n+1,\ z,\ s)$, where

(6.24) $$J_1(n+1,\ z,\ s) := \sum_{\varepsilon,\varepsilon(1)\in\{0,1\}}$$
$$\langle D^+_\varepsilon u\otimes W\left(z\int_{S/z^2}^{T/z^2} S_u Q_+ f du\right)\Phi_F \otimes W\left(z^2\int_{S/z^2}^{T/z^2} Q^- S_u f du\right)\Phi'_F,$$
$$\int_0^{s/z^2} dt_1\cdots\int_0^{t_n} dt_{n+1} D_{\varepsilon(1)}\otimes\frac{1}{z}\left[A(S_{s/z^2}Q_+ g_{1-\varepsilon}),\ A^+(S_{t_1}Q_+ g_{\varepsilon(1)})\right]$$
$$A(S_{t_1}Q_+ g_{1-\varepsilon(1)})\otimes 1 V(t_2)\cdots V(t_{n+1})(-i)^n$$
$$v\otimes W\left(z\int_{S'/z^2}^{T'/z^2} S_u Q_+ f' du\right)\Phi_F \otimes W\left(z^2\int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\right)\Phi'_F\rangle$$

and

(6.25) $$J_2(n+1,\ z,\ s) := \sum_{\varepsilon,\varepsilon(1)\in\{0,1\}}$$

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

$$\langle D_\varepsilon^+ u \otimes W\left(z\int_{S/z^2}^{T/z^2} S_u Q_+ f du\right)\Phi_F \otimes W\left(z^2\int_{S/z^2}^{T/z^2} Q_- S_u f du\right)\Phi_F^\zeta,$$

$$\int_0^{s/z^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_n} dt_{n+1} D_{\varepsilon(1)} \otimes \left[A(S_{s/z^2}Q_+ g_{1-\varepsilon}),\ A^+(S_{t_1}Q_+ g_{\varepsilon(1)})\right]$$

$$1 \otimes 1 \otimes A^+(Q_- S_{t_1} g_{1-\varepsilon(1)}) V(t_2) \cdots V(t_{n+1}) (-i)^n$$

$$v \otimes W\left(z\int_{S'/z^2}^{T'/z^2} S_u Q_+ f' du\right)\Phi_F \otimes W\left(z^2\int_{S'/z^2}^{T'/z^2} Q_- S_u f' du\right)\Phi_F^\zeta \rangle.$$

In (6.25), we apply the creator $A^+(Q_- S_{t_1} g_{1-\varepsilon(1)})$ to the coherent vector

$$W\left(z^2\int_{S/z^2}^{T/z^2} Q_- S_u f du\right)\Phi_F^\zeta$$

and change of variable

$$t_1 - s/z^2 =: t' \tag{6.26}$$

one finds that the module of $J_2(n+1, z, s)$ is less than or equal to

$$\sum_{\varepsilon,\varepsilon(1)\in\{0,1\}} \Big| \langle D_{\varepsilon(1)}^+ D_\varepsilon^+ u \otimes W\left(z\int_{S/z^2}^{T/z^2} S_u Q_+ f du\right)\Phi_F \tag{6.27}$$

$$\otimes W\left(z^2\int_{S/z^2}^{T/z^2} Q_- S_u f du\right)\Phi_F^\zeta,\ \int_{-s/z^2}^0 dt' \langle Q_+ g_{1-\varepsilon},\ S_{t'} Q_+ g_{\varepsilon(1)} \rangle$$

$$z_2 \int_{S/z^2}^{T/z^2} \langle Q_- S_u f,\ Q_- S_{t'-s/z^2} g_{1-\varepsilon(1)} \rangle_\zeta \int_0^{t'+s/z^2} dt_2 \cdots \int_0^{t_n} dt_{n+1} V(t_2) \cdots V(t_{n+1})$$

$$v \otimes W\left(z\int_{S'/z^2}^{T'/z^2} S_u Q_+ f' du\right)\Phi_F \otimes W\left(z^2\int_{S'/z^2}^{T'/z^2} Q_- S_u f' du\right)\Phi_F^\zeta \rangle \Big|$$

$$= z^2 \cdot O(1) \longrightarrow 0.$$

In the formula (6.24), the term $[A(S_{s/z^2}Q_+ g_{1-\varepsilon}),\ A^+(S_{t_1}Q_+ g_{\varepsilon(1)})]$ is equal to the scalar product $\langle S_{s/z^2}Q_+ g_{1-\varepsilon},\ S_{t_1}Q_+ g_{\varepsilon(1)} \rangle$ and the operator

$$1 \otimes A(S_{t_1}Q_+ g_{1-\varepsilon(1)}) \otimes 1 V(t_2) \cdots V(t_{n+1})$$

is equal to the sum of the following two terms:

$$V(t_2) \cdots V(t_{n+1}) 1 \otimes A(S_{t_1}Q_+ g_{1-\varepsilon(1)}) \otimes 1 \tag{6.28a}$$

and

$$[1 \otimes A(S_{t_1}Q_+ g_{1-\varepsilon(1)}) \otimes 1,\ V(t_2) \cdots V(t_{n+1})] \tag{6.28b}$$

therefore

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

(6.29)
$$J_1(n+1, z, s) = \sum_{\varepsilon,\varepsilon(1)\in\{0,1\}}$$
$$\langle D_\varepsilon^+ u \otimes W\Big(z\int_{S/z^2}^{T/z^2} S_uQ_+fdu\Big)\Phi_F \otimes W\Big(z^2\int_{S/z^2}^{T/z^2} Q^- S_ufdu\Big)\Phi_F',$$
$$(-i)^n\int_0^{s/z^2}dt_1\cdots\int_0^{tn}dt_{n+1}D_{\varepsilon(1)}\otimes\frac{1}{z}\langle S_{s/z^2}Q_+g_{1-\varepsilon}, S_{t_1}Q_+g_{\varepsilon(1)}\rangle$$
$$\Big(V(t_2)\cdots V(t_{n+1})1\otimes A(S_{t_1}Q_+g_{1-\varepsilon(1)})\otimes 1 +$$
$$+[1\otimes A(S_{t_1}Q_+g_{1-\varepsilon(1)})\otimes 1, V(t_2)\cdots V(t_{n+1})]\Big)$$
$$v\otimes W\Big(z\int_{S'/z^2}^{T'/z^2} S_uQ_+f'du\Big)\Phi_F\otimes W\Big(z^2\int_{S'/z^2}^{T'/z^2} Q^- S_uf'du\Big)\Phi_F'\rangle.$$

By applying the annihilation operator $A(S_{t_1}Q_+g_{1-\varepsilon(1)})$ on the coherent vector
$$W\Big(z\int_{S'/z^2}^{T'/z^2} S_uQ_+f'du\Big)\Phi_F$$
the first term of (6.29) becomes
(6.30)
$$\sum_{\varepsilon,\varepsilon(1)\in\{0,1\}}\langle D_{\varepsilon(1)}^+D_\varepsilon^+u\otimes W\Big(z\int_{S/z^2}^{T/z^2} S_uQ_+fdu\Big)\Phi_F\otimes W\Big(z^2\int_{S/z^2}^{T/z^2} Q^- S_ufdu\Big)\Phi_F',$$
$$\int_0^{s/z^2}dt_1\cdots\int_0^{tn}dt_{n+1}\frac{1}{z}\langle S_{s/z^2}Q_+g_{1-\varepsilon}, S_{t_1}Q_+g_{\varepsilon(1)}\rangle$$
$$(-i)^nV(t_2)\cdots V(t_{n+1})z\int_{S'/z^2}^{T'/z^2}\langle S_{t_1}Q_+g_{1-\varepsilon(1)}, S_uQ_+f'\rangle du$$
$$v\otimes W\Big(z\int_{S'/z^2}^{T'/z^2} S_uQ_+f'du\Big)\Phi_F\otimes W\Big(z^2\int_{S'/z^2}^{T'/z^2} Q^- S_uf'du\Big)\Phi_F'\rangle.$$

With the change of variable (6.26) and using Lemma (6.3) of [2], (6.30) becomes
(6.31)
$$\sum_{\varepsilon,\varepsilon(1)\in\{0,1\}}\langle D_{\varepsilon(1)}^+D_\varepsilon^+u\otimes W\Big(z\int_{S/z^2}^{T/z^2} S_uQ_+fdu\Big)\Phi_F\otimes W\Big(z^2\int_{S/z^2}^{T/z^2} Q^- S_ufdu\Big)\Phi_F',$$
$$\int_{-s/z^2}^0 dt'\,\langle Q_+g_{1-\varepsilon}, S_{t'}Q_+g_{\varepsilon(1)}\rangle\int_0^{s/z^2}dt_2\cdots\int_0^{tn}dt_{n+1}V(t_2)\cdots V(t_{n+1})(-i)^n$$
$$\int_{s'/z^2}^{T'/z^2}\langle S_{t'+s/z^2}Q_+g_{1-\varepsilon(1)}, S_uQ_+f'\rangle du$$
$$v\otimes W\Big(z\int_{S'/z^2}^{T'/z^2} S_uQ_+f'du\Big)\Phi_F\otimes W\Big(z^2\int_{S'/z^2}^{T'/z^2} Q^- S_uf'du\Big)\Phi_F'\rangle.$$

Thus the conclusion of Section 4 and the fact that

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

$$\int_{-s/z^2}^{0} dt' \langle Q_+ g_{1-\varepsilon},\ S_{t'} Q_+ g_{\varepsilon(1)} \rangle \int_{s'/z^2}^{T'/z^2} \langle S_{t'+3/z^2} Q_+ g_{1-\varepsilon(1)},\ S_u Q_+ f' \rangle du$$

$$\longrightarrow \int_{-\infty}^{0} dt' \langle g_{1-\varepsilon},\ S_{t'} g_{\varepsilon(1)} \rangle \chi_{[S',T']}(s) (g_{1-\varepsilon(1)} \mid f'), \text{ a.e.}$$

shows that the limit of (6.30) exists.

Similar arguments show that by induction the limit of the second term of (6.30) exists.

Summing up we finish the proof.

LEMMA (6.3).

(6.33)

$$\lim_{z\to 0} \sum_{\varepsilon\in\{0,1\}} \langle D_\varepsilon^+ u \otimes W\Big(z\int_{S/z^2}^{T/z^2} S_u Q_+ f du\Big)\Phi_F \otimes W\Big(z^2\int_{S/z^2}^{T/z^2} Q^- S_u f du\Big)\Phi_F^\zeta \rangle$$

$$\Big[1 \otimes 1 \otimes A(Q^- S_{s/z^2} g_\varepsilon), U_{s/z^2}\Big]$$

$$v \otimes W\Big(z\int_{S'/z^2}^{T'/z^2} S_u Q_+ f' du\Big)\Phi_F \otimes W\Big(z^2\int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\Big)\Phi_F^\zeta \rangle = 0.$$

Proof. By the uniform estimate (Theorem (5.2)) it is enough to show that for each $n \in \mathbf{N}$,

$$(6.34)\quad \sum_{\varepsilon\{0,1\}} \langle D_\varepsilon^+ u \otimes W\Big(z\int_{S/z^2}^{T/z^2} S_u Q_+ f du\Big)\Phi_F \otimes W\Big(z^2\int_{S/z^2}^{T/z^2} Q^- S_u f du\Big)\Phi_F^\zeta,$$

$$\int_0^{s/z^2} dt_1 \cdots \int_0^{t_{n-1}} dt_n (-i)^n \Big[1 \otimes 1 \otimes A(Q^- S_{s/z^2} g_\varepsilon),\ V(t_1)\cdots V(t_n)\Big]$$

$$v \otimes W\Big(z\int_{S'/z^2}^{T'/z^2} S_u Q_+ f' du\Big)\Phi_F \otimes W\Big(z^2\int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\Big)\Phi_F^\zeta \rangle \longrightarrow 0,$$

The left hand side of (6.34) is equal to

(6.35)

$$\sum_{\varepsilon,\varepsilon(1)\in\{0,1\}} \langle D_{\varepsilon(1)}^+ D_\varepsilon^+ u \otimes W\Big(z\int_{S/z^2}^{T/z^2} S_u Q_+ f du\Big)\Phi_F \otimes W\Big(z^2\int_{S/z^2}^{T/z^2} Q^- S_u f du\Big)\Phi_F^\zeta,$$

$$\int_0^{s/z^2} dt_1 \cdots \int_0^{t_{n-1}} dt_n (-i)^{n-1}\Big(z \langle Q^- S_{s/z^2} g_\varepsilon,\ Q^- S_{t_1} g_{\varepsilon(1)} \rangle_\zeta$$

$$1 \otimes A^+(Q^- S_{t_1} g_{1-\varepsilon(1)}) \otimes 1\ V(t_2)\cdots V(t_n) +$$

$$+ z^2 \langle Q^- S_{s/z^2} g_\varepsilon,\ Q^- S_{t_1} g_{1-\varepsilon(1)} \rangle_\zeta V(t_2)\cdots V(t_n) 1 \otimes A(Q^- S_{t_1} g_{\varepsilon(1)} \otimes 1 +$$

$$+ z^2 \langle Q^- S_{s/z^2} g_\varepsilon,\ Q^- S_{t_1} g_{1-\varepsilon(1)} \rangle_\zeta \Big[1 \otimes 1 \otimes A(Q^- S_{t_1} g_{\varepsilon(1)}) V(t_2)\cdots V(t_n)\Big]\Big)$$

$$v \otimes W\Big(z\int_{S'/z^2}^{T'/z^2} S_u Q_+ f' du\Big)\Phi_F \otimes W\Big(z^2 \int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\Big)\Phi_F^{\zeta} \rangle.$$

The first two terms of (6.35) are both equal to

(6.36) $$z^2 \cdot O(1) \longrightarrow 0$$

and moreover the same arguments as those in the proof of Lemma (6.2) show that the third term of (6.35) is $z^2 O(1)$, therefore we end our proof.

We can obtain more from the proof of Lemma (6.2), in fact the proof shows how does get the limit (6.14). Now let us realize it.

By definition

(6.37) $$\mathrm{I}_z(b) = \sum_{n=1}^{\infty} \sum_{\varepsilon \in \{0,1\}}$$

$$\langle D_\varepsilon^+ u \otimes W\Big(z\int_{S/z^2}^{T/z^2} S_u Q_+ f du\Big)\Phi_F \otimes W\Big(z^2\int_{S/z^2}^{T/z^2} Q^- S_u f du\Big)\Phi_F^{\zeta},$$

$$\int_0^{s/z^2} dt_1 \cdots \int_0^{t_{n-1}} dt_n \int_{S/z^2}^{T/z^2} \langle S_u Q_+ f,\ S_{s/z^2} Q_+ g_\varepsilon \rangle du$$

$$\frac{1}{z}(-i)^n \Big[1 \otimes A(S_{s/z^2} Q_+ g_{1-\varepsilon}) \otimes 1,\ V(t_1) \cdots V(t_n)\Big]$$

$$v \otimes W\Big(z\int_{S'/z^2}^{T'/z^2} S_u Q_+ f' du\Big)\Phi_F \otimes W\Big(z^2 \int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\Big)\Phi_F^{\zeta} \rangle.$$

By Lemma (6.2), the fact that

(6.38) $$\int_{S/z^2}^{T/z^2} \langle S_u Q_+ f,\ S_{s/z^2} Q_+ g_\varepsilon \rangle du \longrightarrow \chi_{[S,T]}(s) \cdot (f | g_\varepsilon), \text{ a.e.}$$

and an argument similar to the one used in (6.24), ... ,(6.30), we know that the limit of $\mathrm{I}_z(b)$ is equal to limit of

(6.39)

$$\sum_{n=0}^{\infty} \sum_{\varepsilon, \varepsilon(1) \in \{0,1\}} \langle D_{\varepsilon(1)}^+ D_\varepsilon^+ u \otimes W\Big(z\int_{S/z^2}^{T/z^2} S_u Q_+ f du\Big)\Phi_F \otimes W\Big(z^2 \int_{S/z^2}^{T/z^2} Q^- S_u f du\Big)\Phi_F^{\zeta},$$

$$\chi_{[S,T]}(s) \cdot (f | g_\varepsilon) \int_{-s/z^2}^{0} dt' \langle Q_+ g_{1-\varepsilon},\ S_{t'} Q_+ g_{\varepsilon(1)} \rangle$$

$$(-i)^n \int_0^{s/z^2} dt_2 \cdots \int_0^{t_n} dt_{n+1} V(t_2) \cdots V(t_{n-1})$$

$$\times \int_{S'/z^2}^{T'/z^2} \langle S_{t'+s/z^2} Q_+ g_{1-\varepsilon(1)},\ S_u Q_+ f' \rangle du$$

$$v \otimes W\Big(z\int_{S'/z^2}^{T'/z^2} S_u Q_+ f' du\Big)\Phi_F \otimes W\Big(z^2 \int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\Big)\Phi_F^{\zeta} \rangle.$$

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

$$+\sum_{n=1}^{\infty}\sum_{\varepsilon,\varepsilon(1)\in\{0,1\}}\langle D^{+}_{\varepsilon(1)}D^{+}_{\varepsilon}u\otimes W\left(z\int_{S/z^2}^{T/z^2}S_uQ_+fdu\right)\Phi_F\otimes W\left(z^2\int_{S/z^2}^{T/z^2}Q^-S_ufdu\right)\Phi_F^{\zeta},$$

$$\chi_{[S,T]}(s)\cdot(f\mid g_\varepsilon)\int_0^{s/z^2}dt_2\cdots\int_0^{t_n}dt_{n+1}\int_{-s/z^2}^{0}dt'\,\langle Q_+g_{1-\varepsilon},\,S_{t'}Q_+g_{\varepsilon(1)}\rangle$$

$$(-i)^n\frac{1}{z}[1\otimes A(S_{t'+s/z^2}Q_+g_{1-\varepsilon(1)})\otimes 1,V(t_2)\cdots V(t_{n+1})]$$

$$v\otimes W\left(z\int_{S'/z^2}^{T'/z^2}S_uQ_+f'du\right)\Phi_F\otimes W\left(z^2\int_{S'/z^2}^{T'/z^2}Q^-S_uf'du\right)\Phi_F^{\zeta}\rangle.$$

Using Lemma (6.3) of [2], we know that the limit of (6.39) is equal to the limit of

(6.40)

$$\sum_{n=0}^{\infty}\sum_{\varepsilon,\varepsilon(1)\in\{0,1\}}\langle D^{+}_{\varepsilon(1)}D^{+}_{\varepsilon}u\otimes W\left(z\int_{S/z^2}^{T/z^2}S_uQ_+fdu\right)\Phi_F\otimes W\left(z^2\int_{S/z^2}^{T/z^2}Q^-S_ufdu\right)\Phi_F^{\zeta},$$

$$\chi_{[S,T]}(s)\cdot(f\mid g_\varepsilon)\int_{-\infty}^{0}dt'\,\langle g_{1-\varepsilon},\,S_{t'}g_{\varepsilon(1)}\rangle$$

$$(-i)^n\int_0^{s/z^2}dt_2\cdots\int_0^{t_n}dt_{n+1}V(t_2)\cdots V(t_{n+1})\chi_{[S',T']}(g_{1-\varepsilon(1)}\mid f')$$

$$v\otimes W\left(z\int_{S'/z^2}^{T'/z^2}S_uQ_+f'du\right)\Phi_F\otimes W\left(z^2\int_{S'/z^2}^{T'/z^2}Q^-S_uf'du\right)\Phi_F^{\zeta}\rangle+$$

$$+\sum_{n=1}^{\infty}\sum_{\varepsilon,\varepsilon(1)\in\{0,1\}}\langle D^{+}_{\varepsilon(1)}D^{+}_{\varepsilon}u\otimes W\left(z\int_{S/z^2}^{T/z^2}S_uQ_+fdu\right)\Phi_F\otimes W\left(z^2\int_{S/z^2}^{T/z^2}Q^-S_ufdu\right)\Phi_F^{\zeta},$$

$$\chi_{[S,T]}(s)\cdot(f\mid g_\varepsilon)\int_{-\infty}^{0}dt'\,\langle g_{1-\varepsilon},\,S_{t'}g_{\varepsilon(1)}\rangle$$

$$(-i)^n\frac{1}{z}\int_0^{s/z^2}dt_2\cdots\int_0^{t_n}dt_{n+1}[1\otimes A(S_{s/z^2}Q_+g_{1-\varepsilon(1)})\otimes 1,V(t_2)\cdots V(t_{n+1})]$$

$$v\otimes W\left(z\int_{S'/z^2}^{T'/z^2}S_uQ_+f'du\right)\Phi_F\otimes W\left(z^2\int_{S'/z^2}^{T'/z^2}Q^-S_uf'du\right)\Phi_F^{\zeta}\rangle$$

$$=\sum_{\varepsilon,\varepsilon(1)\in\{0,1\}}\langle D^{+}_{\varepsilon(1)}D^{+}_{\varepsilon}u\otimes W\left(z\int_{S/z^2}^{T/z^2}S_uQ_+fdu\right)\Phi_F\otimes W\left(z^2\int_{S/z^2}^{T/z^2}Q^-S_ufdu\right)\Phi_F^{\zeta},$$

$$\chi_{[S,T]}(s)\cdot(f\mid g_\varepsilon)\chi_{[S',T']}(g_{1-\varepsilon(1)}\mid f')(g_{1-\varepsilon}\mid g_{\varepsilon(1)})_-$$

$$U_{s/z^2}v\otimes W\left(z\int_{S'/z^2}^{T'/z^2}S_uQ_+f'du\right)\Phi_F\otimes W\left(z^2\int_{S'/z^2}^{T'/z^2}Q^-S_uf'du\right)\Phi_F^{\zeta}\rangle$$

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

$$+\sum_{\varepsilon\in\{0,1\}}\langle D^+_{1-\varepsilon}D^+_\varepsilon u\otimes W\left(z\int_{S/z^2}^{T/z^2}S_uQ_+fdu\right)\Phi_F\otimes W\left(z^2\int_{S/z^2}^{T/z^2}Q^-S_ufdu\right)\Phi'_F,$$

$$\frac{1}{z}\chi_{[S,T]}(s)\cdot(f\,|\,g_\varepsilon)(g_{1-\varepsilon}\,|\,g_{1-\varepsilon})_-[1\otimes A(S_{s/z^2}Q_+g_{1-\varepsilon(1)})\otimes 1,\ U_{s/z^2}]$$

$$v\otimes W\left(z\int_{S'/z^2}^{T'/z^2}S_uQ_+f'du\right)\Phi_F\otimes W\left(z^2\int_{S'/z^2}^{T'/z^2}Q^-S_uf'du\right)\Phi'_F\rangle.$$

As $z\to 0$, the first term of the right hand side of (6.40) goes to

(6.41)

$$\sum_{\varepsilon,\varepsilon(1)\in\{0,1\}}\langle D^+_{\varepsilon(1)}D^+_\varepsilon u,\ G(s)\rangle\,\chi_{[S,T]}(s)\cdot(f\,|\,g_\varepsilon)\chi_{[S',T']}(s)(g_{1-\varepsilon(1)}\,|\,f')(g_{1-\varepsilon}\,|\,g_{\varepsilon(1)})_-$$

and the second term of the right hand side of (6.40) is of the same type as $I_z(b)$. The only difference is the new factor $(g_{1-\varepsilon}\,|\,g_{\varepsilon(1)})_-$. So by repeating the same arguments about $I_z(b)$ on the second term of the right hand side of (6.40), we obtain the following result:

LEMMA (6.4).

(6.42)
$$\lim_{z\to 0}I_z(b)=\sum_{\varepsilon\in\{0,1\}}\sum_{n=0}^{\infty}\sum_{\sigma\in\{0,1\}^n}\langle D^+_\varepsilon D^+_{\sigma(n)}\cdot\cdots\cdot D^+_{\sigma(1)}D^+_\varepsilon u,\ G(s)\rangle$$

$$(g_{1-\varepsilon}\,|\,g_{\sigma(1)})_-\cdot(g_{1-\sigma(1)}\,|\,g_{\sigma(2)})_-\cdot\cdots\cdot(g_{1-\sigma(n)}\,|\,g_\varepsilon)_-$$

$$\chi_{[S,T]}(s)\cdot(f\,|\,g_\varepsilon)\chi_{[S',T']}(s)\cdot(g_{1-\varepsilon}\,|\,f')$$

$$+\sum_{\varepsilon\in\{0,1\}}\sum_{n=0}^{\infty}\sum_{\sigma\in\{0,1\}^n}\langle D^+_{1-\varepsilon}D^+_{\sigma(n)}\cdot\cdots\cdot D^+_{\sigma(1)}D^+_\varepsilon u,\ G(s)\rangle$$

$$(g_{1-\varepsilon}\,|\,g_{\sigma(1)})_-\cdot(g_{1-\sigma(1)}\,|\,g_{\sigma(2)})_-\cdot\cdots\cdot(g_{1-\sigma(n)}\,|\,g_{1-\varepsilon})_-$$

$$\chi_{[S,T]}(s)\cdot(f\,|\,g_\varepsilon)\chi_{[S',T']}(s)\cdot(g_\varepsilon\,|\,f').$$

Proof. By the above arguments, one has

(6.43)
$$\lim_{z\to 0}I_z(b)=\sum_{\varepsilon,\varepsilon(1)\in\{0,1\}}\langle D^+_{\varepsilon(1)}D^+_\varepsilon u,\ G(s)\rangle$$

$$\chi_{[S,T]}(s)\cdot(f|g_\varepsilon)\chi_{[S',T']}(g_{1-\varepsilon(1)}\,|\,f')(g_{1-\varepsilon}\,|\,g_{\varepsilon(1)})_-+$$

$$+\lim_{z\to 0}\sum_{\varepsilon,\varepsilon(1)\in\{0,1\}}\langle D^+_{\varepsilon(1)}D^+_\varepsilon u\otimes W\left(z\int_{S/z^2}^{T/z^2}S_uQ_+fdu\right)\Phi_F\otimes$$

$$W\left(z^2\int_{S/z^2}^{T/z^2}Q^-S_ufdu\right)\Phi'_F,$$

$$\frac{1}{z}\chi_{[S,T]}(s)\cdot(f\,|\,g_\varepsilon)(g_{1-\varepsilon}\,|\,g_{\varepsilon(1)})_-\,[1\otimes A(S_{s/z^2}Q_+g_{1-\varepsilon(1)})\otimes 1,\;U_{s/z^2}]$$

$$v\otimes W\Big(z\int_{S'/z^2}^{T'/z^2}S_uQ_+f'du\Big)\Phi_F\otimes W\Big(z^2\int_{S'/z^2}^{T'/z^2}Q^-S_uf'du\Big)\Phi_F^\zeta\rangle$$

$$=\sum_{\varepsilon,\varepsilon(1)\in\{0,1\}}\langle D_{\varepsilon(1)}^+D_\varepsilon^+u,\,G(s)\rangle\,\chi_{[S,T]}(s)\cdot(f\,|\,g_\varepsilon)(g_{1-\varepsilon}\,|\,g_{\varepsilon(1)})_-\chi_{[S',T']}(s)\,(g_{1-\varepsilon(1)}\,|\,f')\;+$$

$$+\sum_{\varepsilon,\varepsilon(1),\varepsilon(2)\in\{0,1\}}\langle D_{\varepsilon(2)}^+D_{\varepsilon(1)}^+D_\varepsilon^+u,\,G(s)\rangle\,(g_{1-\varepsilon}\,|\,g_{\varepsilon(1)})_-(g_{1-\varepsilon(1)}\,|\,g_{\varepsilon(2)})_-$$

$$\chi_{[S,T]}(s)\cdot(f\,|\,g_\varepsilon)\chi_{[S',T']}(s)\,(g_{1-\varepsilon(2)}\,|\,f')\;+$$

$$+\lim_{z\to 0}\sum_{\varepsilon,\varepsilon(1),\varepsilon(2)\in\{0,1\}}$$

$$\langle D_{\varepsilon(2)}^+D_{\varepsilon(1)}^+D_\varepsilon^+u\otimes W\Big(z\int_{S/z^2}^{T/z^2}S_uQ_+fdu\Big)\Phi_F\otimes W\Big(z^2\int_{S/z^2}^{T/z^2}Q^-S_ufdu\Big)\Phi_F^\zeta,$$

$$\chi_{[S,T]}(s)\cdot(f\,|\,g_\varepsilon)(g_{1-\varepsilon}\,|\,g_{\varepsilon(1)})_-\cdot(g_{1-\varepsilon(1)}\,|\,g_{\varepsilon(2)})_-$$

$$\frac{1}{z}\,[1\otimes A(S_{s/z^2}Q_+g_{1-\varepsilon(2)})\otimes 1,\;U_{\varepsilon/z^2}]$$

$$v\otimes W\Big(z\int_{S'/z^2}^{T'/z^2}S_uQ_+f'du\Big)\Phi_F\otimes W\Big(z^2\int_{S'/z^2}^{T'/z^2}Q^-S_uf'du\Big)\Phi_F^\zeta\rangle.$$

Iterating, we obtain that

$$\lim_{z\to 0}\mathrm{I}_z(b)=\sum_{\varepsilon,\varepsilon(1)\in\{0,1\}}\langle D_{\varepsilon(1)}^+D_\varepsilon^+u,\,G(s)\rangle \tag{6.43}$$

$$\chi_{[S,T]}(s)\cdot(f\,|\,g_\varepsilon)(g_{1-\varepsilon}\,|\,g_{\varepsilon(1)})_-\chi_{[S',T']}(s)\,(g_{1-\varepsilon(1)}\,|\,f')\;+$$

$$+\sum_{\varepsilon,\varepsilon(1),\varepsilon(2)\in\{0,1\}}\langle D_{\varepsilon(2)}^+D_{\varepsilon(1)}^+D_\varepsilon^+u,\,G(s)\rangle(g_{1-\varepsilon}\,|\,g_{\varepsilon(1)})_-(g_{1-\varepsilon(1)}\,|\,g_{\varepsilon(2)})_-$$

$$\chi_{[S,T]}(s)\cdot(f\,|\,g_\varepsilon)\chi_{[S',T']}(s)\,(g_{1-\varepsilon(2)})\,|\,f')\;+$$

$$+\sum_{\varepsilon,\varepsilon(1),\varepsilon(2),\varepsilon(3)\in\{0,1\}}\langle D_{\varepsilon(3)}^+D_{\varepsilon(2)}^+D_{\varepsilon(1)}^+D_\varepsilon^+u,\,G(s)\rangle$$

$$(g_{1-\varepsilon}\,|\,g_{\varepsilon(1)})_-\cdot(g_{1-\varepsilon(1)}\,|\,g_{\varepsilon(2)})_-\cdot(g_{1-\varepsilon(2)}\,|\,g_{\varepsilon(3)})_-$$

$$\chi_{[S,T]}(s)\cdot(f\,|\,g_\varepsilon)\cdot\chi_{[S',T']}(s)\,(g_{1-\varepsilon(3)})\,|\,f')\;+$$

$$+\cdots\cdots+$$

$$=\sum_{n=1}^{\infty}\sum_{\varepsilon,\varepsilon(1),\cdots,\varepsilon(n)\in\{0,1\}}\langle D_{\varepsilon(n)}^+\cdot\cdots\cdot D_{\varepsilon(1)}^+D_\varepsilon^+u,\,G(s)\rangle$$

$$(g_{1-\varepsilon}\,|\,g_{\varepsilon(1)})_-\cdot(g_{1-\varepsilon(1)}\,|\,g_{\varepsilon(2)})_-\cdot\cdots\cdot(g_{1-\varepsilon(n-1)}\,|\,g_{\varepsilon(n)})_-$$

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

$$\chi_{[S,T]}(s)\cdot(f\,|\,g_\varepsilon)\cdot\chi_{[S',T']}(s)(g_{1-\varepsilon(n)}\,|\,f')$$

where, $\varepsilon(0):=\varepsilon$.

We split the right hand side of (6.43) into two terms according to

(6.44a) $$\varepsilon(n)=\varepsilon$$

or

(6.44b) $$\varepsilon(n)=1-\varepsilon$$

then the right hand side of (6.43) is equal to

(6.45)
$$\sum_{n=1}^{\infty}\sum_{\varepsilon,\varepsilon(1),\cdots,\varepsilon(n-1)\in\{0,1\}}\langle D_\varepsilon^+D_{\varepsilon(n-1)}^+\cdot\cdots\cdot D_{\varepsilon(1)}^+D_\varepsilon^+u,\ G(s)\rangle$$
$$(g_{1-\varepsilon}\,|\,g_{\varepsilon(1)})_-\cdot(g_{1-\varepsilon(1)}\,|\,g_{\varepsilon(2)})_-\cdot\cdots\cdot(g_{1-\varepsilon(n-1)}\,|\,g_\varepsilon)_-$$
$$\chi_{[S,T]}(s)\cdot(f\,|\,g_\varepsilon)\chi_{[S',T']}(s)\cdot(g_{1-\varepsilon}\,|\,f')$$
$$+\sum_{n=1}^{\infty}\sum_{\varepsilon,\varepsilon(1),\cdots,\varepsilon(n-1)\in\{0,1\}}\langle D_{1-\varepsilon}^+D_{\varepsilon(n-1)}^+\cdot\cdots\cdot D_{\varepsilon(1)}^+D_\varepsilon^+u,\ G(s)\rangle$$
$$(g_{1-\varepsilon}\,|\,g_{\varepsilon(1)})_-\cdot(g_{1-\varepsilon(1)}\,|\,g_{\varepsilon(2)})_-\cdot\cdots\cdot(g_{1-\varepsilon(n-1)}\,|\,g_{1-\varepsilon})_-$$
$$\chi_{[S,T]}(s)\cdot(f\,|\,g_\varepsilon)\cdot\chi_{[S',T']}(s)\cdot(g_{1-\varepsilon}\,|\,f')$$
$$=\sum_{\varepsilon\in\{1,0\}}\sum_{n=0}^{\infty}\sum_{\sigma\in\{0,1\}^n}\langle D_\varepsilon^+D_{\sigma(n)}^+\cdot\cdots\cdot D_{\sigma(1)}^+D_\varepsilon^+u,\ G(s)\rangle$$
$$(g_{1-\varepsilon}\,|\,g_{\varepsilon(1)})_-\cdot(g_{1-\sigma(1)}\,|\,g_{\sigma(2)})_-\cdot\cdots\cdot(g_{1-\sigma(n)}\,|\,g_\varepsilon)_-$$
$$\chi_{[S,T]}(s)\cdot(f\,|\,g_\varepsilon)\cdot\chi_{[S',T']}(s)\cdot(g_{1-\varepsilon}\,|\,f')$$
$$+\sum_{\varepsilon\in\{1,0\}}\sum_{n=1}^{\infty}\sum_{\sigma\in\{0,1\}^n}\langle D_{1-\varepsilon}^+D_{\sigma(n)}^+\cdot\cdots\cdot D_{\sigma(1)}^+D_\varepsilon^+u,\ G(s)\rangle$$
$$(g_{1-\varepsilon}\,|\,g_{\sigma(1)})_-\cdot(g_{1-\sigma(1)}\,|\,g_{\sigma(2)})_-\cdot\cdots\cdot(g_{1-\sigma(n)}\,|\,g_{1-\varepsilon})_-$$
$$(\chi_{[S,T]}(s)\cdot(f\,|\,g_\varepsilon)\cdot\chi_{[S',T']}(s)\cdot(g_\varepsilon\,|\,f')$$

where, $\sigma(0):=\varepsilon$. This finish the proof of (6.42).

LEMMA (6.5).

(6.46) $$\lim_{z\to0}\frac{1}{z}\mathrm{II}_z=0.$$

Proof.

(6.47)
$$\frac{1}{z}\mathrm{II}_z=\sum_{\varepsilon\in\{0,1\}}\frac{1}{z}\langle D_\varepsilon^+u\otimes W\left(z\int_{S/z^2}^{T/z^2}S_uQ_+fdu\right)\Phi_F\otimes W\left(z^2\int_{S/z^2}^{T/z^2}Q^-S_ufdu\right)\Phi_F',$$

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

$$1 \otimes A^{+}(S_{s/z^2}Q_{+} g_{\varepsilon}) \otimes A^{+}(Q^{-} S_{s/z^2}g_{1-\varepsilon})\, U_{s/z^2}$$

$$v \otimes W\Big(z\int_{S'/z^2}^{T'/z^2} S_u Q_{+} f' du\Big)\Phi_F \otimes W\Big(z^2 \int_{S'/z^2}^{T'/z^2} Q^{-} S_u f' du\Big)\Phi_F^{\zeta} \rangle$$

$$= \langle D_{\varepsilon}^{+} u \otimes W\Big(z\int_{S/z^2}^{T/z^2} S_u Q_{+} f du\Big)\Phi_F \otimes W\Big(z^2 \int_{S/z^2}^{T/z^2} Q^{-} S_u f du\Big)\Phi_F^{\zeta},$$

$$\sum_{\varepsilon\in\{0,1\}} z^2 \int_{S/z^2}^{T/z^2} \langle S_u Q_{+} f,\, S_{s/z^2}Q_{+} g_{\varepsilon} \rangle\, du \cdot$$

$$\int_{S/z^2}^{T/z^2} \langle Q^{-} S_u f,\, Q^{-} S_{s/z^2} g_{1-\varepsilon} \rangle_{\zeta}\, du \cdot U_{s/z^2}$$

$$v \otimes W\Big(z\int_{S'/z^2}^{T'/z^2} S_u Q_{+} f' du\Big)\Phi_F \otimes W\Big(z^2 \int_{S'/z^2}^{T'/z^2} Q^{-} S_u f' du\Big)\Phi_F^{\zeta} \rangle$$

$$= z^2 \cdot O(1) \longrightarrow 0 \text{ a.e.}$$

LEMMA (6.6).

$$\lim_{z\to 0} \mathrm{IV}_z = \sum_{\varepsilon\in\{0,1\}} \langle D_{\varepsilon}^{+} u,\, G(s) \rangle \langle g_{1-\varepsilon},\, e^{-\frac{1}{2}\beta H} g_{\varepsilon} \rangle. \tag{6.48}$$

Proof. It is enough to prove that the quantity

(6.49)

$$\mathrm{IV}(z, s) := \Big| \langle D_{\varepsilon}^{+} u \otimes W\Big(z\int_{S/z^2}^{T/z^2} S_u Q_{+} f du\Big)\Phi_F \otimes W\Big(z^2 \int_{S/z^2}^{T/z^2} S_u Q^{-} f du\Big)\Phi_F^{\zeta},$$

$$1 \otimes 1 \otimes A^{+}(S_{s/z^2}Q^{-} g_{1-\varepsilon}) A(S_{s/z^2}Q^{-} g_{\varepsilon})\, U_{s/z^2}$$

$$v \otimes W\Big(z\int_{S'/z^2}^{T'/z^2} S_u Q_{+} f' du\Big)\Phi_F \otimes W\Big(z^2 \int_{S'/z^2}^{T'/z^2} S_u Q^{-} f' du\Big)\Phi_F^{\zeta} \rangle \Big|$$

tends to zero (a.e.) as $z \to 0$.

Since the integral

$$\int_0^t ds\, \mathrm{IV}(z, s) \tag{6.50}$$

is a term of type II in the sense of Section 3 ($k'' = 0$), one gets that

$$\lim_{z\to 0} \int_0^t ds\, \mathrm{IV}(z, s) = 0. \tag{6.51}$$

By the uniform estimate, we obtain

$$0 = \int_0^t ds \lim_{z\to 0} \mathrm{IV}(z, s). \tag{6.52}$$

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

This result and the positivity of $\mathrm{IV}(z,s)$ concludes the proof.

Our next step is to investigate the limit

(6.53) $$\lim_{z\to 0}\frac{1}{z}\mathrm{III}_z.$$

By definition

(6.54)
$$\frac{1}{z}\mathrm{III}_z=\frac{1}{z}\sum_{\varepsilon\in\{0,1\}}\langle D_\varepsilon^+u\otimes W\left(z\int_{S/z^2}^{T/z^2}S_uQ_+fdu\right)\Phi_F\otimes W\left(z^2\int_{S/z^2}^{T/z^2}Q^-S_ufdu\right)\Phi_F^\zeta,$$
$$1\otimes A(S_{s/z^2}Q_+g_{1-\varepsilon})\otimes A(Q^-S_{s/z^2}g_\varepsilon)\cdot U_{s/z^2}$$
$$v\otimes W\left(z\int_{S'/z^2}^{T'/z^2}S_uQ_+f'du\right)\Phi_F\otimes W\left(z^2\int_{S'/z^2}^{T'/z^2}Q^-S_uf'du\right)\Phi_F^\zeta\rangle$$

where, one can write the product

$$1\otimes A(S_{s/z^2}Q_+g_{1-\varepsilon})\otimes A(Q^-S_{s/z^2}g_\varepsilon)\cdot U_{s/z^2}$$

to a sum of four terms:

(6.55)
$$U_{s/z^2}1\otimes A(S_{s/z^2}Q_+g_{1-\varepsilon})\otimes A(Q^-S_{s/z^2}g_\varepsilon)+$$
$$+[1\otimes A(S_{s/z^2}Q_+g_{1-\varepsilon})\otimes 1,\ U_{s/z^2}]1\otimes 1\otimes A(Q^-S_{s/z^2}g_\varepsilon)+$$
$$+[1\otimes 1\otimes A(Q^-S_{s/z^2}g_\varepsilon),\ U_{s/z^2}]1\otimes A(S_{s/z^2}Q_+g_{1-\varepsilon})\otimes 1+$$
$$+[1\otimes A(S_{s/z^2}Q_+g_{1-\varepsilon})\otimes A(Q^-S_{s/z^2}g_\varepsilon)\cdot U_{s/z^2}].$$

Respectively, $\frac{1}{z}\mathrm{III}_z$ can be written as a sum of four terms, namely

(6.56) $$\mathrm{III}_z(\mathrm{a})+\mathrm{III}_z(\mathrm{b})+\mathrm{III}_z(\mathrm{c})+\mathrm{III}_z(\mathrm{b})$$

where,

(6.57a)
$$\mathrm{III}_z(\mathrm{a}):=\frac{1}{z}\sum_{\varepsilon\in\{0,1\}}\langle D_\varepsilon^+u\otimes W\left(z\int_{S/z^2}^{T/z^2}S_uQ_+fdu\right)\Phi_F\otimes$$
$$W\left(z^2\int_{S/z^2}^{T/z^2}Q^-S_ufdu\right)\Phi_F^\zeta,$$
$$U_{s/z^2}1\otimes A(S_{s/z^2}Q_+g_{1-\varepsilon})\otimes A(Q^-S_{s/z^2}g_\varepsilon)$$
$$v\otimes W\left(z\int_{S'/z^2}^{T'/z^2}S_uQ_+f'du\right)\Phi_F\otimes W\left(z^2\int_{S'/z^2}^{T'/z^2}Q^-S_uf'du\right)\Phi_F^\zeta\rangle$$

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

(6.57b)

$$\mathrm{III}_z(\mathrm{b}) := \frac{1}{z}\sum_{\varepsilon\in\{0,1\}} \langle D_\varepsilon^+ u \otimes W\Big(z\int_{S/z^2}^{T/z^2} S_u Q_+ f du\Big)\Phi_F \otimes W\Big(z^2\int_{S/z^2}^{T/z^2} Q^- S_u f du\Big)\Phi_F^\zeta,$$

$$[1\otimes A(S_{s/z^2}Q_+ g_{1-\varepsilon})\otimes 1,\ U_{s/z^2}]1\otimes 1\ \otimes A(Q^- S_{s/z^2} g_\varepsilon)$$

$$v\otimes W\Big(z\int_{S'/z^2}^{T'/z^2} S_u Q_+ f' du\Big)\Phi_F \otimes W\Big(z^2\int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\Big)\Phi_F^\zeta\rangle$$

(6.57c)

$$\mathrm{III}_z(\mathrm{c}) := \frac{1}{z}\sum_{\varepsilon\in\{0,1\}} \langle D_\varepsilon^+ u \otimes W\Big(z\int_{S/z^2}^{T/z^2} S_u Q_+ f du\Big)\Phi_F \otimes W\Big(z^2\int_{S/z^2}^{T/z^2} Q^- S_u f du\Big)\Phi_F^\zeta,$$

$$[1\otimes 1\otimes A(Q^- S_{s/z^2} g_{1-\varepsilon}),\ U_{s/z^2}]1\otimes A(S_{s/z^2}Q_+ g_{1-\varepsilon})\otimes 1$$

$$v\otimes W\Big(z\int_{S'/z^2}^{T'/z^2} S_u Q_+ f' du\Big)\Phi_F \otimes W\Big(z^2\int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\Big)\Phi_F^\zeta\rangle$$

and

(6.57d)
$$\mathrm{III}_z(\mathrm{d}) := \frac{1}{z}\sum_{\varepsilon\in\{0,1\}} \langle D_\varepsilon^+ u \otimes W\Big(z\int_{S/z^2}^{T/z^2} S_u Q_+ f du\Big)\Phi_F \otimes$$

$$W\Big(z^2\int_{S/z^2}^{T/z^2} Q^- S_u f du\Big)\Phi_F^\zeta,$$

$$[1\otimes A(S_{s/z^2}Q_+ g_{1-\varepsilon})\otimes A(Q^- S_{s/z^2} g_\varepsilon)\cdot U_{s/z^2}]$$

$$v\otimes W\Big(z\int_{S'/z^2}^{T'/z^2} S_u Q_+ f' du\Big)\Phi_F \otimes W\Big(z^2\int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\Big)\Phi_F^\zeta\rangle.$$

For the four terms we have that

LEMMA (6.7).

(6.58)
$$\mathrm{III}_z(a) = z^2\cdot O(1) \longrightarrow 0.$$

Proof. In the right hand side of (6.57a), applying the two annihilation operators on the coherent vectors and we obtain that

(6.59)

$$\mathrm{III}_z(\mathrm{a}) = \frac{1}{z}\sum_{\varepsilon\in\{0,1\}} \langle D_\varepsilon^+ u \otimes W\Big(z\int_{S/z^2}^{T/z^2} S_u Q_+ f du\Big)\Phi_F \otimes W\Big(z^2\int_{S/z^2}^{T/z^2} Q^- S_u f du\Big)\Phi_F^\zeta,$$

$$U_{s/z^2}\, z\int_{S'/z^2}^{T'/z^2}\langle S_{s/z^2}Q_+ g_{1-\varepsilon},\ SuQ_+ f'\rangle\, du\ \cdot$$

$$z^2\int_{S'/z^2}^{T'/z^2}\langle Q^- S_{s/z^2} g_\varepsilon,\ Q^- Suf'\rangle_\zeta\, du$$

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

$$v \otimes W\left(z\int_{S'/z^2}^{T'/z^2} S_u Q_+ f' du\right)\Phi_F \otimes W\left(z^2 \int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\right)\Phi_F^\zeta \rangle$$

$$= z^2 \cdot \sum_{\varepsilon\in\{0,1\}} \int_{S'/z^2}^{T'/z^2} \langle S_{s/z^2} Q_+ g_{1-\varepsilon}, S_u Q_+ f' \rangle du \cdot$$

$$\cdot \int_{S'/z^2}^{T'/z^2} \langle Q^- S_{s/z^2} g_\varepsilon, Q^- Suf' \rangle_\zeta du$$

$$\langle D_\varepsilon^+ u, G_z(s) \rangle .$$

This gives (6.58).

LEMMA (6.8).

$$\text{III}_z(\text{b}) = z^2 \cdot O(1) \longrightarrow 0. \tag{6.60}$$

Proof. In the right hand side of (6.57b), apply the annihilation operator $A(Q^- S_{s/z^2} g_\varepsilon)$ on the coherent vector

$$W\left(z^2 \int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\right)\Phi_F^\zeta$$

one finds that

$$\text{III}_z(\text{b}) = \sum_{\varepsilon\in\{0,1\}} z^2 \int_{S/z^2}^{T/z^2} \langle Q^- S_{s/z^2} g_\varepsilon, Q^- S_u f' \rangle_\zeta du \tag{6.61}$$

$$\langle D_\varepsilon^+ u \otimes W\left(z\int_{S/z^2}^{T/z^2} S_u Q_+ f du\right)\Phi_F \otimes W\left(z^2 \int_{S/z^2}^{T/z^2} Q^- S_u f du\right)\Phi_F^\zeta,$$

$$\frac{1}{z}[1 \otimes A(S_{s/z^2} Q_+ g_{1-\varepsilon}) \otimes 1, U_{s/z^2}]$$

$$v \otimes W\left(z\int_{S'/z^2}^{T'/z^2} S_u Q_+ f' du\right)\Phi_F \otimes W\left(z^2 \int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\right)\Phi_F^\zeta \rangle .$$

This formula and Lemma (6.2) imply (6.60).

LEMMA (6.9).

$$\text{III}_z(\text{c}) \longrightarrow 0. \tag{6.62}$$

Proof. This Lemma is a direct result of Lemma (6.3).
Our next goal is to obtain the limit of $\text{III}_z(\text{d})$

LEMMA (6.10). *For g_0, g_1, t satisfying the condition (*), the limit*

$$\lim_{z\to 0} \int_0^t \text{III}_z(\text{d}) ds \tag{6.63}$$

exists and is equal to

$$\sum_{\varepsilon\in\{0,1\}}\int_0^t dt_0 \langle u, D_3(\varepsilon)G(t_0)\rangle \tag{6.64}$$

where,

$$D_3(\varepsilon) := \sum_{k=1}^{\infty}\sum_{\sigma\in\{0,1\}^k} D_\varepsilon D_{\sigma(k)}\cdots D_{\sigma(1)}\int_{-\infty}^0 ds_1\cdots\int_{-\infty}^0 ds_k \langle g_{1-\varepsilon}, S_{s_1}g_{\sigma(1)}\rangle \tag{6.65}$$

$$\langle g_{1-\sigma(1)}, S_{s_2}g_{\sigma(2)}\rangle\cdots\langle g_{1-\sigma(k-1)}, S_{s_k}g_{1-\sigma(k)}\rangle\cdot\overline{\langle g_\varepsilon, S_{s_k+\cdots+s_1}g_{\sigma(k)}\rangle}.$$

Proof. With change of variable

$$s/z^2 = t_0 \tag{6.66}$$

one gets

$$\int_0^t \mathrm{III}_z(\mathrm{d})ds = \frac{1}{z}\sum_{\varepsilon\in\{0,1\}} \tag{6.67}$$

$$\langle D_\varepsilon^+ u\otimes W\Big(z\int_{S/z^2}^{T/z^2} S_uQ_+fdu\Big)\Phi_F\otimes W\Big(z^2\int_{S/z^2}^{T/z^2} Q^- S_u fdu\Big)\Phi_F^\zeta,$$

$$\int_0^t ds\Big[1\otimes A(S_{s/z^2}Q_+g_{1-\varepsilon})\otimes A(Q^- S_{s/z^2}g_\varepsilon), U_{s/z^2}\Big]$$

$$v\otimes W\Big(z\int_{S'/z^2}^{T'/z^2} S_uQ_+f'du\Big)\Phi_F\otimes W\Big(z^2\int_{S'/z^2}^{T'/z^2} Q^- S_u f'du\Big)\Phi_F^\zeta\rangle$$

$$=\sum_{\varepsilon\in\{0,1\}}\langle D_\varepsilon^+ u\otimes W\Big(z\int_{S/Z^2}^{T/z^2} S_uQ_+fdu\Big)\Phi_F\otimes W\Big(z_2\int_{S/z^2}^{T/z^2} Q^- S_u fdu\Big)\Phi_F^\zeta,$$

$$\int_0^{t/z^2} dt_0 z\Big[1\otimes A(S_{t_0}Q_+g_{1-\varepsilon})\otimes A(Q^- S_{t_0}g_\varepsilon), U_{t_0}\Big]$$

$$v\otimes W\Big(z\int_{S'/z^2}^{T'/z^2} S_uQ_+f'du\Big)\Phi_F\otimes W\Big(z^2\int_{S'/z^2}^{T'/z^2} Q^- S_u f'du\Big)\Phi_F^\zeta\rangle$$

$$=\sum_{\varepsilon\in\{0,1\}}\sum_{n=0}^{\infty}\langle D_\varepsilon^+ u\otimes W\Big(z\int_{S/Z^2}^{T/z^2} S_uQ_+fdu\Big)\Phi_F\otimes W\Big(z_2\int_{S/z^2}^{T/z^2} Q^- S_u fdu\Big)\Phi_F^\zeta,$$

$$(-i)^n z\int_0^{t/z^2} dt_0\int_0^{t_0} dt_1\cdots\int_0^{t_{n-1}} dt_n$$

$$\times\Big[1\otimes A(S_{t_0}Q_+g_{1-\varepsilon})\otimes A(Q^- S_{t_0}g_\varepsilon), V(t_1)\cdots V(t_n)\Big]$$

$$v\otimes W\Big(z\int_{S'/z^2}^{T'/z^2} S_uQ_+f'du\Big)\Phi_F\otimes W\Big(z^2\int_{S'/z^2}^{T'/z^2} Q^- S_u f'du\Big)\Phi_F^\zeta\rangle.$$

Thus by the uniform estimate, we can exchange the order of the limit of $z\rightarrow 0$ and the sum over $n\in\mathbf{N}$.

Notice that for any operators A, B, X,

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

(6.68) $$AB = BA \Longrightarrow [AB, X] = [A, [B, X]]$$

therefore

(6.69) $$\left[1 \otimes A(S_{t_0}Q_+ g_{1-\varepsilon}) \otimes A[Q^- S_{t_0}g_\varepsilon),\ V(t_1)\cdots V(t_n)\right]$$

$$= \left[1 \otimes A(S_{t_0}Q_+ g_{1-\varepsilon}) \otimes 1, [1 \otimes 1 \otimes A[Q^- S_{t_0}g_\varepsilon),\ V(t_1)\cdots V(t_n)]\right]$$

$$= \sum_{k=1}^{n} \left[1 \otimes A(S_{t_0}Q_+ g_{1-\varepsilon}) \otimes 1,\ V(t_1)\cdots V(t_{k-1})\right.$$

$$\left.[1 \otimes 1 \otimes A(Q^- S_{t_0}g_\varepsilon),\ V(t_k)]V(t_{k+1})\cdots V(t_n)\right]$$

$$= \sum_{k=1}^{n} \sum_{\varepsilon(k)\in\{0,1\}} \left[1 \otimes A(S_{t_0}Q_+ g_{1-\varepsilon}) \otimes 1,\ V(t_1)\cdots V(t_{k-1})\right.$$

$$[1 \otimes 1 \otimes A(Q^- S_{t_0}g_\varepsilon), i\Big(zA^+(S_{t_k}Q_+ g_{1-\varepsilon(k)}) \otimes A^+(Q^- S_{t_k}g_{\varepsilon(k)})$$

$$\left. + z^2 1 \otimes 1 \otimes A^+(Q^- S_{t_k}g_{1-\varepsilon(k)})A(Q^- S_{t_k}g_{\varepsilon(k)})\Big)]V(t_{k+1})\cdots V(t_n)\right].$$

Respectively, the right hand side of (6.67) becomes

(6.70) $$\sum_{\varepsilon\in\{0,1\}} \sum_{n=0}^{\infty} \langle D_\varepsilon^+ u \otimes W\Big(z\int_{S/z^2}^{T/z^2} S_u Q_+ f du\Big)\Phi_F \otimes W\Big(z_2\int_{S/z^2}^{T/z^2} Q^- S_u f du\Big)\Phi_F',$$

$$(-i)^{n-1} z\int_0^{t/z^2} dt_0 \int_0^{t_0} dt_1 \cdots \int_0^{t_{n-1}} dt_n \sum_{k=1}^{n} \sum_{\varepsilon(k)\in\{0,1\}} \langle Q^- S_{t_0}g_\varepsilon,\ Q^- S_{t_k}g_{\varepsilon(h)} \rangle_\iota$$

$$\left[1 \otimes A(S_{t_0}Q_+ g_{1-\varepsilon}) \otimes 1,\ V(t_1)\cdots V(t_{k-1})z \cdot 1 \otimes A^+(S_{t_k}Q_+ g_{1-\varepsilon(k)}) \otimes 1\right]$$

$$V(t_{k-1})\cdots V(t_n)v \otimes W\Big(z\int_{S'/z^2}^{T'/z^2} S_u Q_+ f' du\Big)\Phi_F \otimes W\Big(z^2\int_{S'/z}^{T'/z^2} Q^- S_u f' du\Big)\Phi_F' \rangle$$

$$+ \sum_{\varepsilon\in\{0,1\}} \sum_{n=0}^{\infty} \langle D_\varepsilon^+ u \otimes W\Big(z\int_{S/z^2}^{T/z^2} S_u Q_+ f du\Big)\Phi_F \otimes W\Big(z_2\int_{S/z^2}^{T/z^2} Q^- S_u f du\Big)\Phi_F',$$

$$(-i)^{n-1} z\int_0^{t/z^2} dt_0 \int_0^{t_0} dt_1 \cdots \int_0^{t_{n-1}} dt_n$$

$$\sum_{k=1}^{n} \sum_{\varepsilon(k)\in\{0,1\}} \langle Q^- S_{t_0}g_\varepsilon, Q^- S_{t_k}g_{\varepsilon(h)} \rangle_\iota V(t_1)\cdots V(t_{k-1})z \cdot 1 \otimes$$

$$\otimes A^+(S_{t_k}Q_+ g_{1-\varepsilon(k)}) \otimes 1$$

$$\left[1 \otimes A(S_{t_0}Q_+ g_{1-\varepsilon}) \otimes 1,\ V(t_{k-1})\cdots V(t_n)\right]$$

$$v \otimes W\Big(z\int_{S'/z^2}^{T'/z^2} S_u Q_+ f' du\Big)\Phi_F \otimes W\Big(z^2\int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\Big)\Phi_F' \rangle$$

$$+\sum_{\varepsilon\in\{0,1\}}\sum_{n=0}^{\infty}\langle D_\varepsilon^+ u\otimes W\left(z\int_{S/z^2}^{T/z^2} S_u Q_+ f du\right)\Phi_F\otimes W\left(z^2\int_{S/z^2}^{T/z^2} Q^- S_u f du\right)\Phi_F',$$

$$(-i)^{n-1}z\int_0^{t/z^2}dt_0\int_0^{t_0}dt_1\cdots\int_0^{t_{n-1}}dt_n\sum_{k=1}^{n}\sum_{\varepsilon(k)\in\{0.1\}}\Big[1\otimes A(S_{t_0}Q_+ g_{1-\varepsilon})\otimes 1,$$

$$V(t_1)\cdots V(t_{k-1})$$

$$[1\otimes 1\otimes A(Q^- S_{t_0}g_\varepsilon),\ z^2 1\otimes 1\otimes A^+(Q^- S_{t_k}g_{1-\varepsilon(k)})A(Q^- S_{t_k}g_{\varepsilon(k)})]$$

$$\times\ V(t_{k+1})\cdots V(t_n)\Big]$$

$$v\otimes W\left(z\int_{S'/z^2}^{T'/z^2} S_u Q_+ f' du\right)\Phi_F\otimes W\left(z^2\int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\right)\Phi_F'\rangle.$$

Moreover, the third term of (6.70) is a type II term in the sense explained in Section 3 ($k''>0$), so it goes to zero as $z\to 0$. The second term of (6.70) corresponds to the situation in which there exists some p_h or p_h' (here, it is 0) such that $q_h > p_h + 1$ or $q_h' > p_1' + 1$.

Therefore this term tends to zero as $z\to 0$.

Now let us consider the first term of (6.70). In the language of Theorem (4.8), t_0 is some t_i and t_k is some t_i, therefore we know that one needs only to consider the situation

$$i_1 < i_1' < i_2 < i_2' < \cdots < i_k < i_k'$$

and the limit of the first term as $z\to 0$ is equal to

(6.71)

$$\sum_{n=1}^{\infty}\sum_{\varepsilon\in\{0,1\}}\lim_{z\to 0}\langle D_\varepsilon^+ u\otimes W\left(z\int_{S/z^2}^{T/z^2} S_u Q_+ f du\right)\Phi_F\otimes W\left(z^2\int_{S/z^2}^{T/z^2} Q^- S_u f du\right)\Phi_F',$$

$$(-i)^{n-1}z\int_0^{t/z^2}dt_0\int_0^{t_0}dt_1\cdots\int_0^{t_{n-1}}dt_n\sum_{k=1}^{n}\sum_{\sigma(k)\in\{0,1\}^k}\langle Q^- S_{t_0}g_\varepsilon,\ Q^- S_{t_k}g_{\sigma(k)}\rangle_\iota$$

$$zD_{\sigma(1)}\cdots D_{\sigma(k)}\otimes\Big[A(S_{t_0}Q_+ g_{1-\varepsilon}),\ A^+(S_{t_1}Q_+ g_{\sigma(1)})A(S_{t_1}Q_+ g_{1-\sigma(1)})\cdots$$

$$A^+(S_{t_{k-1}}Q_+ g_{\sigma(k-1)})A(S_{t_{k-1}}Q_+ g_{1-\sigma(k-1)})A^+(S_{t_k}Q_+ g_{1-\sigma(k)})\Big]\otimes 1V(t_{k-1})\cdots V(t_n)$$

$$v\otimes W\left(z\int_{S'/z^2}^{T'/z^2} S_u Q_+ f' du\right)\Phi_F\otimes W\left(z^2\int_{S'/z^2}^{T'/z^2} Q^- S_u f' du\right)\Phi_F'\rangle.$$

By Theorem (4.8), we know that one needs only to consider the case in which the operator

(6.72) $$\Big[A(S_{t_0}Q_+ g_{1-\varepsilon}),\ A+(S_{t_1}Q_+ g_{\sigma(1)})A(S_{t_1}Q_+ g_{1-\sigma(1)})\cdots$$

$$A^{+}(S_{t_{k-1}}Q_{+}g_{\sigma(k-1)})A(S_{t_{k-1}}Q_{+}g_{1-\sigma(k-1)})A^{+}(S_{t_k}Q_{-}g_{1-\sigma(k)})\Big]$$

is replaced by its pure scalar product term

$$\langle S_{t_0}Q_{+}g_{1-\varepsilon}, S_{t_1}Q_{+}g_{\sigma(1)}\rangle\cdot\langle S_{t_1}Q_{+}g_{1-\sigma(1)}, S_{t_2}Q_{+}g_{\sigma(2)}\rangle\cdots \cdots\langle S_{t_{k-1}}Q_{+}g_{1-\sigma(k-1)}, S_{t_k}Q_{+}g_{1-\sigma(k)}\rangle \tag{6.73}$$

therefore one has

$$\lim_{z\to 0}\int_0^t \mathrm{III}_z(\mathrm{d})ds \tag{6.74}$$

$$= \sum_{n=0}^{\infty}\sum_{\varepsilon\in\{0,1\}}\lim_{z\to 0}\langle D_\varepsilon^{+}u\otimes W\Big(z\int_{S/z^2}^{T/z^2}S_uQ_{+}fdu\Big)\Phi_F\otimes W\Big(z^2\int_{S/z^2}^{T/z^2}Q^{-}S_ufdu\Big)\Phi_F^{\zeta},$$

$$(-i)^{n-1}z^2\int_0^{t/z^2}dt_0\int_0^{t_0}dt_1\cdots\int_0^{t_{n-1}}dt_n\sum_{k=1}^{n}\sum_{\varepsilon(k)\in\{0,1\}^k}\langle Q^{-}S_{t_0}g_\varepsilon, Q^{-}S_{t_k}g_{\sigma(h)}\rangle_\zeta$$

$$D_{\sigma(1)}\cdots D_{\sigma(k)}\otimes 1\,\langle S_{t_0}Q_{+}g_{1-\varepsilon}, S_{t_1}Q_{+}g_{\sigma(1)}\rangle\cdot$$

$$\cdot\langle S_{t_1}Q_{+}g_{1-\sigma(1)}, S_{t_2}Q_{+}g_{\sigma(2)}\rangle\cdots$$

$$\cdots\langle S_{t_{k-1}}Q_{+}g_{1-\sigma(k-1)}, S_{t_k}Q_{+}g_{1-\sigma(k)}\rangle$$

$$V(t_{k+1})\cdots V(t_n)v\otimes W\Big(z\int_{S'/z^2}^{T'/z^2}S_uQ_{+}f'du\Big)\Phi_F\otimes W\Big(z^2\int_{S'/z^2}^{T'/z^2}Q^{-}S_uf'du\Big)\Phi_F^{\zeta}\rangle.$$

In the right hand side of (6.74), changing the order of sum for $n=1, 2, \dots$ and $k=1,\dots, n$ and with the change of variables

$$z^2t_h \hookrightarrow t_h,\ h=0, 1,\dots, k \tag{6.75}$$

we obtain that

$$\lim_{z\to 0}\int_0^t \mathrm{III}_z(\mathrm{d})ds = \sum_{k=1}^{\infty}\sum_{\varepsilon\in\{0,1\}}\sum_{\sigma\in\{0,1\}^k}\lim_{z\to 0} \tag{6.76}$$

$$\langle D_{\sigma(k)}^{+}\cdots D_{\sigma(1)}^{+}D_\varepsilon^{+}u\otimes W\Big(z\int_{S/z^2}^{T/z^2}S_uQ_{+}fdu\Big)\Phi_F\otimes W\Big(z^2\int_{S/z^2}^{T/z^2}Q^{-}S_ufdu\Big)\Phi_F^{\zeta},$$

$$z^{-2k}\int_0^t dt_0\int_0^{t_0}dt_1\cdots\int_0^{t_{k-1}}dt_k\,\langle S_{t_0/z^2}Q_{+}g_{1-\varepsilon}, S_{t_1/z^2}Q_{+}g_{\sigma(1)}\rangle$$

$$\langle S_{t_1/z^2}Q_{+}g_{1-\sigma(1)}, S_{t_2/z^2}Q_{+}g_{\sigma(2)}\rangle\cdots\langle S_{t_{k-1}/z^2}Q_{+}g_{1-\sigma(k-1)}, S_{t_k/z^2}Q_{+}g_{1-\sigma(k)}\rangle$$

$$\langle Q^{-}S_{t_0}g_\varepsilon, Q^{-}S_{t_k}g_{\sigma(k)}\rangle_\zeta$$

$$\sum_{n=k}^{\infty}(-i)^{n-k}\int_0^{t_k/z^2}dt_{k+1}\cdots\int_0^{t_{n-1}}dt_n\,V(t_{k+1})\cdots V(t_n)$$

$$v\otimes W\left(z\int_{S'/z^2}^{T'/z^2} S_uQ_+f'du\right)\Phi_F\otimes W\left(z^2\int_{S'/z^2}^{T'/z^2} Q^-S_uf'du\right)\Phi_F^\zeta\rangle.$$

In the right hand side of (6.76), with the change of variables

(6.77) $$s_h := (t_h - t_{h-1})/z^2,\ h=1,\dots,k$$

and applying Lemma (6.3) of [2], we find that

$$\lim_{z\to0}\int_0^t \mathrm{III}_z(\mathrm{d})ds = \sum_{k=1}^{\infty}\sum_{\varepsilon\in\{0,1\}}\sum_{\sigma\varepsilon\in\{0,1\}^k}\int_0^t dt_0$$

$$\int_{-\infty}^0 ds_1\cdots\int_{-\infty}^0 ds_k\ \langle g_{1-\varepsilon},\ S_{s_1}Q_+g_{\sigma(1)}\rangle$$

$$\langle g_{1-\sigma(1)},\ S_{s_2}Q_+g_{\sigma(2)}\rangle\cdots\langle g_{1-\sigma(k-1)},\ S_{s_k}Q_+g_{1-\sigma(k)}\rangle\cdot\overline{\langle g_\varepsilon,\ S_{s_k+\dots+s_1}g_{\sigma(k)}\rangle}$$

$$\lim_{z\to0}\langle D^+_{\sigma(k)}\cdots D^+_{\sigma(1)}D^+_\varepsilon u\otimes W\left(z\int_{S/z^2}^{T/z^2}S_uQ_+fdu\right)\Phi_F\otimes W\left(z^2\int_{S/z^2}^{T/z^2}Q^-S_ufdu\right)\Phi_F^\zeta,$$

$$U_{t_0/z^2}v\otimes W\left(z\int_{S'/z^2}^{T'/z^2}S_uQ_+f'du\right)\Phi_F\otimes W\left(z^2\int_{S'/z^2}^{T'/z^2}Q^-S_uf'du\right)\Phi_F^\zeta\rangle$$

$$=\sum_{\varepsilon\in\{0,1\}}\sum_{k=1}^{\infty}\sum_{\sigma\in\{0,1\}^k}\int_0^t dt_0\ \langle D^+_{\sigma(k)}\cdots D^+_{\sigma(1)}D^+_\varepsilon u,\ G(t_0)\rangle$$

$$\int_{-\infty}^0 ds_1\cdots\int_{-\infty}^0 ds_k\ \langle g_{1-\varepsilon},\ S_{s_1}g_{\sigma(1)}\rangle$$

$$\langle g_{1-\sigma(1)},\ S_{s_2}Q_+g_{\sigma(2)}\rangle\cdots\langle g_{1-\sigma(k-1)},\ S_{s_k}Q_+g_{1-\sigma(k)}\rangle\cdot\overline{\langle g_\varepsilon,\ S_{s_k+\dots+s_1}g_{\sigma(k)}\rangle}$$

this ends the proof of (6.64).

Finally, combining together (6.7), (6.9), (6.11), (6.12), (6.13), Lemma (6.2)–Lemma (6.10), we get the following

THEOREM (6.11). For any t, g_0, g_1 satisfied the condition (1.15),

(6.79) $$\langle u,\ G(t)\rangle = \langle u,\ G(0)\rangle + \sum_{\varepsilon\{0,1\}}\int_0^t ds$$

$$\Big(\chi_{[S,T]}(s)\cdot\chi_{[S',T']}(s)\cdot(g_\varepsilon\,|\,f')\cdot(f\,|\,g_\varepsilon)\ \langle D_1^+(\varepsilon)u,\ G(s)\rangle +$$

$$+\chi_{[S,T]}(s)\cdot\chi_{[S',T']}(s)\cdot(f\,|\,g_\varepsilon)\cdot(g_{1-\varepsilon}\,|\,f')\cdot\langle D_2^+(\varepsilon)u,\ G(s)\rangle +$$

$$+\langle (D_3(\varepsilon)+D_\varepsilon\langle g_\varepsilon,\ e^{-\frac12\beta H}g_{1-\varepsilon}\rangle)^+u,\ G(s)\rangle\Big)$$

where, $D_3(\varepsilon)$ defined by (6.65),

(6.80) $$D_1(\varepsilon) := D_\varepsilon +$$

$$+\sum_{n=0}^{\infty}\sum_{\sigma\in\{0,1\}^n} D_\varepsilon D_{\sigma(1)}\cdot\dots\cdot D_{\sigma(n)}D_{1-\varepsilon}$$

https://doi.org/10.1017/S0027763000003998 Published online by Cambridge University Press

$$(g_{1-\varepsilon} \mid g_{\sigma(1)})_- \cdot (g_{1-\sigma(1)} \mid g_{\sigma(2)})_- \cdot \cdots \cdot (g_{1-\sigma(n)} \mid g_{1-\varepsilon})_-$$

and

(6.81)
$$D_2(\varepsilon) := \sum_{n=0}^{\infty} \sum_{\sigma\in\{0,1\}^n} D_\varepsilon D_{\sigma(1)} \cdot \cdots \cdot D_{\sigma(n)} D_\varepsilon$$
$$(g_{1-\varepsilon} \mid g_{\sigma(1)})_- \cdot (g_{1-\sigma(1)} \mid g_{\sigma(2)})_- \cdot \cdots \cdot (g_{1-\sigma(n)} \mid g_\varepsilon)_-$$

where, $\sigma(0) := \varepsilon$.

§ 7. The quantum stochastic differential equation

In the section, we consider the quantum stochastic differential equation (q.s.d.e.)

(7.1)
$$U(t) = 1 + \sum_{\varepsilon\in\{0,1\}} \int_0^t \Big[D_1(\varepsilon) \otimes dN_s(g_\varepsilon, g_{1-\varepsilon}) + $$
$$+ D_2(\varepsilon) \otimes dN_s(g_\varepsilon, g_\varepsilon) + (D_3(\varepsilon) + D_\varepsilon \langle g_{1-\varepsilon}, e^{-\frac{1}{2}\beta H} g_\varepsilon \rangle) \otimes 1ds\Big] U(s)$$

on $H_0 \otimes \Gamma\Big(L^2(\mathbf{R}) \otimes (\mathrm{K}, (\cdot|\cdot))\Big)$, where, N is the gauge process and

(7.2)
$$N_s(g, g') := N_s(\chi_{[0,s]} \otimes |g\rangle \langle g'|).$$

$D_1(\varepsilon)$, $D_2(\varepsilon)$ and $D_3(\varepsilon)$ are defined in § 6. By [11], we know that the q.s.d.e. (7.1) has a unique solution $U(t)$ which can be obtained by iterative.

Proof of Theorem (1.1). Theorem (6.1) has shown the low density limit (1.13) exists. Now, we shall prove that it is equal to (1.14). Clearly, (1.14) is continuous for u, $v \in H_0$, so one can write it to

$$\langle u, F(t) \rangle$$

where, $F(t) \in H_0$. Hence one has

(7.3)
$$\langle u, F(0) \rangle = \langle u \otimes W (X_{[S,T]} \otimes f)\Psi, v \otimes W (\chi_{[S',T']} \otimes f')\Psi \rangle$$
$$= \langle u, G(0) \rangle.$$

Moreover, by (1.16)

(7.4)
$$\langle u, F(t) \rangle = \langle u, F(0) \rangle + \sum_{\varepsilon\in\{0,1\}} \int_0^t \langle u \otimes W(\chi_{[S,T]} \otimes f')\Psi,$$
$$\Big(D_1(\varepsilon) \otimes dN_s(g_\varepsilon, g_{1-\varepsilon}) +$$
$$+ D_2(\varepsilon) \otimes dN_s(g_\varepsilon, g_\varepsilon) + (D_3(\varepsilon) + \langle g_{1-\varepsilon}, e^{-\frac{1}{2}\beta H} g_\varepsilon \rangle) \otimes 1ds\Big)$$
$$\times U(s) v \otimes W(\chi_{[S',T']} \otimes f')\Psi \rangle.$$

Apply the theorem (4.2) of [11] to (7.4), one obtains

$$(7.5)\qquad \langle u, F(t)\rangle = \langle u, F(0)\rangle + \sum_{\varepsilon\in\{0,1\}}\int_0^t ds$$

$$\Big(\chi_{[S,T]}(s)\cdot\chi_{[S',T']}(s)\cdot(g_\varepsilon|f')\cdot(f|g_\varepsilon)\langle D_1^+(\varepsilon)u, F(s)\rangle +$$

$$+\chi_{[S,T]}(s)\cdot\chi_{[S',T']}(s)\cdot(f|g_\varepsilon)_-\cdot(g_{1-\varepsilon}|f')\cdot\langle D_2^+(\varepsilon)u, F(s)\rangle +$$

$$+\langle(D_3(\varepsilon)+D_\varepsilon\langle g_{1-\varepsilon}, e^{-\frac{1}{2}\beta H}g_\varepsilon\rangle)^+u, F(s)\rangle\Big).$$

Since (7.5) has unique solution, so (7.8) has a unique solution. Therefore,

$$\langle u, F(t)\rangle = \langle u, G(t)\rangle \quad t\geq 0.$$

REFERENCES

[0] L. Accardi, Y. G. Lu, A. Frigerio and R. Alicki, An invitation to the weak coupling limit and the low density limit, to appear in: Quantum probability and related topics VI, World Scientific, 3–31.

[1] L. Accardi, Y. G. Lu, The number process as low density limit of Hamiltonian models, Commun. Math. Phys., **637** (1991), 1–31.

[2] L. Accardi, A. Frigerio, Y. G. Lu, The weak couplihg limit as a functional central limit theorem, Commun. Math. Phys, **131** (1990), 537–570.

[3] L. Accardi, Y. G. Lu, On the low density limit of Boson Models. Lect. Notes in Math, **1442** (1988), 17–53.

[4] ——. The weak coupling limit for nonlinear interactions, Stoch. Proc. Phys. Geom., World Scientific, 1988, 1–26.

[5] R. Dümcke, The low density limit for *n*-level systems, Lect. Notes in Math, **1136** (1985).

[6] R. Dümcke, The low density limit fot an *N*-level system interacting with a free Bose or fermi gas, Commun. Math. Phys., **97** (1985), 331–359.

[7] A. Frogerio, H. Maassen, Quantum Poisson processes and dilations of dynamical semigroups, Prob. Th. Rel. Fields., **83** (1989), 489–508.

[8] H. Grad, Principles of the knietic theory of Gases, Handbuch der Physik, vol. 12, Springer (1958).

[9] R. L. Hudson, K. R. Pathasarathy, Quantum Ito's formula and stochastic evolution, Commun. Math. Phys., **93** (1984), 301–323.

[10] P. F. Palmer; Thesis, Oxford University.

[11] D. Patz; An invitation to the C^*-algebra of the canonical commutation relation, Leuven Notes in Math. and Theor. Phys., eds. Fannes M. Verbeure A., Leuven Univ. Press 1990.

Centro Matematico V. Volterra
Dipartimento di Matematica
Universita' di Roma II
Italia