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FINITELY GENERATED IDEALS IN THE DISK ALGEBRA

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Let $f_1, \dots, f_N \in A(\mathbb{D})$. It is shown that the ideal $I(f_1, \dots, f_N)$ generated by the functions f_j $(j = 1, \dots, N)$ equals the ideal

$$J(f_1, \dots, f_N) = \left\{ f \in A(\mathbb{D}) : \exists\, C = C(f) : |f| \leqslant C \sum_{j=1}^{N} |f_j| \quad \text{on } \overline{\mathbb{D}} \right\}$$

if and only if the functions f_j have no common zero on the boundary of the unit disk $\mathbb{D}$.

1. INTRODUCTION

Let H^∞ be the algebra of bounded analytic functions on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $A(\mathbb{D})$ be the disk algebra, that is, the subalgebra of all those functions in H^∞ which have a continuous extension to the closure $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leqslant 1\}$ of $\mathbb{D}$.

If R is any commutative algebra, we denote by

$$I = I(f_1, \dots, f_N) = \left\{ \sum_{j=1}^{N} h_j f_j,\ h_j \in R \right\}$$

the ideal generated by the f_j $(f_j \in R,\ j = 1, \dots, N)$.

In [1] Rubel states the following problem: If $R = A(\mathbb{D})$ or H^∞, give necessary and sufficient conditions on a function $f \in R$ which ensure that f belongs to the ideal $I(f_1, \dots, f_N)$.

For example, in the algebra $H(G)$ of all analytic functions in a domain $G \subset \mathbb{C}$ it is well known that a function $f \in H(G)$ belongs to the ideal $I(f_1, \dots, f_N)$ generated by the functions $f_j \in H(G)$ if and only if

$$\operatorname{ord}(f, z_0) \geqslant \min_{j=1,\dots,N} \operatorname{ord}(f_j, z_0)$$

for every $z_0 \in G$, where $\operatorname{ord}(f, z_0)$ is the usual multiplicity of the zero z_0 of f.

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In order to solve Rubel's problem for the disk algebra, the ideals

$$J(f_1,\dots,f_N)=\left\{f\in A(\mathbb{D}):\exists\, C=C(f):|f|\leqslant C\sum_{j=1}^{N}|f_j|\ \text{ on }\ \overline{\mathbb{D}}\right\}$$

have shown to be a valuable tool. Obviously, $I(f_1,\dots,f_N)$ is contained in $J(f_1,\dots,f_N)$ whenever $f_j\in A(\mathbb{D})$. It has been shown by von Renteln [**8**] that, whenever the functions f_j $(j=1,\dots,N)$ have no common zero on the boundary $T=\partial\mathbb{D}=\{z\in\mathbb{C}:|z|=1\}$ of the unit disk, then the ideals $I(f_1,\dots,f_N)$ and $J(f_1,\dots,f_N)$ coincide. On the other hand, it is quite easy to see that, in general, $I(f_1,\dots,f_N)$ is a proper subideal of $J(f_1,\dots,f_N)$, even in the case of one single generator. In fact, the function $(1-z)e^{-(1+z)/(1-z)}$ belongs to $J(1-z)$ but not to $I(1-z)$.

The main result of this note is to show that the converse of von Renteln's result holds. We also include a closely related result on divisors of disk algebra functions.

2. Divisors of disk algebra functions

A major tool in working with bounded analytic functions on the unit disk is the Riesz-Smirnov factorisation theorem (see, for example, [**2**, Section 2]). It states that, for every $f\in H^\infty$, $f\not\equiv 0$, there exists a finite positive Borel measure μ, singular with respect to Lebesgue measure on T such that

$$f(z)=e^{i\theta}B(z)S_\mu(z)F(z),$$

where

$$B(z)=\prod_{n=1}^{\infty}\frac{|a_n|}{a_n}\frac{a_n-z}{1-\bar{a}_n z},\qquad \sum_{n=1}^{\infty}1-|a_n|<\infty,$$

is the Blaschke produkt associated with the zero sequence (a_n) of f,

$$S_\mu(z)=\exp\left(-\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\,d\mu(t)\right)$$

and

$$F(z)=\exp\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log v\left|f\left(e^{it}\right)\right|\,\frac{dt}{2\pi}.$$

The function F is called the outer part of f, the function $\varphi=BS_\mu$ the inner part.

If $f\in A(\mathbb{D})$, then $F\in A(\mathbb{D})$ (see [**2**, p.78]). Let $\operatorname{Sing}\varphi$ denote the set of boundary singularities of the inner function φ and let $Z(f)=\{z\in\overline{\mathbb{D}}:f(z)=0\}$ be the zero set of $f\in A(\mathbb{D})$. Then $\operatorname{Sing}\varphi\subseteq Z(f)\cap T=Z(F)$ (see [**2**, p.78]). Moreover, if $f\in A(\mathbb{D})$ and if φ is an inner function with $\operatorname{Sing}\varphi\subseteq Z(f)\cap T$, then $\varphi f\in A(\mathbb{D})$.

Let $f_j \in H^\infty$ $(j = 1, \dots, N)$. It is well known that the functions f_j have a greatest common divisor in H^∞, denoted by $d = \gcd(f_1, \dots, f_N)$ (see [7]). It can be written in the form

$$d = BS_\mu F, \tag{1}$$

where B is the Blaschke product associated with the common zeros of f_j in $\mathbb{D}$, where S_μ is the singular inner function with measure μ defined by

$$\mu(E) = \inf_{\mathcal{P}} \sum_{k=1}^{N} \inf_{1 \leqslant j \leqslant N} \mu_j(E_k) \tag{2}$$

($\mathcal{P}$ is the set of all finite measurable partitions $\{E_1, \dots, E_k\}$ of the Borel set E (see [**3**, p.85] respectively [**2**, p.84])), and where

$$F(z) = \exp \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \sum_{j=1}^{N} |f_j| \left(e^{it}\right) \frac{dt}{2\pi}. \tag{3}$$

Of course, if h is invertible in H^∞, then hd is another greatest common divisor of the f_j. In particular, this applies for functions $f_j \in A(\mathbb{D})$. In general, however, neither d nor the quotients f_j/d are in $A(\mathbb{D})$ whenever $f_j \in A(\mathbb{D})$.

Our first objective is now to show that, whenever $f_j \in A(\mathbb{D})$, then there exists an outer function h invertible in H^∞ such that $hd \in A(\mathbb{D})$. Hence there exists in H^∞ a smooth greatest common divisor of the f_j.

The proof is based on the following lemma from [**5**, p.11].

Lemma 1.1. *Let u be a continuous positive function on an open interval $I \subseteq \mathbb{R}$. Then there exists a continuously differentiable function v on I such that*

$$|u - v| \leqslant \frac{1}{2} u \quad \text{on } I.$$

We note that the proof follows from Carleman's approximation theorem after having mapped I by a continuously differentiable bijective map onto $\mathbb{R}$.

Proposition 1.2. *Let $f_1, \dots, f_N \in A(\mathbb{D})$. Then there exists, with respect to the algebra H^∞, a greatest common divisor which belongs to $A(\mathbb{D})$.*

Proof: Let $\varphi_j F_j = f_j$ be the inner-outer factorisation of the functions f_j and let $d = (BS_\mu)F = \varphi F$ be the greatest common divisor of the f_j according to (1)(2)(3). Obviously,

$$\operatorname{Sing} \varphi \subseteq \bigcap_{j=1}^{N} Z(f_j) \cap T = \bigcap_{j=1}^{N} Z(F_j)$$

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and

$$|d| = |F| = \sum_{j=1}^{N} |f_j| \text{ almost everywhere on } T. \tag{4}$$

Because

$$\log|F(z)| = \int_0^{2\pi} P(z,t) \log \sum_{j=1}^{N} |f_j| \left(e^{it}\right) \frac{dt}{2\pi} \quad (z \in \mathbb{D})$$

is the Poisson integral of an extended continuous function, we see that $|F(z)|$ has a continuous extension u on $\overline{\mathbb{D}}$ satisfying

$$u = \sum_{j=1}^{N} |f_j| \text{ on } T. \tag{5}$$

Let $E = \bigcap_{j=1}^{N} Z(f_j) \cap T$. Suppose first that $E = \emptyset$. Then, by (4), F is invertible in H^∞. Moreover, $\operatorname{Sing}\varphi = \emptyset$; therefore φ is a finite Blaschke product. Hence we have found a greatest common divisor which belongs to $A(\mathbb{D})$.

Now suppose that $E \neq \emptyset$. Because E is a compact subset of T of measure zero, there exist at most countably many pairwise disjoint open arcs $I_j \subseteq T$ so that $T \setminus E = \bigcup_{j=1}^{\infty} I_j$. Let $I_j = (\alpha_j, \beta_j)$ be any of these arcs. Then $u > 0$ on I_j and $u(\alpha_j) = u(\beta_j) = 0$.

By Lemma 1.1 there exist functions $v_j \in C^1(I_j) \cap C(\overline{I}_j)$, $v_j \geqslant 0$, so that

$$|u - v_j| \leqslant \frac{1}{2} |u| \text{ on } I_j. \tag{6}$$

Let $q(e^{it}) = v_j(e^{it})$ if $e^{it} \in I_j$ and $q = 0$ otherwise. Then (6) implies that q is continuous on T, continuously differentiable on $T \setminus Z(q)$ and $\log q \in L^1(T)$. Hence, by [**6**, p.52], the function

$$Q(z) = \exp \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log q\left(e^{it}\right) \frac{dt}{2\pi}$$

is an outer function in the disk algebra with

$$Z(Q) = E \quad \text{and} \quad |Q| = q \text{ on } T.$$

By (4)–(6) we have

$$\frac{1}{2} \leqslant \frac{|Q|}{|F|} \leqslant \frac{3}{2} \text{ almost everywhere on } T.$$

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Let

$$h(z) = \exp \int \frac{e^{it} + z}{e^{it} - z} \log \left| \frac{Q}{F} \right| (e^{it}) \, \frac{dt}{2\pi}.$$

Then $h \in H^\infty$ and satisfies $1/2 \leqslant |h| \leqslant 3/2$ on $\mathbb{D}$. Hence h is invertible in H^∞. Moreover,

$$Q = Fh.$$

Let $f = \varphi Q$. Then $f \in A(\mathbb{D})$ and f is, with respect to H^∞, a greatest common divisor of the functions f_j. $\square$

REMARK. Note that the quotients f_j/f do not belong to $A(\mathbb{D})$ in general. If this happens, then the functions f_j obviously have a greatest common divisor with respect to the algebra $A(\mathbb{D})$. In that case, the ideal generated by the f_j is a principal ideal (see [**6**]).This highly contrasts with the situation in H^∞.

It is now a natural question to ask whether any countable set of functions in $A(\mathbb{D})$ has, with respect to the algebra H^∞, a greatest common divisor. For example, it is easy to see that any set of inner functions has this property (see [**3**,p. 85] or [**2**, p. 84]).

It is now rather surprising that this is no longer true for outer functions. The following example has been communicated to me by H.-M. Lingenberg. I want to thank him for allowing me to include his example in this paper.

EXAMPLE. Let $f_j = (1 - z)^{1/2^j}$ $(j = 0, 1, 2, \dots)$. Assume that the functions f_j have a greatest common divisor $d \in H^\infty$. Then d^2 divides f_j^2 for every $j = 0, 1, 2, \dots$. In particular, d^2 divides f_k for $k = 0, 1, 2, \dots$. Since d is a greatest common divisor, we see that d^2 divides d. Hence d is invertible in H^∞.

But on the other hand, the function

$$g(z) = \exp \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \sum_{j=1}^{\infty} \frac{1}{2^j} \left| f_j(e^{it}) \right| \, \frac{dt}{2\pi}$$

is a proper divisor of the f_j. This contradiction shows that the f_j do not admit a greatest common divisor.

3. THE IDEALS $I(f_1, \dots, f_N)$ AND $J(f_1, \dots, f_N)$

For an ideal $I \subseteq A(\mathbb{D})$ let $Z(I) = \bigcap_{f \in I} Z(f)$ denote its zero set. Ideals in $A(\mathbb{D})$ whose zero sets are contained in the open unit disk are easily described. In fact, we have the following result (see [**4**, p. 13]).

PROPOSITION 2.1. *Let $I \neq (0)$ be an ideal in $A(\mathbb{D})$ such that*

$$Z(I) \cap T = \emptyset.$$

Then I is a principal ideal generated by a finite Blaschke product.

For the reader's convenience, we present the proof.

PROOF: Because $Z(I) \cap T = \emptyset$ and $I \neq (0)$, the set $Z(I)$ is either finite or empty. In the latter case, $I = A(\mathbb{D}) = (1)$, and the assertion is trivial.

Now let B be the finite Blaschke product associated with the common zeros of the functions in I (including multiplicities). Obviously, B is a common divisor of all functions in I. Hence $I \subseteq (B)$. But by construction we have $\bigcap_{f \in I} Z(f/B) = \emptyset$.

By compactness, there exist finitely many $f_j \in I$ so that

$$\bigcap_{j=1}^{N} Z\left(\frac{f_j}{B}\right) = \emptyset.$$

Hence, by the corona theorem for $A(\mathbb{D})$, we have

$$1 \in \left(\frac{f_1}{B}, \dots, \frac{f_N}{B}\right).$$

This implies that $B \in (f_1, \dots, f_N) \subseteq I$. Altogether we have $I = (B)$. ∎

Next we recall some definitions from the theory of bounded analytic functions.

1. Let $f \in H^\infty$. Then, for $\alpha \in T$, the cluster set of f at α will be denoted by $\mathrm{Cl}\,(f, \alpha)$, that is,

$$\mathrm{Cl}\,(f, \alpha) = \{\beta \in \mathbb{C} : \exists\, z_n \in \mathbb{D},\ z_n \to \alpha,\ f(z_n) \to \beta\}.$$

We note that f has a continuous extension to $\alpha \in T$ if and only if $\mathrm{Cl}\,(f, \alpha)$ is a singleton.

3. A sequence (z_n) is called an interpolating sequence if

$$\inf_{k \in \mathbb{N}} \prod_{n: n \neq k} \left| \frac{z_n - z_k}{1 - \bar{z}_n z_k} \right| \geqslant \delta > 0.$$

It is well known [**3**, p.204] that any sequence in $\mathbb{D}$ converging to the boundary has an interpolating subsequence. Blaschke products associated with interpolating sequences are called interpolating Blaschke products.

We are now ready to prove the main result of this paper.

THEOREM 2.2. *Let $f_1, \dots, f_N$ be functions in the disk algebra. Then the ideal*

$$I(f_1, \dots, f_N) = J(f_1, \dots, f_N)$$

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if and only if the functions f_j have no common zero on the boundary of the unit disk.

PROOF: 1. Let $I = I(f_1, \dots, f_N)$ and $J = J(f_1, \dots, f_N)$. Suppose that the functions f_j have no common zero on the unit circle T. Then $Z(I) = Z(J) \subseteq \mathbb{D}$. Hence, by Proposition 2.1, $I = J = (B)$, where B is the finite Blaschke product associated with the common zeros of the f_j (including multiplicities).

2. Now let $I = J$. We have to show that $Z(I) \cap T = \emptyset$. Assume the contrary. Without loss of generality, let $1 \in Z(I) \cap T$. According to Proposition 1.2, let $d \in A(\mathbb{D})$ be a greatest common divisor of the functions f_j (with respect to the algebra H^∞). Then $f_j = dg_j$ for some $g_j \in H^\infty$. We claim that there exists $j_0 \in \{1, \dots, N\}$ so that

$$(\star) \qquad \operatorname{Cl}(g_{j_0}, 1) \neq \{0\}.$$

Otherwise, each of the g_j would have a continuous extension to $z = 1$ with value 0 there. Then the function

$$G(z) = \exp \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \sum_{j=1}^N |g_j| \left(e^{it}\right) \frac{dt}{2\pi}$$

would be a noninvertible common divisor of the g_j, which is a contradiction to the fact that any common divisor of the g_j is invertible.

Now choose a sequence (z_n) in $\mathbb{D}$ with $\lim z_n = 1$ so that $g_j(z_n)$ converges to some $\alpha_j \in \mathbb{C}$ for every $j \in \{1, \dots, N\}$. Without loss of generality let (z_n) be an interpolating sequence. By $(\star)$ we may also assume that $\alpha_{i_0} \neq 0$ for some $i_0 \in \{1, \dots, N\}$. Let b be the interpolating Blaschke product associated with the sequence (z_{2n}). Then

$$|b(z_{2\ell+1})| = \prod_{n=1}^\infty \left| \frac{z_{2n} - z_{2\ell+1}}{1 - \overline{z}_{2n} z_{2\ell+1}} \right| \geqslant \delta > 0$$

for all $\ell \in \mathbb{N}$.

Obviously, $\operatorname{Sing} b = \{1\}$. Hence $bf_{i_0} \in A(\mathbb{D})$. Moreover, $|bf_{i_0}| \leqslant |f_{i_0}|$. Therefore $bf_{i_0} \in J$. The hypothesis $I = J$ implies that there exists $x_j \in A(\mathbb{D})$ such that

$$bf_{i_0} = \sum_{j=1}^N x_j f_j = \sum_{j=1}^N x_j d g_j.$$

Dividing by d, we obtain

$$bg_{i_0} = \sum_{j=1}^N x_j g_j.$$

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Let $x_j(1) = \beta_j$. Then

$$0 = \lim_{l \to \infty} bg_{i_0}(z_{2l}) = \sum_{j=1}^{N} \beta_j \alpha_j$$

on the one hand, but

$$0 \neq \delta \alpha_{i_0} \leqslant \liminf_{l \to \infty} |bg_{i_0}(z_{2l+1})| = \left| \sum_{j=1}^{N} \beta_j \alpha_j \right|$$

on the other hand.

This contradiction shows that $Z(I) \cap T$ must be empty. Hence the functions f_j have no common zero on the boundary of the unit disk. □

COROLLARY 2.3. *Let $f_j \in A(\mathbb{D})$ $(j = 1, \dots, N)$. Then $I(f_1, \dots, f_N) = J(f_1, \dots, f_N)$ if and only if I (respectively J) is a principal ideal generated by a finite Blaschke product.*

PROOF: Let $I = J$. Then by Theorem 2.2, $Z(I) \cap T = \emptyset$. Hence by Proposition 2.1, $I = (B)$, where B is a finite Blaschke product.

The converse of the assertion has already been proved by von Renteln [8]. It also follows immediately from our theorem. □

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