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ELIMINATION FROM HOMOGENEOUS POLYNOMIALS OVER A POLYNOMIAL RING

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1. Preliminaries. Let Ω be a field and Γ a parameter. We designate the set of all polynomials homogeneous in $(X) = (X_1, \ldots, X_n)$ with coefficients in $\Omega[\Gamma]$ by $H\Omega_\Gamma[X]$ and write such polynomials as F, $F(X)$, or $F(X, \Gamma)$. The degree of a polynomial in $H\Omega_\Gamma[X]$ shall mean the degree in (X). Let $I = (F_1, \ldots, F_r)$ be a fixed ideal in $H\Omega_\Gamma[X]$ generated by $F_1, \ldots, F_r$. With $W^{(l)}$ denoting the space of polynomials in $H\Omega_\Gamma[X]$ of degree l, we investigate the quotient $I:W^{(l)}$ of I by $W^{(l)}$ in $\Omega[\Gamma]$. This quotient is the set of all polynomials $\rho(\Gamma)$ in $\Omega[\Gamma]$ satisfying

(1) $\rho(\Gamma)G \equiv 0 \mod I$

for all G in $W^{(l)}$. In particular, when $I:W^{(l)}$ is not the zero ideal, we study the monic generator of $I:W^{(l)}$ which we call the *minimal quotient for degree l of I*, written $R_l(\Gamma)$. As $I:W^{(l)}$ is contained in $I:W^{(l+1)}$, the union of all such ideals is an ideal with monic generator $R(\Gamma)$ which we call the *minimal quotient of I*, also denoted $R(F_1, \ldots, F_r)(\Gamma)$. In that $R(\Gamma) = R_\lambda(\Gamma)$ for some λ, $R(\Gamma)$ is the minimal monic polynomial in $\Omega[\Gamma]$ satisfying (1) for all G of degree $l \geqq \lambda$ and thus for all l sufficiently large. Further we use $\omega_t^{(i)}$ to denote the monomials in (X) of degree t. There are $m(t, n) = (t + n - 1)!/t!(n - 1)!$ such monomials. If the above is carried out for a single polynomial F in $H\Omega_\Gamma[X]$ instead of $W^{(l)}$ we speak of the *minimal quotient of F relative to I*, denoted $R_F(\Gamma)$, which generates $I:(F)$ and is the minimal monic polynomial in $\Omega[\Gamma]$ satisfying $\rho(\Gamma)F \equiv 0 \mod I$. If $\deg F = l$ then $R_F(\Gamma)$ exists if $R_l(\Gamma)$ does and $R_F(\Gamma) | R_l(\Gamma)$.

We state without proof certain algebraic properties of minimal quotients which follow directly from elimination theory and the theory of elementary divisors [**3**; **4**].

THEOREM 1.1. *For the ideal I generated by $F_1, \ldots, F_r$ in $H\Omega_\Gamma[X]$ with* $\deg F_i = l_i$ *we have the following:*

(i) *Let (α_{km}) be the coefficient matrix of the polynomials $P_k = \omega^{(j)}{}_{l-l_i}F_i = \sum \alpha_{km}\omega_l{}^{(m)}$ with $1 \leqq k \leqq s$ where $s = \sum_{i=1}^{r} m(l - l_i, n)$. $R_l(\Gamma)$ exists if and only if $s \geqq m(l, n)$ and at least one minor of order $m(l, n)$ of the coefficient matrix is nonzero. A necessary condition is $r \geqq n$. Then $R_l(\Gamma)$ is the maximal invariant factor of the coefficient matrix (α).*

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(ii) $R(a) = 0$ *if and only if there exists a nontrivial zero of the specialized system* $F_1(X, a) = \ldots = F_r(X, a) = 0$ *where* $F_i(X, a)$ *denotes* F_i *with the coefficient polynomials in* $\Omega[\Gamma]$ *evaluated at* a.

(iii) *If the classical resultant system for homogeneous polynomials specialized for* $F_i(X, \Gamma)$ *satisfies Equation* (1) *for all* G *of degree* l^* *but not for* $l^* - 1$ (*if* $r = n$, $l^* = 1 + \sum_{i=1}^n (l_i - 1)$) *then* $R_{l^*}(a) = 0$ *implies* $R(a) = 0$. *Thus if* $R_{l^*}(\Gamma)$ *does not exist, neither does* $R(\Gamma)$.

In the present paper, we study the multiplicity of the zeros of $R(\Gamma)$ and $R_F(\Gamma)$. That the multiplicity of a as a root of $R(\Gamma)$ is not simply related to multiplicity in the specialized system $F_1(X, a) = \ldots = F_r(X, a) = 0$ can be seen in the following example.

Example. Let

$$f(X, \Gamma) = X_1^{d+1} + X_2^{d+1} + \Gamma((X_1 + X_2)^{d+1} - X_1^{d+1} - X_2^{d+1})$$

with

$$F_1 = f_{X_1}/(d+1) = (1 - \Gamma)X_1^d + \Gamma(X_1 + X_2)^d$$

and

$$F_2 = f_{X_2}/(d+1) = (1 - \Gamma)X_2^d + \Gamma(X_1 + X_2)^d.$$

A routine argument shows $R(F_1, F_2)(\Gamma) = R_{2d-1}(\Gamma)$ which equals

$$(1 - \Gamma) \prod \{1 + \Gamma((\epsilon_i + 1)^d - 1)\},$$

the product taken over the dth roots of unity ϵ_i. In particular, although $(1, -1)$ is projectively a zero of multiplicity d for $F_1(X, 1) = F_2(X, 1) = 0$, if d is odd $R(\Gamma)$ has only simple roots; but for d even, $(1 - \Gamma)^2 | R(\Gamma)$. We note that this example provides a negative answer to the question posed by Dwork [**1**, p. 484] whether $R(\Gamma)$ has only simple roots for the case $F_i = \partial f(X, \Gamma)/\partial X_i$ (or $= X_i(\partial f(X, \Gamma)/\partial X_i)$ for which $R(\Gamma)$ is also as given above) with $f(X, \Gamma)$ of the form $\sum_{i=1}^n X_i^d + \Gamma h(X)$.

Specializing now to the case in which Ω is algebraically closed, we shall use multiplicity varieties as conceived by Ehrenpreis to give geometric content to the multiplicities involved and to develop criteria for determining them. A *differential operator* shall mean a linear combination of differentiations $\partial^m/\partial X_1^{m_1} \ldots \partial X_n^{m_n}$ with polynomials in (X) as coefficients. A *multiplicity variety* is an ordered collection V of varieties in Ω^n, not necessarily irreducible, and differential operators associated with the varieties, written $\bar{V} = (V_1, \partial_1; \ldots; V_a, \partial_a)$ where ∂_i is the differential operator associated with V_i. We call (V_i, ∂_i) a *component* of $\bar{V}$. For P in $\Omega[X]$, we mean by $P|_V$ the restriction of P to the variety V. By the restriction of P to a multiplicity variety $\bar{V}$, $P|_{\bar{V}}$, is meant $(\partial_1 P|_{V_1}, \ldots, \partial_a P|_{V_c})$. We shall use the following theorem due to Ehrenpreis; the proof is given in a somewhat different setting in [**2**, Chapter II].

MULTIPLICITY VARIETY THEOREM. *For any ideal I in $\Omega[X]$ with variety V in Ω^n, there is a multiplicity variety $\bar{V} = (V_1, \partial_1; \ldots ; V_a, \partial_a)$ such that $V = V_1 \cup \ldots \cup V_a$ and any polynomial P in $\Omega[X]$ is in I if and only if P restricted to $\bar{V}$ is zero, i.e., $P|_{\bar{V}} = (0, \ldots, 0)$.*

In our application of multiplicity varieties we take the variables to be $(X_1, \ldots, X_n, \Gamma)$ and designate zeros of $F(X, \Gamma)$ by $(\xi_1, \ldots, \xi_n, a)$ or simply (ξ, a). Suppose $R(F_1, \ldots, F_r)(\Gamma) = \prod (\Gamma - a_\nu)^{k_\nu}$. We know from Theorem 1.1(ii) and the homogeneity in (X) that the common zeros of $F_1, \ldots, F_r$ are of the form $(t(\xi), a) = (t\xi_1, \ldots, t\xi_n, a)$ where a is one of the a_ν, t takes on all values in Ω, and one or more such rays belong to each of the a_ν. From the homogeneity in (X), $(0, a)$ is a zero for all a in Ω.

In the sequel we assume $\bar{V}$ to be normalized by eliminating from any differential operator ∂_k of a component (V_k, ∂_k) all terms

$$c_i(X, \Gamma)\partial^m/\partial X_1^{m_1} \ldots \partial X_n^{m_n} \partial \Gamma^{m_\Gamma}$$

for which $c_i(X, \Gamma)|_{V_k} = 0$.

If ∂ is a linear combination of differentiations of the above form we call the maximum of the m_Γ ranging over the differentiations the *order of* Γ-*differentiation of* ∂ and the maximum of the sums $m_1 + \ldots + m_n$ the *order of* (X)-*differentiation of* ∂. For fixed a in Ω let $V\{a\} = \{(\xi, a) | F_1(\xi, a) = \ldots = F_r(\xi, a) = 0\}$. For $\bar{V}$ let $\partial\{a\} = \{\partial_i | (V_i, \partial_i)$ is a component of $\bar{V}$ with V_i contained in $V\{a\}$ and $V_i \neq \{(0, a)\}\}$. We define the Γ-multiplicity of $\bar{V}$ at a to be one plus the maximum of the orders of Γ-differentiation of ∂_i as ∂_i ranges over $\partial\{a\}$, if $\partial\{a\}$ is nonempty; and to be 0, if $\partial\{a\}$ is empty.

2. Main results.

THEOREM 2.1. *Let $\bar{V}$ be the normalized multiplicity variety for the ideal $(F_1, \ldots, F_r)$ in $H\Omega_\Gamma[X]$. Then k is the Γ-multiplicity of $\bar{V}$ at a if and only if a is a zero of $R(F_1, \ldots, F_r)(\Gamma)$ of multiplicity k.*

We prepare with the following lemma.

LEMMA 2.2. *If ∂ is a differential operator with differentiations in (X) alone, normalized on the variety $(t(\xi), a)$ with $\xi_k \neq 0$ for some k, then there is a monomial M of arbitrarily high degree such that $\partial M|_{(t(\xi), a)} \neq 0$.*

Proof. Choose a differentiation $\partial^m/\partial X_1^{m_1} \ldots \partial X_n^{m_n}$ in ∂ with maximum m from among those differentiations with maximum m_k. The coefficient of the differentiation does not vanish at (ξ, a) from the normalization of ∂. The desired monomial is then $M = X_1^{m_1} \ldots X_k^{j} \ldots X_n^{m_n}$ with $j \geqq m_k$ as all other differentiations in ∂ applied to M are zero at (ξ) and

$$(\partial^m(X_1^{m_1} \ldots X_k^{j} \ldots X_n^{m_n})/\partial X_1^{m_1} \ldots \partial X_k^{m_k} \ldots \partial X_n^{m_n})|_{(\xi)} \neq 0.$$

Proof of Theorem 2.1. Let k be the Γ-multiplicity of $\bar{V}$ at a and $R(\Gamma) =$

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$(\Gamma - a)^\nu Q(\Gamma)$, $Q(a) \neq 0$. Suppose $\nu < k$. Let ∂ be a differential operator in $\partial\{a\}$ with $k - 1$ as its order of Γ-differentiation and W as its variety. Then

$$\partial = \partial_X \frac{\partial^{k-1}}{\partial \Gamma^{k-1}} + \partial_1$$

where ∂_X is a differential operator in (X) alone and the order of Γ-differentiation of ∂_1 is less than $k - 1$. In that W is composed of rays of the form $(t(\xi), a)$, we can choose by Lemma 2.1 a homogeneous polynomial M of arbitrarily high degree such that $\partial_X M|_W \neq 0$. $R(\Gamma)M$ and thus $M_1 = (\Gamma - a)^{(k-1)-\nu} R(\Gamma) M$ are in the ideal and must vanish on $\bar{V}$. However,

$$\partial M_1|_W = \partial_X \frac{\partial^{k-1}}{\partial \Gamma^{k-1}} M_1|_W + \partial_1 M_1|_W = Q(a)(k-1)!\partial_X M|_W \neq 0$$

as $\partial_1 M_1|_W = 0$. Thus $\nu \geqq k$. Consider now the polynomial $R_1(\Gamma) = \prod (\Gamma - \alpha)^{k(\alpha)}$ where $k(\alpha)$ is the Γ-multiplicity of $\bar{V}$ at α. Now $R_1(\Gamma)P(X)$ is in the ideal for all polynomials $P(X)$ homogeneous in (X) of sufficiently high degree. This follows in that $R_1(\Gamma)P(X)$ satisfies $\partial_i R_1(\Gamma)P(X)|_{V_i} = 0$ where $V_i \subset V\{\alpha\}$ for any α as $k(\alpha)$ is at least one greater than the order of Γ-differentiation of ∂_i. Thus we need only check $\partial_j R_1(\Gamma)P(X)|_{V_j} = 0$ for $V_j \subset \{(X) = 0\}$. However for $P(X)$ of degree greater than the orders of (X)-differentiation of any such ∂_j, $\partial_j R_1 P$ will still have (X) dependence and will vanish on V_j. Thus $R(\Gamma)|R_1(\Gamma)$ which implies $\nu \leqq k$ and completes the proof.

COROLLARY 2.3. *If λ is the maximum of the orders of (X)-differentiation of differential operators which have associated varieties contained in $\{(X) = 0\}$, then $R(\Gamma) = R_{\lambda+1}(\Gamma)$.*

Proof. Any homogeneous polynomial of degree $\lambda + 1$ will satisfy all the differential operators described in the hypothesis. Further, Γ-multiplicity associated with varieties not contained in $\{(X) = 0\}$ perseveres from the above proof.

Analogously, we have the following theorem for the minimal quotient for a polynomial F, which we state in the case that F is a polynomial in (X) alone i.e., having coefficients in Ω.

THEOREM 2.4. *Let $\partial_j = \sum_{i=0}^{k} \partial_{ji}\partial^i/\partial\Gamma^i$ be the differential operators in $\partial\{a\}$ where (V_j, ∂_j) are components of the multiplicity variety for the ideal $(F_1, \ldots, F_r)$ and ∂_{ji} are differentiations in (X) alone. If $F(X)$ is a polynomial in (X) alone, then $\partial_{ji}F(X)|_{V_j} = 0$ for all j, for $i \geqq k$, and $\partial_{j,(k-1)}F(X)|_{V_j} \neq 0$ for some j if and only if a is a zero of $R_F(\Gamma)$ of multiplicity k.*

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Proof. Suppose $F(X)$ satisfies the differential relations. Considering

$$\partial_j p(\Gamma)(\Gamma - a)^{k-1}F|_{V_j} = \sum_{i=0}^{k-2} (\partial_{ji}F)\left(\frac{\partial^i}{\partial \Gamma^i} p(\Gamma)(\Gamma - a)^{k-1}\right)|_{V_j}$$
$$+ (\partial_{j,k-1}F)\left(\frac{\partial^i}{\partial \Gamma^i} p(\Gamma)(\Gamma - a)^{k-1}\right)|_{V_j}$$
$$+ \sum_{i=k}^{k_j} (\partial_{ij}F)\left(\frac{\partial^i}{\partial \Gamma^i} p(\Gamma)(\Gamma - a)^{k-1}\right)|_{V_j}$$

we see that the sums on the right hand side are zero while the middle term is zero if and only if $p(a) = 0$ or $\partial_{j,k-1}F|_{V_k} = 0$. Letting $R_F(\Gamma) = (\Gamma - a)^\nu Q(\Gamma)$, $Q(a) \neq 0$, it follows that $\nu \leqq k$ as otherwise $(\Gamma - a)^{k-1}Q(\Gamma)F$ would be in the ideal and yet for the choice of j such that $\partial_{j,k-1}F|_{V_j} \neq 0$, we would have the middle term of the above not equal to zero, a contradiction. Now let $P(\Gamma) = \prod (\Gamma - a_\nu)^{k_\nu}$ where $\partial\{a_\nu\} \neq \emptyset$ and k_ν is defined as is k in the statement of the theorem. Then $P(\Gamma)F|_{\bar{V}} = 0$ and so $P(\Gamma)F$ is in the ideal. Thus $R_F(\Gamma)|P(\Gamma)$, giving $k \leqq \nu$ and thus $k = \nu$. The converse follows similarly in that $(\Gamma - a)^iQ(\Gamma)F$ is in the ideal if $i \geqq k$ and thus $\partial_{ji}F|_{V_k} = 0$ for all j, and $(\Gamma - a)^{k-1}Q(\Gamma)F$ is not in the ideal giving $\partial_j(\Gamma - a)^{k-1}Q(\Gamma)F|_{V_j} \neq 0$ for some j and thus $\partial_{j,k-1}F|_{V_j} \neq 0$. This completes the proof.

We now apply the above theorem to obtain a criterion for multiplicity in $R_F(\Gamma)$ in the case in which the generating polynomials depend linearly on Γ which does not explicitly depend on a representation for the multiplicity variety but rather on the coefficient matrix discussed in Theorem 1.1(i). First observe upon letting $G(X, \Gamma) = \sum_{i=1}^m b_i(\Gamma)\omega_l^{(i)}$ where $\omega_l^{(i)}$ are the monomials in (X) of degree l in a specified order, that if ∂ is a differential operator with differentiations in (X) alone, then

$$\partial G(X, \Gamma)|_{(\xi,a)} = \sum_{i=1}^{n} b_i(\Gamma)\partial\omega_l^{(i)}|_{(\xi,a)} = \langle \mathbf{G}, \boldsymbol{\partial}\rangle,$$

where $\mathbf{G} = (b_1(a), \ldots, b_m(a))$ and $\boldsymbol{\partial} = (\partial\omega_l^{(1)}|_{(\xi)}, \ldots, \partial\omega_l^{(m)}|_{(\xi)})$. Note that $\mathbf{G}$ depends on a; $\boldsymbol{\partial}$ depends on (ξ); and both depend on l and the order for $\omega_l^{(i)}$. This dependence is clear in context and thus is suppressed in the notations $\mathbf{G}$ and $\boldsymbol{\partial}$. Also we use the notation $H^{(p)}$ for the $p \times p$ matrix with ones in the superdiagonal and zeros elsewhere.

THEOREM 2.5. *Let $(F_1, \ldots, F_r)$ be an ideal with F_i in $H\Omega_\Gamma[X]$ where each of the F_i have as coefficients linear polynomials in Γ and* $\deg F_i = l_i$. *Suppose $R_l(\Gamma) = (\Gamma - a)^\nu Q(\Gamma)$, $Q(a) \neq 0$, with $R_l(0) \neq 0$. Then the coefficient matrix of the polynomials $\omega^{(j)}{}_{l-l_i}F_i$ relative to a fixed order for $\omega_l^{(i)}$ can be transformed by linear combinations of its rows to the form*

$$\left[\frac{1_{m\times m}}{0_{m'\times m}}\right] + \Gamma\left[\frac{(\beta)_{m\times m}}{(\beta^*)_{m'\times m}}\right].$$

Further, letting $(\alpha(\Gamma)) = 1_{m\times m} + \Gamma(\beta)$, suppose $\boldsymbol{\delta}_i$ are column vectors such that

$(\alpha(a))\mathbf{\delta}_i = \mathbf{\delta}_{i+1}$, $0 \leqq i \leqq \nu - 2$; and $(\alpha(a))\mathbf{\delta}_{\nu-1} = 0$ with no vector other than $\mathbf{\delta}_{\nu-1}$ being a null vector in the range of $(\alpha(a))$. Then for any polynomial $F(X)$ in (X) alone, $\mathbf{F}$ is orthogonal to $\mathbf{\delta}_i$, $\mu \leqq i \leqq \nu - 1$ with $\mu \geqq 1$ but not orthogonal to $\mathbf{\delta}_{\mu-1}$ if and only if a is a zero of $R_F(\Gamma)$ of multiplicity μ.

Proof. As the F_i are linear in Γ and $R_l(0) \neq 0$ the coefficient matrix for degree l is of the form $(\rho) + \Gamma(\sigma)$ with (ρ) of maximal rank. Thus there is a unit (τ) such that

$$(\tau)(\rho) = \left[\frac{1_{m\times m}}{0_{m'\times m}}\right]$$

and $(\tau)((\rho) + \Gamma(\sigma))$ is of the desired form. We can take a to be 1 by formally substituting $a\Gamma$ for Γ and $(1/a)(\sigma)$ for (σ). Let $\bar{V}$ be a multiplicity variety for $(F_1, \ldots, F_r)$. Because $(\Gamma - 1)^\nu | R_l(\Gamma)$ we must have a component (V, ∂) in $\bar{V}$ with $(t(\xi), 1) \subset V$ and

$$\partial = \sum_{i=0}^{k} (1/i!)\partial_i \frac{\partial^i}{\partial \Gamma^i},$$

∂_i differentiations in (X) alone and $k \geqq \nu - 1$. Further, for degree l, $\mathbf{\partial}_i = 0$ for $i \geqq \nu$ and $\mathbf{\partial}_{\nu-1} \neq 0$ by Theorem 3.3. As all the polynomials of the form $\omega^{(j)}{}_{l-l_i}F_i$ are in the ideal we have $\partial\omega^{(j)}{}_{l-l_i}F_i|_V = 0$ or equivalently

$$(\alpha(1))\mathbf{\partial}_0 + (\beta)\mathbf{\partial}_1 = \mathbf{\partial}_0 + (\beta)\mathbf{\partial}_0 + (\beta)\mathbf{\partial}_1 = 0.$$

In general, from $\partial(\Gamma - 1)^m\omega^{(j)}{}_{l-l_i}F_i|_V = 0$, $m \leqq \nu - 2$ we obtain

$$(\alpha(1))\mathbf{\partial}_m + (\beta)\mathbf{\partial}_{m+1} = \mathbf{\partial}_m + (\beta)\mathbf{\partial}_m + (\beta)\mathbf{\partial}_{m+1} = 0.$$

From $\partial(\Gamma - 1)^{\nu-1}\omega^{(j)}{}_{l-l_i}F_i|_V = 0$ and $\mathbf{\partial}_\nu = 0$ we have

$$(\alpha(1))\mathbf{\partial}_{\nu-1} = \mathbf{\partial}_{\nu-1} + (\beta)\mathbf{\partial}_{\nu-1} = 0.$$

Using $(\beta)\mathbf{\partial}_{\nu-1} = -\mathbf{\partial}_{\nu-1}$ in $(\alpha(1))\mathbf{\partial}_{\nu-2} + (\beta)\mathbf{\partial}_{\nu-1} = 0$, we obtain $(\alpha(1))\mathbf{\partial}_{\nu-2} = \mathbf{\partial}_{\nu-1}$. Thus it follows that $(\beta)\mathbf{\partial}_{\nu-2} = \mathbf{\partial}_{\nu-1} - \mathbf{\partial}_{\nu-2}$ and that $(\alpha(1))\mathbf{\partial}_{\nu-3} = \mathbf{\partial}_{\nu-2} - \mathbf{\partial}_{\nu-1}$ and in general

$$(\alpha(1))\mathbf{\partial}_i = \sum_{j=i+1}^{\nu-1} (-1)^{i+j+1}\mathbf{\partial}_j.$$

Defining

$$\mathbf{d}_i = \sum_{j=i}^{\nu-2} \binom{\nu - i - 2}{j - i}\mathbf{\partial}_j, \quad \text{if } i < \nu - 1 \text{ and } \mathbf{d}_{\nu-1} = \mathbf{\partial}_{\nu-1}$$

we have

$$\begin{aligned}(\alpha(1))\mathbf{d}_i &= \sum_{j=i}^{\nu-2} \binom{\nu - i - 2}{j - i}(\alpha(1))\mathbf{\partial}_j \\ &= \sum_{j=i}^{\nu-2} \binom{\nu - i - 2}{j - i}\left(\sum_{k=j+1}^{\nu-1} (-1)^{j+k+1}\mathbf{\partial}_k\right) \\ &= \sum_{k=i+1}^{\nu-1} (-1)^k\left(\sum_{j=i}^{k-1} (-1)^{j+1}\binom{\nu - i - 2}{j - i}\right)\mathbf{\partial}_k.\end{aligned}$$

As

$$\sum_{j=i}^{k-i} (-1)^{j+1}\binom{\nu-i-2}{j-i} = (-1)^k\binom{\nu-i-3}{k-i-1} \quad \text{for } k < \nu - 1$$

equals 0 if $k = \nu - 1$, we have

$$(\alpha(1))\mathbf{d}_i = \sum_{k=i+1}^{\nu-2}\binom{\nu-i-3}{k-i-1}\boldsymbol{\partial}_k = \mathbf{d}_{i+1}$$

and $(\alpha(1))\mathbf{d}_{\nu-1} = 0$.

Now suppose $\mathbf{D}_i$ are vectors such that $(\alpha(1))\mathbf{D}_i = \mathbf{D}_{i+1}$, $1 \leqq i \leqq \nu - 2$, and $(\alpha(1))\mathbf{D}_{\nu-1} = 0$. Then for some choice of λ_i with $\lambda_i = 0$ for $i > \nu - 1$ and $\lambda_{\nu-1} \neq 0$ we have

$$\mathbf{D}_i = \sum_{i=0}^{\nu-1} \lambda_{(\nu-1)+i-j}\boldsymbol{\delta}_j, \quad 0 \leqq i \leqq \nu - 1,$$

where a null vector $\mathbf{v}$ of $(\alpha(1))$ may be added to $\mathbf{D}_0$. This can be seen in that $\{\mathbf{D}_{\nu-1}, \mathbf{D}_{\nu-2}, \ldots, \mathbf{D}_0\}$ is the most general basis preserving $H^{(\nu)}$ (by direct computation or solving $XH^{(\nu)} = H^{(\nu)}X$ for X general nonsingular matrix). Then $(\alpha(1))(\mathbf{D}_0 + \mathbf{v}) = \mathbf{D}_1$ and yet a null vector of $(\alpha(1))$ can be added to no other $\mathbf{D}_i$ as multiples of $\boldsymbol{\delta}_{\nu-1}$ are the only null vectors in the range of $(\alpha(1))$. Hence for some choice of λ_i and $\mathbf{v}$, $\mathbf{D}_i = \mathbf{d}_i$, $0 \leqq i \leqq \nu - 1$.

Observe now that the matrices relating $\mathbf{D}_i$ to $\boldsymbol{\delta}_j$ and $\mathbf{d}_i$ to $\boldsymbol{\partial}_j$ are nonsingular upper triangular matrices thus giving

$$\boldsymbol{\partial}_i = \sum_{j=i}^{\nu-1} c_{ij}\boldsymbol{\delta}_j \quad \text{with } c_{ii} \neq 0.$$

Hence $\mathbf{F}$ is orthogonal to $\boldsymbol{\delta}_i$, $\mu \leqq i \leqq \nu - 1$, but not to $\boldsymbol{\delta}_{\mu-1}$ if and only if $\mathbf{F}$ is orthogonal to $\boldsymbol{\partial}_i$, $\mu \leqq i \leqq \nu - 1$, but not to $\boldsymbol{\partial}_{\mu-1}$. Suppose now that ∂ in the form given above is any differential in $\partial\{a\}$. An analysis similar to the above applies. That is, if p is the largest integer such that $\boldsymbol{\partial}_p \neq 0$ then p must be less than ν and $\boldsymbol{\partial}_p$ is a null vector in the range of $(\alpha(1))$ and must thus be a multiple of $\boldsymbol{\delta}_{\nu-1}$. It follows as above that

$$\boldsymbol{\partial}_i = \sum_{j=(\nu-1)-p+i}^{\nu-1} c_{ij}\boldsymbol{\delta}_j, \quad c_{ii} \neq 0.$$

Hence $\mathbf{F}$ is orthogonal to $\boldsymbol{\partial}_i$ for $\mu \leqq i \leqq \nu - 1$ where $\boldsymbol{\partial}_{p+1} = \ldots = \boldsymbol{\partial}_{\nu-1} = 0$ Thus F satisfies $\partial_i F|_V = 0$, $i \geqq \mu$, for any component (V, ∂) with ∂ in $\partial\{a\}$ and there exists a differential operator in $\partial\{a\}$ such that $\partial_{\mu-1}F|_V \neq 0$ if and only if $\mathbf{F}$ is orthogonal to $\boldsymbol{\delta}_i$, $\mu \leqq i \leqq \nu - 1$, but not to $\boldsymbol{\delta}_{\mu-1}$. The result follows from Theorem 2.4.

3. Concluding remarks and example. The restriction in Theorem 2.5 that Γ be linearly embedded in the polynomials F_i can be lifted with a slight modification of the proof required. This follow in that in the general case as

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well, the coefficient matrix acts on $\boldsymbol{\partial}_i$ to give a linear combination of the form

$$\sum_{j=i+1}^{\nu-1} \gamma_{ij}\boldsymbol{\partial}_j, \quad \gamma_{i,i+1} \neq 0,$$

with the proof continuing from that point essentially as before. Further, if the polynomial F is not in (X) alone but has Γ-dependence, the proof of Theorem 2.5 reveals that conditions are then required on F and on $\partial^i F/\partial \Gamma^i$ at $\Gamma = a$. As we are primarily interested in the linear case, we shall not pursue these or other possible generalizations at this time, but conclude with the following example.

Example. We consider the example given in the first section with $F_i = (1 - \Gamma)X_i^d + \Gamma(X_1 + X_2)^d$, $i = 1, 2$, for which $(1 - \Gamma)^2|R(\Gamma)$ when d is even, while $R(\Gamma)$ has only simple roots when d is odd. We examine the case when d is even and investigate conditions on F equivalent to $R_F(\Gamma)$ having only simple roots, when F is homogeneous of degree $2d - 1$ and in (X) alone. The coefficient matrix for (F_1, F_2) of degree $2d - 1$ relative to the order $X_1^{2d-1}, X_1^{2d-2}X_2, \ldots, X_2^{2d-1}$ with rows listed in the order $X_1^{d-1}F_1, X_1^{d-2}X_2F_1, \ldots, X_2^{d-1}F_1, X_1^{d-1}F_2, \ldots, X_2^{d-1}F_2$ is of the form described in Theorem 2.5 by $(\alpha(\Gamma))$. Furthermore routine calculations give that $\boldsymbol{\delta}_1 = (1, -1, 1, -1, \ldots, 1, -1)$ is, up to a multiple, the only null vector in the range of $(\alpha(1))$. Thus by Theorem 2.5, $R_F(\Gamma)$ has simple roots if and only if $\mathbf{F}$ is orthogonal to $\boldsymbol{\delta}_1$. As $(X_1^{2d-1}, X_1^{2d-2}X_2, \ldots, X_2^{2d-2})$ at $(1, -1)$ equals $\boldsymbol{\delta}_1$ the criterion becomes simply that $F(X)$ vanish on $(1, -1)$ and thus is applicable for $F(X)$ of arbitrary degree $\geqq 2d - 1$.

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