INVERSE SEMIGROUPS WITH CERTAIN TYPES OF PARTIAL AUTOMORPHISM MONOIDS

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0. Abstract. For an inverse semigroup S, the set of all isomorphisms between inverse subsemigroups of S is an inverse monoid under composition which is denoted by $\mathcal{PA}(S)$ and called the partial automorphism monoid of S. Kirkwood [**7**] and Libih [**8**] determined which groups have Clifford partial automorphism monoids. Here we investigate the structure of inverse semigroups whose partial automorphism monoids belong to certain other important classes of inverse semigroups. First of all, we describe (modulo so called "exceptional" groups) all inverse semigroups S such that $\mathcal{PA}(S)$ is completely semisimple. Secondly, for an inverse semigroup S, we find a convenient description of the greatest idempotent-separating congruence on $\mathcal{PA}(S)$, using a well-known general expression for this congruence due to Howie, and describe all those inverse semigroups whose partial automorphism monoids are fundamental.

1. Introduction. Let S be an inverse semigroup. For any subset X of S, let $\langle X\rangle$ denote the inverse subsemigroup generated by X. To express the fact that a subset $H \subseteq S$ is an inverse subsemigroup of S, we write $H \leqslant S$. It will be assumed that $\emptyset \leqslant S$. The semilattice of idempotents of S will be denoted by E_S and the order of an element $x \in S$ by $o(x)$. A *partial automorphism* of S is any isomorphism between inverse subsemigroups of S. The set of all partial automorphisms of S is denoted by $\mathcal{PA}(S)$. Note that $\emptyset \in \mathcal{PA}(S)$ since $\emptyset$ can be considered as an isomorphism of the empty inverse subsemigroup of S onto itself. It is easy to see that with respect to composition $\mathcal{PA}(S)$ is an inverse semigroup; moreover, it is an inverse subsemigroup of $\mathcal{I}(S)$, the symmetric inverse semigroup on the set S. In particular, the idempotents of $\mathcal{PA}(S)$ are precisely the identity mappings ι_H for every $H \leqslant S$. Clearly $\iota_\emptyset(=\emptyset)$ is the zero and ι_S the identity of $\mathcal{PA}(S)$. Thus $\mathcal{PA}(S)$ is an inverse monoid with zero; it is called the *partial automorphism monoid of* S. The group of units of $\mathcal{PA}(S)$ is the automorphism group of S. Since $\iota_H \circ \iota_K = \iota_{H\cap K}$ for any H, $K \leqslant S$, the semilattice of idempotents of $\mathcal{PA}(S)$ is, in fact, a lattice isomorphic to the lattice of all inverse subsemigroups of S.

In [**3**] the author studied conditions under which an inverse semigroup S is determined by $\mathcal{PA}(S)$. Here we will investigate how certain natural restrictions imposed on $\mathcal{PA}(S)$ reflect upon the structure of S. It should be noted that a similar problem for lattices of substructures of mathematical structures such as groups, rings, semigroups, inverse semigroups, etc., was the topic of numerous publications (see, for example, [**14**, Chapter I], [**12**, Chapter V] or [**13**, Chapter III]). However relatively few results on the above-mentioned problem for partial automorphism monoids have been established (see [**4**] for a brief survey). In fact, the author is aware of only one result which describes how the structure of an *inverse semigroup* S is influenced by imposing a certain restriction on $\mathcal{PA}(S)$: the determination by Kirkwood [**7**] of the structure of all *groups* whose partial automorphism monoids are Clifford semigroups. Two years earlier Libih [**8**] described all the *groups* G for which the inverse semigroup of isomorphisms between *subsemigroups of*

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G is Clifford. It turns out that the two classes of groups coincide (see [**4**, Theorem 3.3]). Moreover, the results of Libih and Kirkwood can be easily generalized to a description of all *inverse semigroups* S for which the inverse semigroups of all isomorphisms between the [inverse] subsemigroups of S are Clifford [**4**, Corollary 3.4].

In addition to the class of Clifford semigroups, some of the other important special classes of inverse semigroups consist of fundamental, E-unitary and completely semisimple ones. The main result of Section 2 (Theorem 5) provides a description of all inverse semigroups with completely semisimple partial automorphism monoids. In Section 3 we investigate how the requirements of being fundamental or E-unitary (or a somewhat weaker requirement than the latter), imposed on $\mathcal{PA}(S)$, influence the structure of an inverse semigroup S. In our main result here (Theorem 8) we find a convenient description of the greatest idempotent-separating congruence on $\mathcal{PA}(S)$, using a well-known general expression for this congruence due to Howie, and characterize all those inverse semigroups whose partial automorphism monoids are fundamental. The main results of the paper were announced at the 1988 Lisbon Conference on Universal Algebra, Lattices and Semigroups; they are stated without proofs in [**4**].

2. Restricting the ideal structure of $\mathcal{PA}(S)$. Green's relations on a semigroup provide one of the most important tools for clarifying its structure. A useful description of Green's relations on any inverse subsemigroup of the symmetric inverse semigroup on a set was given by Munn [**10**]. Since $\mathcal{PA}(S) \leqslant \mathcal{I}(S)$ for an inverse semigroup S, we can apply Munn's result to $\mathcal{PA}(S)$.

Result 1 (a corollary to Munn [**10**, Lemma 1.2]). *Let S be an inverse semigroup and let $\alpha, \beta \in \mathcal{PA}(S)$. Then*

(i) $(\alpha, \beta) \in \mathcal{L}$ *iff* $\operatorname{ran} \alpha = \operatorname{ran} \beta$;

(ii) $(\alpha, \beta) \in \mathcal{R}$ *iff* $\operatorname{dom} \alpha = \operatorname{dom} \beta$;

(iii) $(\alpha, \beta) \in \mathcal{D}$ *iff there exists* $\gamma \in \mathcal{PA}(S)$ *such that* $\operatorname{dom} \alpha = \operatorname{dom} \gamma$ *and* $\operatorname{ran} \gamma = \operatorname{ran} \beta$;

(iv) $(\alpha, \beta) \in \mathcal{J}$ *iff there exist* $\gamma, \delta \in \mathcal{PA}(S)$ *such that* $\operatorname{dom} \alpha = \operatorname{dom} \gamma$, $\operatorname{ran} \gamma \leqslant \operatorname{ran} \beta$, $\operatorname{dom} \beta = \operatorname{dom} \delta$, *and* $\operatorname{ran} \delta \leqslant \operatorname{ran} \alpha$. *In particular, $\mathcal{PA}(S)$ is 0-simple iff for all nonempty $\alpha, \beta \in \mathcal{PA}(S)$, there exists $\gamma \in \mathcal{PA}(S)$ such that* $\operatorname{dom} \alpha = \operatorname{dom} \gamma$ *and* $\operatorname{ran} \gamma \leqslant \operatorname{ran} \beta$.

An immediate consequence of Result 1(iv) is the following.

Corollary 2. *Let S be an inverse semigroup. Then the following conditions are equivalent*:

(i) *$\mathcal{PA}(S)$ is 0-simple*;

(ii) *$\mathcal{PA}(S)$ is 0-bisimple*;

(iii) *$\mathcal{PA}(S)$ is completely 0-simple*;

(iv) *S is trivial (i.e. $|S| = 1$).*

Thus, if S is an inverse semigroup, the condition that $\mathcal{PA}(S)$ be 0-simple is so strong that it causes S to shrink to a singleton. More interesting results can be obtained if the restrictions imposed on the ideal structure of $\mathcal{PA}(S)$ are not so severe.

Let U be an inverse semigroup. It is well known that any principal factor of U is either 0-simple or simple. If each of the principal factors of U is either completely 0-simple or completely simple, then U is said to be *completely semisimple* [**2**, §6.6]. It is also well known (see, for example, [**5**, Result 6]) that U is completely semisimple iff no two distinct

$\mathcal{D}$-related idempotents of U are comparable (with respect to the natural partial order). Using this fact and Result 1(iii), we obtain the following

LEMMA 3. *Let S be an inverse semigroup. Then $\mathcal{PA}(S)$ is completely semisimple iff for any pair of isomorphic inverse subsemigroups of S, neither of the two is properly contained in the other.*

Proof. As noted above, $\mathcal{PA}(S)$ is not completely semisimple iff there exist two distinct comparable $\mathcal{D}$-related idempotents of $\mathcal{PA}(S)$, that is, iff there exist $M, N \leqslant S$ such that

$$\iota_M \subset \iota_N \quad \text{and} \quad (\iota_M, \iota_N) \in \mathcal{D}. \tag{1}$$

By Result 1(iii), (1) holds iff $M \subset N$ and there exists an isomorphism of $M(=\operatorname{dom} \iota_M)$ onto $N(=\operatorname{ran} \iota_N)$. The proof is complete.

REMARK. Since the subject of this paper is the study of partial automorphism monoids of *inverse semigroups,* we formulated and proved Lemma 3 for an inverse semigroup S. However a much more general result is valid. Let (S, Σ) be any mathematical structure in the sense of Bourbaki (that is, S is a set endowed with a certain structure of the species Σ of structures; for a comprehensive discussion of these and related notions including that of an isomorphism between two sets endowed with the structures of the species Σ see [**1**, pp. 262–266 and 383–385]). Subsets of S, distinguished by a certain property $\tau = \tau(\Sigma)$ are called the *τ-subsets* of S. If the intersection of any two τ-subsets of S is again a τ-subset, then with respect to composition the set of all isomorphisms between τ-subsets of S is an inverse semigroup which is denoted by $\mathcal{PA}_\tau(S)$ and called the *partial τ-automorphism semigroup* of S (see [**4**]). The above-mentioned more general result is obtained from Lemma 3 by replacing the words "an inverse semigroup" and "inverse subsemigroups" by "a mathematical structure" and "τ-subsets", respectively, and the notation "$\mathcal{PA}(S)$" by "$\mathcal{PA}_\tau(S)$". This result is established by virtually the same proof as in Lemma 3.

LEMMA 4. *A semilattice is finite iff it has no pair of isomorphic subsemilattices one of which is properly contained in the other.*

Proof. The 'only if' part is obvious. To prove the 'if' part, take an arbitrary infinite semilattice E. There are two possibilities.

Case 1. Either ACC or DCC does not hold in E.

Suppose that E does not satisfy ACC [DCC]. Then E contains an infinite ascending [descending] chain $e_0 < e_1 < e_2 < \ldots$ [$e_0 > e_1 > e_2 > \ldots$]. Let $H = \{e_0, e_1, e_2, \ldots\}$ and $K = H \setminus \{e_0\}$. Then H and K are subsemilattices of E, $K \subset H$ and the mapping $e_n \mapsto e_{n-1}$, $n \in \mathbb{N}$, is an isomorphism of K onto H.

Case 2. E satisfies both ACC and DCC.

As usual, for $x \in E$, set $[x) = \{z \in E : z \geqslant x\}$. Recall that $y \in [x)$ is an *atom* of $[x)$ if y covers x, that is, for no $z \in E$, $y > z > x$.

Since E is a semilattice satisfying DCC, it contains a zero element 0, and for any nonmaximal $x \in E$, the set of atoms of $[x)$ is nonempty. Construct by induction the following sequence of elements of E. Set $a_0=0$. Note that $[a_0)$ coincides with E and hence is infinite. Suppose that for $n\geqslant 0$, we have already selected the elements $a_0,\ldots,a_n\in E$ such that $[a_n)$ is infinite. If $[a_n)$ has only finitely many atoms, we can choose one of them, b, say, such that $[b)$ is infinite, and set $a_{n+1}=b$; otherwise $a_{n+1}=a_n$. It is clear that $a_0\leqslant a_1\leqslant a_2\leqslant\ldots$. Since E satisfies ACC, there exists $m\geqslant 0$ such that $a_0<\ldots<a_m=a_{m+1}=\ldots$. By construction, the set A of atoms of $[a_m)$ is infinite. Let $H=A\cup\{a_m\}$. Choose any $a\in A$ and set $K=H\backslash\{a\}$, so that $K\subset H$. It is clear that H and K are subsemilattices of E. They both are primitive semilattices of the same cardinality and hence are isomorphic.

Thus, in each of the two possible cases, E has a pair of isomorphic subsemilattices one of which is properly contained in the other. This completes the proof.

For convenience of reference we distinguish a class of groups in which no subgroup is isomorphic to a proper subgroup of itself and call such groups *exceptional.* It is clear that every subgroup of an exceptional group is itself exceptional and that the class of exceptional groups is properly contained in the class of all periodic groups. Examples of exceptional groups include, of course, all finite groups, quasicyclic groups, direct sums of quasicyclic groups, infinite *nonabelian* groups all of whose subgroups are finite **[11]**, etc. However the author is not aware of any complete description of exceptional groups, nor of any publications in which these groups were distinguished and studied as a class. It might be of interest to note that exceptional groups form a proper subclass of the class of cohopfian groups (a group is called *cohopfian* [**9**, §V. 10] if it is not isomorphic to any of its proper subgroups). It is clear that a group is exceptional precisely when each of its subgroups is cohopfian.

The main result of this section is the following.

THEOREM 5. *Let S be an inverse semigroup. Then $\mathcal{PA}(S)$ is completely semisimple iff E_S is finite and all (maximal) subgroups of S are exceptional.*

Proof. Suppose that $\mathcal{PA}(S)$ is completely semisimple. By Lemma 3, S has no pair of isomorphic inverse subsemigroups one of which is properly contained in the other. Then, in particular, each (maximal) subgroup of S is exceptional and, according to Lemma 4, E_S is finite.

Conversely, let E_S be finite and let every (maximal) subgroup of S be exceptional. Suppose that $\mathcal{PA}(S)$ is not completely semisimple. Then, by Lemma 3, there exist M, $N\leqslant S$ such that $N\subset M$ and $M\cong N$. Note that E_M is finite, $E_N\subseteq E_M$ and $E_N\cong E_M$. It follows that $E_N=E_M$. Let α be an isomorphism of M onto N. Since E_M is finite and $\alpha|_{E_M}$ is an automorphism of E_M, there exists $k\geqslant 1$ such that $(\alpha|_{E_M})^k=\iota_{E_M}$. Set $P=M\alpha^k$ and $\beta=\alpha^k$. Then P is a proper inverse subsemigroup of M and β an isomorphism of M onto P such that $e\beta=e$ for every $e\in E_M$ $(=E_P)$.

To distinguish the Green's relations on M from those on P, we use the superscripts M and P, respectively. Take any $a\in M\backslash P$ and set $b=a\beta$. Then for some $e,f\in E_M$, $e\mathcal{R}^M a\mathcal{L}^M f$ and hence $e\beta\mathcal{R}^P a\beta\mathcal{L}^P f\beta$, that is, $e\mathcal{R}^P b\mathcal{L}^P f$. It follows that $H_b^P\subset H_b^M=H_a^M$. Note that $H_x^M\beta=H_{x\beta}^P$ for any $x\in M$. Since $e\beta=e$, we conclude that $\beta\mid H_e^M$ is an isomorphism of H_e^M onto $H_e^P(\subseteq H_e^M)$. However, H_e^S is an exceptional group. Therefore, $H_e^M=H_e^P$. Since $e\mathcal{R}^P b$, there exists $s\in P$ such that $es=b$. For any $x\in H_e^P$, set $x\gamma=xs$. By Green's lemma, γ is a bijection of H_e^P onto H_b^P and, at the same time, a bijection of

$H_e^M (= H_e^P)$ onto H_b^M. Thus $H_b^P = H_b^M$; a contradiction. Therefore $\mathcal{PA}(S)$ is completely semisimple.

Recall that an inverse semigroup is said to be *combinatorial* if all its subgroups are trivial. An immediate consequence of Theorem 5 is the following.

COROLLARY 6. *Let S be an inverse semigroup. If all subgroups of S are finite (in particular, if S is combinatorial), then $\mathcal{PA}(S)$ is completely semisimple iff S is finite.*

3. The greatest idempotent-separating congruence and the fundamentality of $\mathcal{PA}(S)$. Let U be an inverse semigroup. A nonempty subset A of U is called *closed* if A contains every element $u \in U$ such that $u \geq a$ for some $a \in A$, and U is said to be *E-unitary* (also *proper* or *reduced*) if E_U is a closed subset of U (see, for example, [**6**, pp. 139 and 181–182]). It is obvious that an inverse semigroup with zero is E-unitary iff it is a semilattice. Now let S be an inverse semigroup such that $\alpha^2 = \alpha$ for every $\alpha \in \mathcal{PA}(S)$. It follows immediately that no two distinct inverse subsemigroups of S are isomorphic. Therefore S is a periodic group. Moreover, for every $H \leq S$, ι_H is the only automorphism of H. It follows that S is a cyclic group of order ≤ 2 (cf. the argument used below in the proof of Proposition 7; see also [**8**, pp. 26–27]). Thus for an inverse semigroup S, *$\mathcal{PA}(S)$ is E-unitary iff $\mathcal{PA}(S)$ is a semilattice iff S is a cyclic group of order* ≤ 2. One might hope that the situation will become less trivial if the requirement that $\mathcal{PA}(S)$ be E-unitary (i.e., that $E_{\mathcal{PA}(S)}$ be closed) is replaced by a less restrictive one that $E_{\mathcal{PA}(S)} \setminus \{\emptyset\}$ be a closed subset of $\mathcal{PA}(S)$. It turns out, however, that even this assumption severely limits the structure of S.

PROPOSITION 7. *Let S be an inverse semigroup. Then $E_{\mathcal{PA}(S)} \setminus \{\emptyset\}$ is a closed subset of $\mathcal{PA}(S)$ iff S is a Clifford semigroup with at most two idempotents in which each maximal subgroup is a cyclic group of order* ≤ 2.

Proof. Let $E_{\mathcal{PA}(S)} \setminus \{\emptyset\}$ be a closed subset of $\mathcal{PA}(S)$. Suppose that $|E_S| \geq 3$. Then we can choose three distinct idempotents e, f and g in S such that $e < f$ and $e < g$. Set $e\alpha = e$ and $f\alpha = g$. Then α, being an isomorphism of $\{e, f\}$ onto $\{e, g\}$, belongs to $\mathcal{PA}(S)$, $\iota_{\{e\}} \subset \alpha$ and $\alpha^2 \neq \alpha$; a contradiction. Hence E_S is a chain of cardinality ≤ 2 and, by [**2**, Theorem 7.5], S is a Clifford semigroup. Let $e \in E_S$ and H_e be the group component of S corresponding to e (that is, the $\mathcal{H}$-class of S containing e). Take any $x \in H_e$ and assume that $o(x) > 2$. Then $\langle x \rangle$, the cyclic subgroup of H_e generated by x, has a nontrivial automorphism α defined by $x\alpha = x^{-1}$. Thus $\iota_{\{e\}} \subset \alpha \neq \alpha^2$; a contradiction. It follows that $o(x) \leq 2$. Suppose that H_e contains a pair of distinct elements, y and z, say, each of order 2. Set $e\beta = e$ and $y\beta = z$. Then β is an isomorphism of $\{e, y\}$ onto $\{e, z\}$ and thus belongs to $\mathcal{PA}(S)$, $\iota_{\{e\}} \subset \beta$ and $\beta^2 \neq \beta$; a contradiction. Hence H_e is a cyclic group of order ≤ 2.

The converse is easily established by considering a Clifford semigroup S with exactly two idempotents and each maximal subgroup being a cyclic group of order 2 and verifying that no $\alpha \in \mathcal{PA}(S) \setminus E_{\mathcal{PA}(S)}$ is such that for some $x \in \operatorname{dom} \alpha$, $\iota_{\langle x \rangle} = \alpha \mid \langle x \rangle$. We omit the details.

Let U be an inverse semigroup. A congruence ρ on U is called *idempotent-separating* if for any $e, f \in E_U$, $e \rho f$ implies $e = f$, and U is said to be *fundamental* if ι_U is the only idempotent-separating congruence on U [**6**, pp. 140–141]. In general, there exists a greatest idempotent-separating congruence μ on U given by the formula [**6**, Theorem

V.3.2]:

$$\mu = \{(a, b) \in U \times U : (\forall e \in E_U) a^{-1}ea = b^{-1}eb\}, \tag{2}$$

so that U is fundamental iff $\mu = \iota_U$. This characterization of fundamental inverse semigroups and formula (2) will be applied to the case when $U = \mathcal{PA}(S)$ for some inverse semigroup S. Let A be any subset of S, $x \in S$ and $\alpha \in \mathcal{PA}(S)$. In accordance with **[6]** (see formulae (4.8) and (4.9) on p. 15 and the definition of a partial mapping on p. 16) we have $A\alpha = (A \cap \operatorname{dom} \alpha)\alpha$ and $x\alpha = \{x\}\alpha$. As usual, if $x \in \operatorname{dom} \alpha$, we can identify $x\alpha$ with the unique element of $\{x\}\alpha$ (see [**6**, p. 16, a remark following (4.15)]). Since for all $\alpha \in \mathcal{PA}(S)$ and $H \leqslant S$, $\alpha^{-1} \circ \iota_H \circ \alpha = \iota_{H\alpha}$ (recall that $\emptyset\alpha = \emptyset$ and $\iota_\emptyset = \emptyset$), formula (2) implies that for any $\alpha, \beta \in \mathcal{PA}(S)$,

$$(\alpha, \beta) \in \mu \Leftrightarrow (\forall H \leqslant S)\ H\alpha = H\beta \tag{3}$$

and, in particular,

$$(\alpha, \beta) \in \mu \Rightarrow (\forall e \in E_S)\ e\alpha = e\beta. \tag{4}$$

If an element $x \in S$ does not belong to a subgroup of S, we say that x is a *nongroup* element. It is obvious that $x \in S$ is a nongroup element iff $\langle x \rangle$ is not a group.

We can now prove the main result of this section.

THEOREM 8. *Let S be an inverse semigroup. Then*

(i) *for any $\alpha, \beta \in \mathcal{PA}(S)$, $(\alpha, \beta) \in \mu$ iff $x\alpha = x\beta$ for each nongroup element x of S and $G\alpha = G\beta$ for every subgroup G of S*;

(ii) *$\mathcal{PA}(S)$ is fundamental iff each (maximal) subgroup of S is of exponent* $\leqslant 2$; *in particular, $\mathcal{PA}(S)$ is fundamental if S is combinatorial.*

Proof. (i) Let $\alpha, \beta \in \mathcal{PA}(S)$. Suppose that $(\alpha, \beta) \in \mu$. Then (3) implies that for every subgroup G of S, $G\alpha = G\beta$. Take an arbitrary nongroup element x of S. If $x \notin \operatorname{dom} \alpha \cup \operatorname{dom} \beta$, then $x\alpha = \emptyset = x\beta$. Now let $x \in \operatorname{dom} \alpha \cup \operatorname{dom} \beta$. Without loss of generality we may assume that $x \in \operatorname{dom} \alpha$, so that $\langle x \rangle \subseteq \operatorname{dom} \alpha$. Formula (3) implies that $\langle x\alpha \rangle = \langle x \rangle\alpha = \langle x \rangle\beta$. Then for some $y \in \langle x \rangle$, $x\alpha = y\beta$, and therefore $(xx^{-1})\alpha = (yy^{-1})\beta$ and $(x^{-1}x)\alpha = (y^{-1}y)\beta$. However by (4) we have $(xx^{-1})\alpha = (xx^{-1})\beta$ and $(x^{-1}x)\alpha = (x^{-1}x)\beta$. It follows that $xx^{-1} = yy^{-1}$ and $x^{-1}x = y^{-1}y$, that is, $x\mathcal{H}y$. Since $\langle x \rangle$ is not a group, from the structure of monogenic inverse semigroups it follows that either $\langle x \rangle$ is the bicyclic semigroup, or $\{x, x^{-1}, xx^{-1}, x^{-1}x\}$ is a $\mathcal{D}$-class in $\langle x \rangle$. In both cases the $\mathcal{D}$-class $D_x^{\langle x \rangle}$ is $\mathcal{H}$-trivial (i.e., each $\mathcal{H}$-class contained in $D_x^{\langle x \rangle}$ is a singleton). Since $y \in \langle x \rangle$ and $y\mathcal{H}x$, we have $x = y$, so that $x\alpha = x\beta$.

Conversely, assume that for any subgroup G of S and any nongroup element $x \in S$, we have $G\alpha = G\beta$ and $x\alpha = x\beta$. Let H be an arbitrary inverse subsemigroup of S. Suppose that $a \in H\alpha$. Hence $a = h\alpha$ for some $h \in H$. If h is a nongroup element, then by our assumption $h\alpha = h\beta$, so that $a \in H\beta$. Otherwise $\langle h \rangle$ is a cyclic subgroup of $H \cap \operatorname{dom} \alpha$. Again by the assumption $\langle h \rangle\alpha = \langle h \rangle\beta$. Hence for some integer n, $h\alpha = (h^n)\beta \in H\beta$, that is, $a \in H\beta$. Therefore $H\alpha \subseteq H\beta$, and similarly $H\beta \subseteq H\alpha$. Thus $H\alpha = H\beta$ and, according to (3), $(\alpha, \beta) \in \mu$.

(ii) Suppose that every subgroup of S is of exponent $\leqslant 2$. Let $\alpha, \beta \in \mathcal{PA}(S)$ be such that $(\alpha, \beta) \in \mu$. Take any $x \in S$. If x is an idempotent or a nongroup element, then $x\alpha = x\beta$ according to (4) or to (i), respectively. Now suppose that $x \neq x^2$ and $\langle x \rangle$ is a group. By our assumption $o(x)=2$, so that $\langle x\rangle=\{e,x\}$ for some $e\in E_S$. According to (i), $\{e,x\}\alpha=\{e,x\}\beta$. Suppose, first, that $x\in\operatorname{dom}\alpha$. Then $\{e,x\}\subseteq\operatorname{dom}\alpha$ and $|\{e,x\}\alpha|=|\{e\alpha,x\alpha\}|=2$. Hence $\{e,x\}\subseteq\operatorname{dom}\beta$ and $x\beta=x\alpha$. Now assume that $x\notin\operatorname{dom}\alpha$. Then $|\{e,x\}\alpha|\leq 1$. It follows that $x\notin\operatorname{dom}\beta$ and $x\alpha=\emptyset=x\beta$. Thus $\alpha=\beta$ and therefore $\mathcal{PA}(S)$ is fundamental.

Conversely, suppose that S contains an element a such that $\langle a\rangle$ is a group and $o(a)>2$, i.e. $a\neq a^{-1}$. Let $\alpha=\iota_{\langle a\rangle}$ and let β be an automorphism of $\langle a\rangle$ defined by $a\beta=a^{-1}$. Then $\alpha,\beta\in\mathcal{PA}(S)$, $\operatorname{dom}\alpha=\operatorname{dom}\beta=\langle a\rangle$ and $\operatorname{ran}\alpha=\operatorname{ran}\beta=\langle a\rangle$. Moreover, for any subgroup G of S, $G\alpha=(G\cap\langle a\rangle)\alpha=G\cap\langle a\rangle=(G\cap\langle a\rangle)\beta=G\beta$, and for any nongroup element $x\in S$, $x\alpha=\emptyset=x\beta$. Hence, according to (i), $(\alpha,\beta)\in\mu$. However $a\alpha\neq a\beta$, that is, $\alpha\neq\beta$. Therefore $\mathcal{PA}(S)$ is not fundamental.

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