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RANDERS METRICS OF SCALAR FLAG CURVATURE

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Abstract

We study an important class of Finsler metrics, namely, Randers metrics. We classify Randers metrics of scalar flag curvature whose S-curvatures are isotropic. This class of Randers metrics contains all projectively flat Randers metrics with isotropic S-curvature and Randers metrics of constant flag curvature.

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1. Introduction

In Finsler geometry, there are several important geometric quantities. In this paper, our main focus is on the flag curvature, the S-curvature and their interaction.

For a Finsler manifold (M, F), the flag curvature $\mathbf{K}$ at a point x is a function of tangent planes $P \subseteq T_x M$ and nonzero vectors $y \in P$. This quantity tells us how curved the space is. When F is Riemannian, $\mathbf{K}$ depends only on the tangent plane $P \subseteq T_x M$ and is just the sectional curvature in Riemannian geometry. Thus the flag curvature is the analogue of sectional curvature in Riemannian geometry (see [4, 6]). A Finsler metric F is said to be of *scalar flag curvature* if the flag curvature $\mathbf{K}$ at a point x is independent of the tangent plane $P \subseteq T_x M$, that is, the flag curvature $\mathbf{K}$ is a scalar function on the slit tangent bundle $TM \backslash \{0\}$. Throughout this paper we denote a point in M by x and a point in TM by (x, y), where $y \in T_x M$. A Finsler metric F is said to be of *almost isotropic flag curvature* if, in local coordinates,

$$\mathbf{K} = \frac{3c_m y^m}{F} + \sigma, \tag{1.1}$$

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where σ and c are scalar functions on M, and c_m denotes the partial derivative $\partial c/\partial x^m$. If $\mathbf{K} = \sigma$, we say that F is of *isotropic flag curvature*. In this case, σ is constant if the dimension $n \geq 3$, by Schur's lemma. One of the important problems in Finsler geometry is to study and characterize Finsler metrics of scalar, almost isotropic, and constant flag curvature [6, 9].

There is another important quantity closely related to the flag curvature, the so-called S-curvature $\mathbf{S}$. The S-curvature, a scalar function on TM, was introduced by the second author to study volume comparison in Riemann–Finsler geometry [12, 13]. The Finsler metric F is said to be of *isotropic S-curvature* if $\mathbf{S} = (n+1)cF$, where c is a scalar function on M. Further, if c is a constant, then F is said to be of *constant S-curvature*. It is proved that for a Finsler metric F of scalar flag curvature, if it is of isotropic S-curvature, then F must be of almost isotropic flag curvature with flag curvature in the form (1.1) (see [4]). Thus the flag curvature and the S-curvature are closely related.

Randers metrics were introduced by the physicist Randers in 1941 in the context of general relativity. These metrics were used in the theory of the electron microscope in 1957 by Ingarden, who first named them Randers metrics. Randers metrics form an important and ubiquitous class of Finsler metrics with a strong presence in both the theory and applications of Finsler geometry, and studying Randers metrics is an important step in understanding general Finsler metrics. A Randers metric on a manifold M is a Finsler metric that can be expressed in the following special form:

$$F = \alpha + \beta,$$

where α is a Riemannian metric and β is a 1-form on M such that the norm of β with respect to α satisfies $\|\beta\|_\alpha < 1$; in local coordinates, $\alpha = \sqrt{a_{ij}y^i y^j}$ and $\beta = b_i y^i$. Randers metrics also arise naturally from the navigation problem on a manifold M with a Riemannian metric h under the influence of an external force field W (in local coordinates, $h = \sqrt{h_{ij}y^i y^j}$ and $W = W^i \partial/\partial x^i$). The least time path from one point to another is a geodesic of the Randers metric F defined by

$$F = \frac{\sqrt{\lambda h^2 + W_0^2}}{\lambda} - \frac{W_0}{\lambda}, \tag{1.2}$$

where

$$W_0 := W_i y^i, \quad W_i := h_{ij} W^j, \quad \lambda := 1 - W_i W^i.$$

See [3, 14, 18]. It is easy to see that every Randers metric can be expressed in the form (1.2). The pair (h, W) is called the *navigation data* of F.

Our main result concerns Randers metrics of scalar flag curvature with isotropic S-curvature.

THEOREM 1.1. *Let F be a Randers metric on an n-dimensional manifold M given by (1.2) with navigation data (h, W). Assume that $n \geq 3$. Then F is of scalar flag*

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curvature and isotropic S-curvature (that is, $\mathbf{K} = K(x, y)$ *and* $\mathbf{S} = (n+1)c(x)F$*) if and only if at every point, there is a local coordinate system in which h, c and W are given by*

$$h = \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 + \mu|x|^2} \tag{1.3}$$

$$c = \frac{\delta + \langle a, x\rangle}{\sqrt{1 + \mu|x|^2}} \tag{1.4}$$

$$\begin{aligned} W = -2\bigg(\Big(\delta\sqrt{1+\mu|x|^2} + \langle a, x\rangle\Big)x - \frac{|x|^2 a}{\sqrt{1+\mu|x|^2}+1}\bigg) \\ + xQ + b + \mu\langle b, x\rangle x, \end{aligned} \tag{1.5}$$

where δ *and* μ *are constants, Q is a fixed antisymmetric matrix, and a and b in* $\mathbf{R}^n$ *are constant vectors. In this case, the flag curvature is given by*

$$\mathbf{K} = \frac{3c_m y^m}{F} + \sigma, \tag{1.6}$$

where $\sigma = \mu - c^2 - 2c_m W^m$.

Locally projectively flat Randers metrics are always of scalar flag curvature, hence Theorem 1.1 generalizes [4, Theorem 1.3], which classifies locally projectively flat Randers metrics of isotropic S-curvature. Furthermore, since every Einstein–Randers metric F must be of constant S-curvature [1, 2], the class of Randers metrics of scalar flag curvature with isotropic S-curvature contains all Randers metrics of constant flag curvature. Therefore, Theorem 1.1 also generalizes the classification theorem on Randers metrics of constant flag curvature [3].

Let us consider a special example. In (1.3)–(1.5), take $\mu = 0$, $\delta = 0$, $Q = 0$ and $b = 0$. Then

$$h = |y|, \quad c = \langle a, x\rangle, \quad W = -2\langle a, x\rangle x + |x|^2 a.$$

With the above navigation data (h, W), the Randers metric in (1.2) is given by

$$\begin{aligned} F = \frac{\sqrt{(1 - |a|^2|x|^4)|y|^2 + (|x|^2\langle a, y\rangle - 2\langle a, x\rangle\langle x, y\rangle)^2}}{1 - |a|^2|x|^4} \\ - \frac{|x|^2\langle a, y\rangle - 2\langle a, x\rangle\langle x, y\rangle}{1 - |a|^2|x|^4}. \end{aligned}$$

By direct computation, one can easily verify that F is of isotropic S-curvature and scalar flag curvature, and, more precisely,

$$\mathbf{S} = (n+1)cF, \quad \mathbf{K} = \frac{3c_m y^m}{F} + \sigma,$$

where $c = \langle a, x\rangle$ and $\sigma = 3\langle a, x\rangle^2 - 2|a|^2|x|^2$. This example was constructed by the second author in [15].

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2. Preliminaries

Consider a Randers metric F on a manifold M. We can express it in the form (1.2),

$$F = \frac{\sqrt{\lambda h^2 + W_0^2}}{\lambda} - \frac{W_0}{\lambda},$$

where h is a Riemannian metric and W is a vector field on M. The geodesics of F are characterized locally by a system of second order ordinary differential equations [6]:

$$\frac{d^2 x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where

$$G^i = \tfrac{1}{4} g^{il}\{[F^2]_{x^m y^l} y^m - [F^2]_{x^l}\}.$$

The G^i are called the *spray coefficients* of F.

Let $\nabla W = W_{i;j}\, dx^i \otimes dx^j$ denote the covariant derivative of W with respect to h. Let

$$\mathcal{R}_{ij} := \tfrac{1}{2}(W_{i;j} + W_{j;i}), \quad \mathcal{S}_{ij} := \tfrac{1}{2}(W_{i;j} - W_{j;i}),$$
$$\mathcal{R}_j := W^i \mathcal{R}_{ij}, \quad \mathcal{R} := W^j \mathcal{R}_j, \quad \mathcal{S}_j := W^i \mathcal{S}_{ij}.$$

Denote by $\bar{G}^i$ the spray coefficients of h. Then

$$G^i = \bar{G}^i - \frac{1}{2} F^2(\mathcal{S}^i + \mathcal{R}^i) - F\mathcal{S}^i{}_0 + \frac{1}{2F}(y^i - FW^i)(2\mathcal{R}_0 F - \mathcal{R}_{00} - \mathcal{R}F^2), \tag{2.1}$$

where

$$\mathcal{S}^i := h^{ij}\mathcal{S}_j, \quad \mathcal{R}^i := h^{ij}\mathcal{R}_j, \quad \mathcal{S}^i{}_j := h^{il}\mathcal{S}_{lj},$$
$$\mathcal{S}^i{}_0 := \mathcal{S}^i{}_j y^j, \quad \mathcal{R}_0 := \mathcal{R}_i y^i \quad \text{and} \quad \mathcal{R}_{00} := \mathcal{R}_{ij} y^i y^j.$$

Formula (2.1) may be found in [11].

Denote by dV_F the volume form $\sigma_F dx^1 \cdots dx^n$ of a Finsler metric F, where

$$\sigma_F := \frac{\mathrm{Vol}(\mathrm{B}^n(1))}{\mathrm{Vol}\{y \mid F(x, y) < 1\}}.$$

Here Vol denotes the Euclidean volume and $\mathrm{B}^n(1)$ denotes the unit ball in $\mathbf{R}^n$. Then the S-curvature $\mathbf{S}$ of F is given by

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m}(\log \sigma_F); \tag{2.2}$$

see [5, 6, 13]. The S-curvature $\mathbf{S}$ measures the average rate of change of $(T_x M, F_x)$ in the direction $y \in T_x M$. It is known that $\mathbf{S} = 0$ for Berwald metrics [12, 13].

Lemma 2.1 [17]. *Let F be a Randers metric given by (1.2) with navigation data (h, W) and c be a scalar function on an n-dimensional manifold. Then $\mathbf{S} = (n+1)cF$ if and only if*

$$\mathcal{R}_{00} = -2ch^2. \tag{2.3}$$

Assume that F has isotropic S-curvature, that is, $\mathbf{S} = (n+1)cF$. By Lemma 2.1, W satisfies (2.3). Then the spray coefficients G^i of F in (2.1) reduce to the expression

$$G^i = \bar{G}^i - F\mathcal{S}^i{}_0 - \tfrac{1}{2}F^2\mathcal{S}^i + cFy^i. \tag{2.4}$$

It is known that a Randers metric F given in the form (1.2) is of constant flag curvature (equal to σ) if and only if h has constant sectional curvature (equal to μ) and W is homothetic, that is, it satisfies (2.3) for a constant c. In this case, $\sigma = \mu - c^2$. Moreover, $c = 0$ if $\mu \neq 0$. This leads to the classification of Randers metrics of constant flag curvature [3]. See also [1, 2, 8] for some early work on Randers metrics of constant flag curvature. Thus a Randers metric of constant flag curvature must have constant S-curvature.

As we know, every Riemannian metric of constant sectional curvature μ is locally isometric to the following metric on the open ball $\mathrm{B}^n(r_\mu)$ in $\mathbf{R}^n$:

$$h = \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 + \mu|x|^2}. \tag{2.5}$$

Here

$$r_\mu := \begin{cases} +\infty & \text{if } \mu \geq 0 \\ 1/\sqrt{-\mu} & \text{if } \mu < 0. \end{cases}$$

For each point $x \in \mathrm{B}^n(r_\mu)$, we can identify the tangent vector $W^i\partial/\partial x^i|_x$ in $T_x\mathbf{R}^n$ with the vector $(W^i) \in \mathbf{R}^n$ in a canonical way.

The following lemma is important for the proof of Theorem 1.1.

Lemma 2.2 [16]. *Let h be the Riemannian metric in (2.5) and W be a vector field on the open ball $\mathrm{B}^n(r_\mu)$ in $\mathbf{R}^n$. Let F be the Randers metric on $\mathrm{B}^n(r_\mu)$ given by (1.2) with navigation data (h, W). Assume that $n \geq 3$. Then F has isotropic S-curvature, that is, $\mathbf{S} = (n+1)cF$ for some scalar function c, if and only if*

$$c = \frac{\delta + \langle a, x\rangle}{\sqrt{1 + \mu|x|^2}} \tag{2.6}$$

and W satisfying (2.3) is given by

$$W = -2\left\{\left(\delta\sqrt{1 + \mu|x|^2} + \langle a, x\rangle\right)x - \frac{|x|^2 a}{\sqrt{1 + \mu|x|^2} + 1}\right\} + xQ + b + \mu\langle b, x\rangle x, \tag{2.7}$$

where δ is a constant, Q is a fixed antisymmetric matrix and a and b in $\mathbf{R}^n$ are constant vectors.

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3. The Riemann curvature

Let F be a Finsler metric on a manifold M with spray coefficients G^i. The Riemann curvature R (in local coordinates $R^i{}_k \partial/\partial x^i \otimes dx^k$) is defined by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^m \partial y^k} y^m + 2G^m \frac{\partial^2 G^i}{\partial y^m \partial y^k} - \frac{\partial G^i}{\partial y^m}\frac{\partial G^m}{\partial y^k}. \tag{3.1}$$

It is known [6] that F is of scalar flag curvature if and only if, in a standard local coordinate system,

$$R^i{}_k = K(x, y)\{F^2 \delta^i{}_k - F F_{y^k} y^i\}. \tag{3.2}$$

From now on, we always assume that F is a Randers metric given by (1.2) with isotropic S-curvature, that is, $\mathbf{S} = (n+1)cF$. We will use (2.4) to express the Riemann curvature in terms of h and W.

Rewrite (2.4) as

$$G^i = \bar{G}^i + Q^i,$$

where

$$Q^i := -F\mathcal{S}^i{}_0 - \tfrac{1}{2}F^2\mathcal{S}^i + cFy^i.$$

Then

$$R^i{}_k = \bar{R}^i{}_k + 2Q^i{}_{;k} - [Q^i{}_{;m}]_{y^k} y^m + 2Q^m [Q^i]_{y^m y^k} - [Q^i]_{y^m}[Q^m]_{y^k}, \tag{3.3}$$

where $\bar{R} = \bar{R}^i{}_k \partial/\partial x^i \otimes dx^k$ denotes the Riemann curvature of h, and the semicolon denotes horizontal covariant differentiation with respect to h (see [6]). We first compute the horizontal and vertical derivatives of Q^i and express them in terms of h, W and the covariant derivatives of W with respect to h. Since W satisfies (2.3),

$$W_{i;j;k} = 2(c_i h_{jk} - c_j h_{ik} - c_k h_{ij}) - \bar{R}_{kpij} W^p, \tag{3.4}$$

where $\bar{R} = \bar{R}_{kpij}\, dx^k \otimes dx^p \otimes dx^i \otimes dx^j$ is the Riemann curvature tensor of h in a standard form. In fact, (2.3) was studied a long time ago [7]. Equation (3.4) is straightforward by the Ricci identity; see in [7, Equation (69.2)].

By (3.4) and the Bianchi identities for the Riemann curvature tensor $\bar{R}$ of h,

$$\mathcal{S}^i{}_{k;0} = 2(h^{im} c_m y_k - c_k y^i) - \bar{R}_k{}^i{}_{mq} W^m y^q,$$

$$\mathcal{S}^i{}_{0;k} = 2(h^{im} c_m y_k - c_m y^m \delta^i_k) + \bar{R}_p{}^i{}_{kq} y^p W^q,$$

$$\mathcal{S}^i{}_{;k} = 2c\mathcal{S}^i{}_k - \mathcal{S}^i{}_m \mathcal{S}^m_k + 2(c_m W^m \delta^i_k - h^{im} c_m W_k) - \bar{R}_p{}^i{}_{kq} W^p W^q,$$

$$\mathcal{S}^i{}_{;0} = 2c\mathcal{S}^i{}_0 - \mathcal{S}^i{}_m \mathcal{S}^m_0 + 2(c_m W^m y^i - h^{im} c_m W_0) - \bar{R}_p{}^i{}_{mq} W^p W^q y^m,$$

$$\mathcal{S}^i{}_{0;0} = 2(h^{im} c_m h^2 - c_{;0} y^i) - \bar{R}_p{}^i{}_{mq} y^p y^q W^m.$$

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Define $A := \sqrt{\lambda h^2 + W_0^2}$. Then $A = \lambda F + W_0$ from (1.2). It is easy to verify that

$$h^2 - 2FW_0 = \lambda F^2.$$

It follows that

$$h^2 - FW_0 - AF = 0. \tag{3.5}$$

Further, by (2.3) and (3.5),

$$F_{;k} = \frac{2cF(y_k - FW_k) + F(F\mathcal{S}_k + \mathcal{S}_{k0})}{A},$$

$$F_{;0} = 2cF^2 + \frac{F^2}{A}\mathcal{S}_0,$$

$$(F_{y^k})_{;0} = \left(\frac{h^2}{A^3}\mathcal{S}_0 + 2c\frac{F}{A}\right)\{y_k - FW_k\} - \frac{F^2}{A^2}\mathcal{S}_0 W_k - \frac{F}{A}\mathcal{S}_{k0}.$$

By (3.3) and the above identities,

$$\begin{aligned} R^i{}_k = {} & \bar{R}_p{}^i{}_{kq} y^p y^q - F\bar{R}_p{}^i{}_{kq} W^p y^q - F\bar{R}_p{}^i{}_{kq} y^p W^q + F^2 \bar{R}_p{}^i{}_{kq} W^p W^q \\ & - F_{y^k} \bar{R}_p{}^i{}_{mq} y^p y^q W^m + F F_{y^k} \bar{R}_p{}^i{}_{mq} y^q W^p W^m \\ & + \left(\frac{3c_m y^m}{F} - c^2 - 2c_m W^m\right)\{F^2\delta^i_k - FF_{y^k} y^i\}. \end{aligned} \tag{3.6}$$

Here we use `Maple` to do the computation of $R^i{}_k$. It is surprising that none of the terms with $\mathcal{S}^i$ or $\mathcal{S}^i{}_k$ occur in (3.6).

Observe that

$$\begin{aligned} & \bar{R}_p{}^i{}_{kq}(y^p - FW^p)(y^q - FW^q) \\ & \quad = \bar{R}_p{}^i{}_{kq} y^p y^q - F\bar{R}_p{}^i{}_{kq} y^p W^q - F\bar{R}_p{}^i{}_{kq} W^p y^q + F^2 \bar{R}_p{}^i{}_{kq} W^p W^q \end{aligned}$$

and

$$\begin{aligned} & \bar{R}_p{}^i{}_{mq}(y^p - FW^p)(y^q - FW^q)W^m \\ & \quad = \bar{R}_p{}^i{}_{mq} y^p y^q W^m - F\bar{R}_p{}^i{}_{mq} W^p y^q W^m \\ & \qquad - F\bar{R}_p{}^i{}_{mq} y^p W^q W^m + F^2 \bar{R}_p{}^i{}_{mq} W^p W^q W^m \\ & \quad = \bar{R}_p{}^i{}_{mq} y^p y^q W^m - F\bar{R}_p{}^i{}_{mq} W^p y^q W^m. \end{aligned}$$

Substituting these into (3.6), we deduce that

$$\begin{aligned} R^i{}_k = {} & \bar{R}_p{}^i{}_{kq}(y^p - FW^p)(y^q - FW^q) \\ & - F_{y^k} \bar{R}_p{}^i{}_{mq}(y^p - FW^p)(y^q - FW^q)W^m \\ & + \left(\frac{3c_m y^m}{F} - c^2 - 2c_m W^m\right)\{F^2\delta^i_k - FF_{y^k} y^i\}. \end{aligned} \tag{3.7}$$

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Let

$$\xi^i := y^i - F(x, y)W^i, \quad \xi_k := h_{ik}\xi^i, \quad \tilde{W}_0 := W_i\xi^i$$

and

$$\tilde{h} := h(x, \xi) = \sqrt{h_{pq}\xi^p\xi^q} = \sqrt{\xi_k\xi^k}.$$

Then

$$\tilde{h}^2 = h_{pq}(y^p - FW^p)(y^q - FW^q) = h^2 - 2FW_0 + F^2h(x, W)^2 = F^2.$$

Thus

$$y^i = \xi^i + \tilde{h}W^i.$$

Observe that

$$\lambda\tilde{h} = \lambda F = A - W_0 = A - W_i(\xi^i + \tilde{h}W^i) = A - \tilde{W}_0 - \tilde{h}(1 - \lambda).$$

This gives

$$A = \tilde{h} + \tilde{W}_0.$$

From the above identities,

$$F_{y^k} = \frac{1}{A}(y_k - FW_k) = \frac{\xi_k}{\tilde{h} + \tilde{W}_0},$$

$$F^2\delta^i_k - FF_{y^k}y^i = \tilde{h}^2\delta^i_k - \xi_k\xi^i - \frac{1}{\tilde{h} + \tilde{W}_0}\xi_k(\tilde{h}^2\delta^i_p - \xi_p\xi^i)W^p,$$

where $y_k := h_{ik}y^i$. Let

$$\tilde{R}^i{}_k := \bar{R}_p{}^i{}_{kq}\xi^p\xi^q.$$

The following lemma follows from (3.7).

LEMMA 3.1. *Let F be a Randers metric given by (1.2) with navigation data (h, W). Suppose that F has isotropic S-curvature, that is, $\mathbf{S} = (n + 1)cF$. Then for any scalar function μ on M,*

$$\begin{aligned} R^i{}_k - \left(\frac{3c_m y^m}{F} + \mu - c^2 - 2c_m W^m\right)\{F^2\delta^i_k - FF_{y^k}y^i\} \\ = \tilde{R}^i{}_k - \mu(\tilde{h}^2\delta^i_k - \xi_k\xi^i) - \frac{\xi_k}{\tilde{h} + \tilde{W}_0}\{\tilde{R}^i{}_p - \mu(\tilde{h}^2\delta^i_p - \xi_p\xi^i)\}W^p. \end{aligned} \tag{3.8}$$

4. The Ricci curvature

In this section we study the Ricci curvature of a Randers metric F with isotropic S-curvature. Express F by (1.2) with navigation data (h, W). Let Ric and $\overline{\mathrm{Ric}}$ denote the Ricci curvature of F and h, respectively. They are defined by

$$\mathrm{Ric} := R^m_{\ m}, \quad \overline{\mathrm{Ric}} := \bar{R}^m_{\ m}.$$

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Let

$$\widetilde{\mathrm{Ric}} := \tilde{R}^m_{\ m} = \bar{R}_{p\ mq}^{\ m}\xi^p\xi^q.$$

Clearly, $\overline{\mathrm{Ric}} = (n-1)\mu h^2$ if and only if $\widetilde{\mathrm{Ric}} = (n-1)\mu\tilde{h}^2$.

First we have the following lemma.

LEMMA 4.1. *Let F be the Randers metric given by (1.2) with navigation data (h, W). Suppose that F has isotropic S-curvature, that is, $\mathbf{S} = (n+1)cF$. Then for any scalar function μ on M,*

$$\mathrm{Ric} - (n-1)\left(\frac{3c_m y^m}{F} + \sigma\right)F^2 = \widetilde{\mathrm{Ric}} - (n-1)\mu\tilde{h}^2, \tag{4.1}$$

where $\sigma := \mu - c^2 - 2c_m W^m$.

PROOF. Observe that

$$\xi_m\tilde{R}^m_{\ p} = \xi_m\bar{R}_{i\ pj}^{\ m}\xi^i\xi^j = \xi^m\bar{R}_{impj}\xi^i\xi^j = 0$$

and

$$\xi_m(\tilde{h}^2\delta^m_p - \xi_p\xi^m) = \tilde{h}^2\xi_p - \xi_p\tilde{h}^2 = 0.$$

Then (4.1) follows from (3.8). □

From Lemma 4.1 we immediately obtain the following result.

THEOREM 4.2. *Let F be a Randers metric on an n-dimensional manifold M given by (1.2) with navigation data (h, W), and let c and μ be scalar functions on M. Suppose that $\mathbf{S} = (n+1)cF$. Then $\overline{\mathrm{Ric}} = (n-1)\mu h^2$ if and only if*

$$\mathrm{Ric} = (n-1)\left(\frac{3c_m y^m}{F} + \sigma\right)F^2. \tag{4.2}$$

where $\sigma := \mu - c^2 - 2c_m W^m$.

COROLLARY 4.3. *Let F be a Randers metric given by (1.2) with navigation data (h, W). If W is an infinitesimal homothety of h (or equivalently, if $\mathbf{S} = (n+1)cF$ for some constant c), and μ is a scalar function, then $\overline{\mathrm{Ric}} = (n-1)\mu h^2$ if and only if $\mathrm{Ric} = (n-1)(\mu - c^2)F^2$.*

Corollary 4.3 was proved in [2, Theorem 9]; see also [10]. In fact, Bao and Robles prove that, for a Randers metric F given by (1.2), F is Einstein (that is, $\mathrm{Ric} = (n-1)\sigma F^2$) if and only if $\mathbf{S} = (n+1)cF$ for some constant c and $\overline{\mathrm{Ric}} = (n-1)\mu h^2$ with $\sigma = \mu - c^2$.

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5. Proof of Theorem 1.1

In order to prove Theorem 1.1, we must prove the following theorem.

THEOREM 5.1. *Let F be a Randers metric on an n-dimensional manifold M given by (1.2) with navigation data (h, W). Suppose that the S-curvature is isotropic, that is, $\mathbf{S} = (n+1)cF$, where c is a scalar function on M. Then F is of scalar flag curvature if and only if the sectional curvature $\bar{\mathbf{K}}$ of h is a scalar, μ (μ is constant when $n \geq 3$). In this case, the flag curvature of F is given by*

$$\mathbf{K} = \frac{3c_m y^m}{F} + \sigma, \tag{5.1}$$

where $\sigma := \mu - c^2 - 2c_m W^m$.

PROOF. Assume that F is of scalar flag curvature. Then by [4, Theorem 1.1], the flag curvature is given by

$$\mathbf{K} = \frac{3c_m y^m}{F} + \sigma,$$

where σ is a scalar function on M. That is,

$$R^i{}_k = \left(\frac{3c_m y^m}{F} + \sigma\right)\{F^2\delta^i_k - FF_{y^k}y^i\}.$$

Let

$$\mu := \sigma + c^2 + 2c_m W^m.$$

It suffices to show that the sectional curvature $\bar{\mathbf{K}}$ of h is equal to μ. It follows from (3.8) that

$$\tilde{R}^i{}_k - \mu(\tilde{h}^2\delta^i_k - \xi_k\xi^i) - \frac{1}{\tilde{h} + \tilde{W}_0}\xi_k\{\tilde{R}^i{}_p - \mu(\tilde{h}^2\delta^i_p - \xi_p\xi^i)\}W^p = 0.$$

Clearly,

$$\tilde{R}^i{}_k = \mu(\tilde{h}^2\delta^i_k - \xi_k\xi^i). \tag{5.2}$$

Thus h has sectional curvature equal to μ. By Schur's lemma, μ is constant when $n \geq 3$.

Conversely, if the sectional curvature $\bar{\mathbf{K}}$ of h is equal to μ, then (5.2) holds. By (3.8) again,

$$R^i{}_k = \left(\frac{3c_m y^m}{F} + \sigma\right)\{F^2\delta^i_k - FF_{y^k}y^i\}, \tag{5.3}$$

where $\sigma = \mu - c^2 - 2c_m W^m$. Thus F is of scalar flag curvature. □

PROOF OF THEOREM 1.1. By assumption, the dimension of M is at least 3. First we assume that F is of isotropic S-curvature and of scalar flag curvature. By Theorem 5.1, the flag curvature of F is given by (5.1) and h has constant sectional curvature. At any

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point, there is a local coordinate system in which h is given by (1.3). By Lemma 2.2, if $\mathbf{S} = (n+1)cF$, then c and W are given by (1.4) and (1.5) in the same local coordinate system.

Conversely, assume that there is a local coordinate system in which h, c and W are given by (1.3), (1.4) and (1.5), respectively. Then $\mathbf{S} = (n+1)cF$ by Lemma 2.2. Since h has constant sectional curvature, F is of scalar curvature with flag curvature given by (5.1) by Theorem 5.1. □

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