Can. J. Math., Vol. XXXII, No. 5, 1980, pp. 1058-1071

FREDHOLM TOEPLITZ OPERATORS AND SLOW OSCILLATION

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The purpose of this paper is to show how Fredholm criteria for Toeplitz operators, whose symbols lie in an algebra A, may often be generalized to cover a larger symbol algebra generated by A and SO, the slowly oscillating functions. Here A and SO are algebras of continuous functions on the real line, so that we are concerned principally with the effect of a single discontinuity in the symbol function.

We shall treat the cases when A is the almost periodic functions, the semi-almost periodic functions and the multiplicatively periodic functions. Sufficient criteria for Fredholmness are obtained in Section 5. The more difficult task of establishing necessary and sufficient criteria is only achieved here for the slowly oscillating almost periodic functions and this is done in Section 6.

Our results are obtained by localisation, and in particular by making frequent use of Douglas's theorem [**4**, 7.47]. Its relevance here is due to a result of Sarason which ensures that a Toeplitz operator with slowly oscillating symbol commutes, modulo the compacts, with all other Toeplitz operators. As we shall see, the localisation method allows us to establish Fredholm criteria whilst knowing precious little about the underlying symbol algebras, in terms of distance estimates and structure.

Our context is the line and so T_f denotes the Toeplitz operator defined on H^2, the Hardy space corresponding to functions analytic in the upper half plane, whose symbol is f, a function on the real line $\mathbf{R}$. Also H^∞ refers to the Hardy space of functions on the real line which are boundary functions for functions analytic in the upper half plane.

1. Asymptotic independance. We first define and discuss some of the algebras of functions that we shall be considering. All functions on the real line are assumed to be continuous unless otherwise indicated. For a function f defined on a set I we shall let $\operatorname{osc}(f, I)$ denote the oscillation of f over I, that is, the supremum of $|f(s) - f(t)|$ for s and t in I.

C_0 : Functions vanishing at $+\infty$ and $-\infty$.
C_∞ : Functions possessing limits at $+\infty$ and $-\infty$.
SO : Functions f such that $g(x) = \operatorname{osc}(f, [x, 2x] \cup [-x, -2x])$ belongs to C_0.

Received November 22, 1978.

AP : The uniformly almost periodic functions.
MP : Functions f such that $f(x) = f(2x)$ for $|x| > 1$.

The letters MP stand for multiplicatively periodic and functions in MP are determined by their restrictions to $[-2, 2]$. If A is a function algebra on $\mathbf{R}$ then we shall let $M(A)$ denote the maximal ideal space of A and, if A contains C_0, let $M_\infty(A)$ denote these characters of $M(A)$ which annihilate C_0. The set $M_\infty(A)$ is a compact subset of $M(A)$ referred to as the fibre of $M(A)$ at (or over) infinity. We may define, without ambiguity, $M_\infty(A)$ for function algebras A, on $\mathbf{R}$, which do not necessarily contain C_0 by setting $M_\infty(A) = M_\infty(A + C_0)$. The fibre at infinity is the interesting part of the maximal ideal space for the function algebras listed above. The following facts are easily verified: The fibre $M_\infty(C_\infty)$ is a two point space; the fibre $M_\infty(SO)$ is a connected space; the fibre $M_\infty(AP)$ is homeomorphic to $M(AP)$, the Bohr compactification of $\mathbf{R}$; the fibre $M_\infty(MP)$ is naturally homeomorphic to the disjoint union of two circles.

Our intuition tells us that the last four function algebras listed above behave quite differently at infinity. This belief is captured by the following concept of asymptotic independence. If A and B are two function algebras then let $[A, B]$ denote the function algebra that they generate.

Definition. Two function algebras A and B on the real line are said to be *asymptotically independent* if $M_\infty([A, B])$ is naturally homeomorphic to $M_\infty(A) \times M_\infty(B)$.

The natural homeomorphism referred to is, of course, the restriction map $x \to (x|A, x|B)$, and asymptotic independence requires that it be onto. It turns out that the function algebras C_∞, SO, AP and MP are (pairwise) asymptotically independent except for the pair $\{C_\infty, MP\}$. In fact we even have, using the obvious notation,

$$M_\infty([SO, AP, MP]) = M_\infty(SO) \times M_\infty(AP) \times M_\infty(MP).$$

We shall content ourselves with the proof of the following lemma and some remarks which illustrate the general method.

LEMMA 1. *The fibre $M_\infty([SO, AP])$ is naturally homeomorphic to $M_\infty(SO) \times M(AP)$.*

Proof. Let x belong to $M_\infty(SO)$, let y belong to $M(AP)$ and let $g_1, g_2, \ldots, g_m$ be almost periodic functions. We first show that there exists z in $M_\infty([SO, AP])$ such that $z|SO = x$ and $g_i(z) = g_i(y)$ for $i = 1, 2, \ldots, m$.

For each positive integer n let L_n be an n^{-1}-almost period for the almost periodic function $h(t) = \sum_{i=1}^{m} |g_i(t) - g_i(y)|$. Since $h(y) = 0$ it follows that each interval of length L_n contains a point t such that $h(t) \leqq 2n^{-1}$. In fact if any interval of length L_n failed to have this property we would

https://doi.org/10.4153/CJM-1980-081-2 Published online by Cambridge University Press

deduce, by translating, that $h(t) \geqq n^{-1}$ everywhere, which contradicts $h(y) = 0$. Now let t_α be any net of points in $\mathbf{R}$ which converge to x in $M(SO)$. Construct, for each n, a perturbed net $t_{\alpha,n}$ where $t_{\alpha,n}$ is any point in an interval of length L_n, containing t_α, such that $h(t_{\alpha,n}) \leqq 2n^{-1}$. It is routine to show that the perturbed net $t_{2,n}$ still converges to x for each n. Now, for each n, let z_n be a limit point in $M([SO, AP])$ for the net $t_{\alpha,n}$ and let z be a limit point in $M([SO, AP])$ for the sequence z_n, $n = 1, 2, \ldots$. It is readily checked that $z|SO = x$ and $h(z) = 0$, that is, $g_i(z) = g_i(y)$ for $i = 1, 2, \ldots, m$ as desired.

For each finite family, G say, of functions in AP let z_G be the character of $M_\infty([SO, AP])$, as constructed above, whose restrictions to SO and G are x and y respectively. Then $\{z_G\}$ is a net in $M_\infty([SO, AP])$ which converges to a character ω such that $\omega|SO = x$ and $\omega|AP = y$. Thus the natural mapping provides the desired homeomorphism since it is onto.

If f is a continuous function on $\mathbf{R}$ then its asymptotic norm, denoted $||f||_a$, is given by

$$||f||_a = \limsup_{x\to\infty} \{|f(s)|; \quad |s| > x\}.$$

It is straightforward to show that the asymptotic norm of f is the norm of the restriction of f to the fibre over infinity of any C^*-algebra of continuous functions which contains it. A relatively straightforward consequence of the Gelfand theory is that if A and B are two commutative C^*-subalgebras of a commutative C^*-algebra then $M([A, B]) = M(A) \times M(B)$ if and only if $||ab|| = ||a||\,||b||$ for all a in A and b in B. Combining these facts, it follows that if C and D are C^*-algebras of continuous functions on the line then C and D are asymptotically independent if and only if $||fg||_a = ||f||_a||g||_a$ for all f in C and g in D. This useful property provides an alternative proof of Lemma 1 and rapidly establishes the asymptotic independence of the pairs (SO, C_∞), (SO, MP), $(SO, [C_\infty, AP])$, (AP, MP) and many more besides.

The space $[C_\infty, AP]$ is referred to as the space of semi-almost periodic functions [**13**] and we shall denote it by SAP.

2. Symbol data. Let φ be a continuous function defined on the real line. A well known fact concerning the Toeplitz operators T_φ is that its essential spectrum contains the range of φ. Consequently we may assume, without loss of generality, that φ is invertible and that arg φ is a continuous function, where, for the sake of definiteness, we take $0 \leqq \arg \varphi(0) < 2\pi$.

The following data (1)–(5) associated with the symbol function φ has proved to be important in establishing Fredholm criteria. In the next section this information will be localized to points in $M_\infty(SO)$, in order to accommodate the introduction of slowly oscillating functions. Assertions involving the symbol $\pm$ should be interpreted as making two statements in the obvious fashion.

(1) Averaging function. $m(\varphi, t) = t^{-1}\int_t^{2t} \varphi(s)ds \; (-\infty < t < +\infty)$.

(2) Means at $\pm\infty$. $m^{\pm}(\varphi) = \lim_{t\to\pm\infty} m(\varphi, t)$.

(3) Mean motion at $\pm\infty$. $\mu^{\pm}(\varphi) = \lim_{t\to\pm\infty} t^{-1}(\arg \varphi(2t) - \arg \varphi(t))$.

(4) Winding function $\omega(\varphi, t) = \arg \varphi(2t) - \arg \varphi(t)$.

(5) Weighted mean function $w(\varphi, t) = \int_t^{2t} \varphi(s) \dfrac{ds}{|s| \log 2}$

For the sake of convenience of comparison, and also for later use, we collect together some of the known criteria for Fredholmness of Toeplitz operators in the following theorem. Let $[\lambda, \nu]$ denote the line segment joining the complex numbers λ, ν, and let $d(\cdot,\cdot)$ denote the usual distance function between points and sets in a metric space.

THEOREM 2. *Let φ be a bounded continuous function on the real line which is bounded away from zero.*

(a) *If φ is in C_∞ then T_φ is Fredholm if and only if*

$$\lim_{x\to\infty} d(0, [\varphi(x), \varphi(-x)]) > 0.$$

(b) *If φ is in $[SO, C_\infty]$ then T_φ is Fredholm if and only if*

$$\liminf_{x\to\infty} d(0, [\varphi(x), \varphi(-x)]) > 0.$$

(c) *If φ is in AP then T_φ is Fredholm if and only if $\mu^+(\varphi) = 0$.*

(d) *If φ is in SAP then T_φ is Fredholm if and only if $\mu^+(\varphi) = \mu^-(\varphi) = 0$ and $d(0, [\exp(m^+(\log \varphi)), \exp(m^-(\log \varphi))]) > 0$.*

(e) *If φ is in MP then T_φ is Fredholm if and only if $\omega(\varphi, 1) = \omega(\varphi, -1)$ and $d(0, [\exp(w(\log \varphi, +1)), \exp(-w(\log \varphi, -1))] > 0$.*

Part (a) is due to Widom [**15**] (see also [**10**], [**5**]), part (b) is due to Sarason and is implicit in [**14**], part (c) is due independently to Coburn and Douglas [**2**] and Gohberg and Feldman [**9**], and part (d) is due to Sarason [**13**]. Of course if φ is in AP then $\mu^+(\varphi) = \mu^-(\varphi)$. If $\mu^+(\varphi)$ and $\mu^-(\varphi)$ exist and vanish for an arbitrary (continuous, invertible) φ we shall say that φ has vanishing asymptotic mean motion.

Part (e) follows, as we now show, from a theorem of Abrahamse [**1**] concerning the invertibility of a Toeplitz operator T_ψ whose symbol ψ is continuous on $(-\infty, 0) \cup (0, \infty)$ and satisfies the identity $\psi(x) = \psi(2x)$. His theorem asserts that T_ψ is invertible if and only if $\omega(\psi, 1) = \omega(\psi, -1)$ and

$$\int_{-2}^{-1} \arg \psi(s) \frac{ds}{|s| \log 2} - \int_1^2 \arg \psi(s) \frac{ds}{|s| \log 2}$$

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is not equal to π, modulo 2π. Since the last condition is equivalent to

$$d(0, [\exp(w(\log \psi, 1), \exp(-w(\log \psi, -1))]) > 0$$

we shall have established Theorem 2(e) if we show that if φ is an invertible function in MP such that $\varphi(x) = \psi(x)$ for $|x| > 1$, then T_φ is Fredholm if and only if T_ψ is invertible.

To see this we shall need the following five lemmas, which, incidentally, will not be required again in this paper. If f and g are functions in $L^\infty(\mathbf{R})$ we let

$$d_{+\infty}(f, g) = \text{ess lim sup}_{t\to+\infty} |f(t) - g(t)|.$$

Also if F is a subset of $L^\infty(\mathbf{R})$ we let $d_{+\infty}(f, F) = \inf d_{+\infty}(f, g)$ as g varies over F. Similarly we define $d_{-\infty}(f, g)$, $d_{-\infty}(f, F)$ and $d_\infty(f, g) = \max \{d_{+\infty}(f, g), d_{-\infty}(f, g)\}$ and also $d_\infty(f, F)$.

LEMMA 3 [**3**]. *If φ is an invertible function in $L^\infty(\mathbf{R})$ then T_φ is invertible (resp. Fredholm) if and only if $T_{\varphi/|\varphi|}$ is invertible (resp. Fredholm).*

LEMMA 4 [**4**]. *If φ is unimodular then T_φ is invertible if and only if*

$$d(\varphi, H^\infty) < 1 \quad \text{and} \quad d(\bar{\varphi}, H^\infty) < 1.$$

LEMMA 5 [**8**]. *If φ is unimodular then T_φ is Fredholm if and only if*

$$d(\varphi, H^\infty + C_0) < 1 \quad \text{and} \quad d(\bar{\varphi}, H^\infty + C_0) < 1.$$

LEMMA 6 [**12**]. *If φ is a bounded continuous function then*

$$d(\varphi, H^\infty + C_0) = d_\infty(\varphi, H^\infty).$$

LEMMA 7. *If ψ is continuous on $(-\infty, 0) \cup (0, \infty)$ and satisfies $\psi(x) = \psi(2x)$ then $d_\infty(\psi, H^\infty) = d(\psi, H^\infty)$.*

Proof. For $\epsilon > 0$ choose h in H^∞ such that

$$\text{ess sup}_{|t|\geqq t_0} |\psi(t) - h(t)| \leqq d_\infty(\psi, H^\infty) + \epsilon$$

for some t_0. Let h_n in H^∞ be given by $h_n(x) = h(2^n x)$ so that, since $\psi(x) = \psi(2^n x)$, we have

$$\text{ess sup}_{|t|\geqq 2^{-n} t_0} |\psi(t) - h_n(t)| \leqq d_\infty(\psi, H^\infty) + \epsilon.$$

Since h_n is a bounded sequence we may choose a weak star limit point g in H^∞ which must satisfy $||\psi - g|| \leqq d_\infty(\psi, H^\infty) + \epsilon$. Thus $d(\psi, H^\infty)$ is dominated by $d_\infty(\psi, H^\infty)$ and so equality holds.

We can now see, using the notation following Theorem 2, that T_φ is Fredholm if and only if T_ψ is invertible and so complete the proof of Theorem 2(e). By Lemma 3 we may assume that φ and ψ are unimodular, and so, by Lemmas 4 and 5, we need only show that $d(\psi, H^\infty) = d(\varphi, H^\infty + C)$. However, by Lemma 7, $d(\psi, H^\infty) = d_\infty(\psi, H^\infty) = d_\infty(\varphi, H^\infty)$ which, by Lemma 6, is $d(\varphi, H^\infty + C)$.

3. Local symbol data.

a) $[SO, AP]$. Since SO and AP are asymptotically independent, we have, for each x in $M_\infty(SO)$, a natural homomorphism α_x from $[SO, AP]$ to AP defined by

$$(\alpha_x(\varphi))(y) = \varphi((x, y)) \quad (y \text{ in } M(AP)).$$

Consequently, if φ is an invertible function in $[SO, AP]$ we may think of the mean motion of φ over the point x as the number $\mu^+(\alpha_x(\varphi))$. This data, which we shall denote by $\mu_x^+(\varphi)$, may be obtained directly from the symbol function as follows. Let t_α be any net of positive real numbers which converges to x in $M(SO)$. Then for φ in $[SO, AP]$ we have

$$(6) \qquad \mu_x^+(\varphi) = \lim_\alpha t_\alpha^{-1}\{\arg \varphi(2t_\alpha) - \arg \varphi(t_\alpha)\}.$$

To see this write $\varphi = f + h$ where $f = \alpha_x(\varphi)$ and $\alpha_x(h) = 0$. We first note that $\lim_\alpha \operatorname{osc}(|h|, [t_\alpha, 2t_\alpha]) = 0$. If this were not so there would exist $\epsilon > 0$ and x_β is $[t_\beta, 2t_\beta]$ for some subnet x_β of x_α, such that $|h|(x_\beta) \geqq \epsilon$, and so any cluster point z in $M([SO, AP])$ of $\{x_\beta\}$ would be such that $|h|(z) \geqq \epsilon$ and $z|SO = x$ contradicting the fact that $\alpha_x(h) = 0$. It now follows that

$$\begin{aligned}
&\lim_\alpha t_\alpha^{-1}\{\arg \varphi(2t_\alpha) - \arg \varphi(t_\alpha)\} \\
&= \lim_\alpha t_\alpha^{-1}\{\arg f(2t_\alpha) - \arg f(t_\alpha)\} \\
&= \mu^+(f) = \mu^+(\alpha^x(\varphi)) = \mu_x^+(\varphi).
\end{aligned}$$

b) $[SO, SAP]$. Since SO and SAP are asymptotically independent, we have, for each x in $M_\infty(SO)$, a natural homomorphism β_x from $[SO, SAP]$ to $SAP|M_\infty(SAP)$ given by

$$(\beta_x(\varphi))(y) = \varphi((x, y)) \quad (y \text{ in } M_\infty(SAP)).$$

If φ is an invertible function in $[SO, SAP]$ and if ψ is an invertible function in SAP such that $\beta_x(\varphi) = \psi|M_\infty(SAP)$ then it can be seen that the numbers

$$\begin{aligned}
\mu_x^\pm(\varphi) &= \mu^\pm(\psi) \\
m_x^\pm(\varphi) &= m^\pm(\psi)
\end{aligned}$$

do not depend on the particular choice of ψ. Moreover, as in (a), this information may be obtained directly from the symbol function. Using the notation of (a) we have, for an invertible function φ in $[SO, SAP]$

$$(7) \qquad \mu_x^+(\varphi) = \lim_\alpha t_\alpha^{-1}\{\arg \varphi(2t_\alpha) - \arg \varphi(t_\alpha)\}$$

$$(8) \qquad \mu_x^-(\varphi) = \lim_\alpha - t_\alpha^{-1}\{\arg \varphi(-2t_\alpha) - \arg \varphi(-t_\alpha)\}$$

$$(9) \qquad m_x^+(\varphi) = \lim_\alpha t_\alpha^{-1}\int_{t_\alpha}^{2t_\alpha} \varphi(s)ds$$

$$(10) \qquad m_x^-(\varphi) = \lim_\alpha - t_\alpha^{-1}\int_{-t_\alpha}^{-2t_\alpha} \varphi(s)ds.$$

The proofs of (7) and (8) are analogous to the proof of (6) given in (a). To see (9) and (10) note that these formulae are true when φ is of the form $f_1g_1 + f_2g_2 \ldots + f_ng_n$ where f_i belongs to SO and g_i belongs to SAP for $i = 1, \ldots, n$. Thus, by approximating, they are true in general.

(c) $[SO, MP]$. Since SO and MP are asymptotically independent we have, for each x in $M_\infty(SO)$, a natural homomorphism γ_x from $[SO, MP]$ to $MP|M_\infty(MP)$ given by

$$(\gamma_x(\varphi))(y) = \varphi((x, y)) \quad (y \text{ in } M_\infty(MP)).$$

If φ is an invertible function in $[SO, MP]$ and ψ is an invertible function in MP such that $\gamma_x(\varphi) = \psi|M_\infty(MP)$ then it is clear that the numbers

(11) $\omega_x^+(\varphi) = \omega(\psi, 1)$

(12) $\omega_x^-(\varphi) = \omega(\psi, -1)$

(13) $w_x^+(\varphi) = w(\psi, 1)$

(14) $w_x^-(\varphi) = w(\psi, -1)$

do not depend on the particular choice of ψ. Moreover this local data may be obtained directly from the symbol function. Let t_α be as in (a), that is, a net of positive real numbers which converge in $M(SO)$ to the character x. Then we have

(15) $\omega_x^+(\varphi) = \lim_\alpha (\arg \varphi(2t_\alpha) - \arg \varphi(t_\alpha))$,

(16) $\omega_x^-(\varphi) = \lim_\alpha (\arg \varphi(-2t_\alpha) - \arg \varphi(-t_\alpha))$,

(17) $w_x^+(\varphi) = \lim_\alpha w(\varphi, t_\alpha)$,

(18) $w_x^-(\varphi) = \lim_\alpha w(\varphi, -t_\alpha)$.

These formulae are verified as in (a) and (b).

4. Localisation over $M_\infty(SO)$. The following two theorems are the key to all the results of this paper. The first is due to Sarason [**14**] and the second is due to Douglas [**4**].

THEOREM 8. *If f belongs to $L^\infty(\mathbf{R})$ and g belongs to SO then $T_fT_g - T_gT_f$ is compact.*

It is well known that the commutator of two Toeplitz operators is compact if the symbol of one of them belongs to the space $QC = (H^\infty + C_0) \cap (\overline{H^\infty} + C_0)$. The theorem is proved by using the relationship between the functions of vanishing mean oscillation VMO and QC to show that QC contains SO.

To state the second theorem we shall require some notation. If A is an algebra of functions on the real line, containing SO, then let $\mathcal{T}_A$ denote

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the corresponding Toeplitz algebra, that is, the C^*-algebra generated by the Toeplitz operators whose symbol belong to A. If x is a point in $M(SO)$ then let $\mathscr{I}_x$ denote the ideal in $\mathscr{T}_A$ generated by those T_f with f in SO and $f(x) = 0$. Each such ideal contains the compact operators.

THEOREM 9. *Let A be an algebra of continuous functions on the real line which contains SO. Then T_φ is Fredholm if and only if $T_\varphi + \mathscr{I}_x$ is invertible in $\mathscr{T}_A/\mathscr{I}_x$ for each x in $M(SO)$.*

Let $\mathscr{I}_\infty$ be the ideal in $\mathscr{T}_A$ generated by those Toeplitz operators with symbols belonging to C_0. In view of Theorem 9 the question of Fredholmness for T_φ will reduce to the question of the invertibility of $T_\varphi + \mathscr{I}_\infty$ in $\mathscr{T}_A/\mathscr{I}_\infty$. We shall need the following distance formulae concerning the ideals $\mathscr{I}_x$ and $\mathscr{I}_\infty$. The coset of an operator T in the Calkin algebra is denoted by $[T]$, and similarly the coset of $\mathscr{T}_A$ is denoted by $[\mathscr{T}_A]$.

LEMMA 10. *Let T be an operator in T_A. Then*
(i) $d(T, \mathscr{I}_\infty) = \inf\{\|[T\,T_f]\|;\ f \text{ in } C_\infty, f(\pm\infty) = 1, 0 \leqq f \leqq 1\}$.
(ii) $d(T, \mathscr{I}_x) = \inf\{\|[T\,T_\varphi]\|;\ \ \varphi \text{ in } SO, \varphi(x) = 1\ \ \ 0 \leqq \varphi \leqq 1\}$.

Proof. (i) Since $\mathscr{I}_\infty$ contains the compact operators we have $d(T, \mathscr{I}_\infty) = d([T], [\mathscr{I}_\infty])$. Since, for f in C_0, $[T_f]$ commutes with $[\mathscr{T}_A]$, we see that

$$d([T], [\mathscr{I}_\infty]) = \inf\{\|[T] + [S]\|\}$$

where the infimum is taken over all operators $[S]$ of the form

$$[S] = \sum_{i=1}^{n} [T_i][T_{f_i}]$$

where $f_1, f_2, \ldots, f_n$ belong to C_0 and $T_1, T_2, \ldots, T_n$ belong to $\mathscr{T}_A$. This follows because the collection of such $[S]$ is a dense subalgebra of $[\mathscr{I}_\infty]$. Given $\epsilon > 0$ choose $[S]$ of the above form so that

$$\|[T] + [S]\| \leqq d(T, \mathscr{I}_\infty) + \epsilon.$$

Now choose g in C_∞ such that $g(+\infty) = g(-\infty) = 1$, $0 \leqq g \leqq 1$, and such that

$$\|[ST_g]\| = \left\|\sum_{i=1}^{n} [T_i][T_{f_i g}]\right\| < \epsilon.$$

Thus

$$\begin{aligned}\|[T][T_g]\| &\leqq \|([T] + [S])[T_g]\| + \|[S][T_g]\| \\ &\leqq d(T, \mathscr{I}_\infty) + 2\epsilon.\end{aligned}$$

Consequently $d(T, \mathscr{I}_\infty)$ dominates the infimum of (i). On the other hand the equation $\|[T] + [T][T_{1-f}]\| = \|[T\,T_f]\|$ immediately shows that $d(T, \mathscr{I}_\infty) = d([T], [\mathscr{I}_\infty])$ is dominated by this infimum, and so equality holds.

(ii) The proof of (ii) is exactly analogous to the proof of (i).

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5. Sufficient criteria.

THEOREM 11. *Let φ be a continuous function on the real line which is invertible. Then T_φ is Fredholm if any one of the following holds.*

(i) *φ lies in $[SO, AP]$, $\mu^+(\varphi)$ exists and is* 0.

(ii) *φ lies in $[SO, SAP]$, $\mu^+(\varphi)$ and $\mu^-(\varphi)$ exist and are* 0, *and*

$$\liminf_{x\to\infty} d(0, [\exp(m(\log \varphi, x)), \quad \exp(m(\log \varphi, -x))]) > 0.$$

(iii) *φ lies in $[SO, MP]$*, $\lim_{x\to\infty} (\omega(\varphi, x) - \omega(\varphi, -x))$ *exists and is zero, and*

$$\liminf_{x\to\infty} d(0, [\exp(w(\log \varphi, x)), \exp(-w(\log \varphi, -x))]) > 0.$$

Proof. The method of proof for each part follows the same pattern. Thus, the hypotheses will ensure that all the local Toeplitz operators $T_\varphi + \mathcal{I}_x$, x in $M(SO)$, are invertible and consequently, by Theorem 9, T_φ is Fredholm. Note that in each case $\mathcal{I}_x$ denotes a different ideal, corresponding to the different Toeplitz algebras, but this notational convenience should not cause any confusion. Since, for x in $\mathbf{R}$, $T_\varphi + \mathcal{I}_x = \varphi(x) + \mathcal{I}_x$ we need only concern ourselves with x in $M_\infty(SO)$.

(i) We first show that if θ is a function in $[SO, AP]$ and $\alpha_x(\theta) = 0$, then T_θ belongs to $\mathcal{I}_x$. Since SO and AP are asymptotically independent it should be clear that given $\epsilon > 0$, there exists a function f in SO such that $0 \leqq f \leqq 1$, $f(x) = 1$ and such that on $M_\infty([SO, AP])$ we have $|f\theta| \leqq \epsilon$. Since this means $||f\theta||_a \leqq \epsilon$ it is clear that we can rechoose f to guarantee $||f\theta||_\infty \leqq \epsilon$ and we assume we have done this. Thus

$$||[T_\theta T_f]|| = ||[T_{\theta f}]|| \leqq \epsilon$$

and so, by Lemma 10, we see that T_θ is in $\mathcal{I}_x$. Thus $T_\varphi + \mathcal{I}_x = T_{\alpha_x(\varphi)} + \mathcal{I}_x$ for all x in $M_\infty(SO)$. Since $\mu^+(\varphi) = 0$ it follows from (6) that $\mu_x^+(\varphi) = 0$ for all x in $M_\infty(SO)$. Thus, by Theorem 2, part (c), $T_\varphi + \mathcal{I}_x = T_{\alpha_x(\varphi)} + \mathcal{I}_x$ is invertible, and Theorem 9 completes the proof.

(ii) As in (i) if θ is a function in $[SO, SAP]$ and $\beta_x(\theta) = 0$ then T_θ belongs to $\mathcal{I}_x$. Fix x in $M_\infty(SO)$ and let ψ be any invertible function in SAP such that $\beta_x(\varphi - \psi) = 0$. Since $\mu^+(\varphi) = \mu^-(\varphi) = 0$ it follows from (7) and (8) that $\mu_x^+(\varphi) = \mu_x^-(\varphi) = 0$ and therefore that $\mu^\pm(\psi) = 0$. Moreover, from (7) and (8) it can be seen that our hypothesis implies that

$$d(0, [\exp(m_x^+(\log \varphi)), \exp(m_x^-(\log \varphi))]) > 0$$

and so since by (9) and (10)

$$m_x^+(\log \varphi) = m^+(\log \psi) \quad \text{and} \quad m_x^-(\log \varphi) = m^-(\log \psi),$$

we see, by Theorem 2 part (d), that T_ψ is Fredholm. Consequently $T_\varphi + \mathcal{I}_x = T_\psi + \mathcal{I}_x$ is invertible and (ii) now follows.

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(iii) Again, as in (i), if θ lies in $[SO, MP]$ and x is a point in $M_\infty(SO)$ then $\gamma_x(\theta) = 0$ implies that T_θ belongs to $\mathscr{I}_x$. Let ψ be any invertible function in MP such that $\gamma_x(\varphi - \psi) = 0$. Since

$$\lim_{x\to\infty} (\omega(\varphi, x) - \omega(\varphi, -x)) = 0$$

it follows from (15) and (16) that $\omega_x^+(\varphi) - \omega_x^-(\varphi) = 0$ and therefore that $\omega(\psi, 1) - \omega(\psi, -1) = 0$. In a similar fashion, since

$$w_x^+(\log \varphi) = w(\log \psi, +1) \quad \text{and} \quad w_x^-(\log \varphi) = w(\log \psi, -1)$$

our hypothesis implies (using (17) and (18)) that

$$d(0, [\exp(w(\log \psi, +1)), \exp(-w(\log \psi, -1))]) > 0$$

and so, by Theorem 2(e), T_ψ is Fredholm. Consequently $T_\varphi + \mathscr{I}_x = T_\psi + \mathscr{I}_x$ is invertible and (iii) follows.

In the next section we show that the condition in (i) above is necessary. It seems likely that the conditions of (ii) and (iii) are also necessary. However, a proof of this, along the lines of the next section, will require a good understanding of the interesting Toeplitz algebras associated with these symbols.

6. Necessary and sufficient conditions for $[SO, AP]$.

THEOREM 12. *Let φ be a function in $[SO, AP]$. Then T_φ is Fredholm if and only if φ is invertible and has vanishing asymptotic mean motion.*

Our proof rests on showing that the natural mapping from $\mathscr{T}_{AP}$ onto $\mathscr{T}_{[SO,AP]}/\mathscr{I}_x$ is an isomorphism for each x in $M_\infty(SO)$. That is, that each local algebra at infinity is isomorphic to the almost periodic Toeplitz algebra. Suppose for the moment that this is so. If T_φ is Fredholm then, as in the proof of Theorem 11(i), $T_\varphi + \mathscr{I}_x = T_{\alpha_x(\varphi)} + \mathscr{I}_x$ is invertible in $\mathscr{T}_{[SO,AP]}/\mathscr{I}_x$ for each x in $M(SO)$. Consequently $T_{\alpha_x(\varphi)}$ is invertible in $\mathscr{T}_{AP}$ for all x in $M_\infty(SO)$, and so, by Theorem 2(c), it follows that

$$\mu_x^+(\varphi) = \mu(\alpha_x(\varphi)) = 0$$

for all x in $M_\infty(SO)$. In view of (6) this implies $\mu^+(\varphi)$ $(= \mu^-(\varphi))$ exists and is 0. That is, that φ has vanishing asymptotic mean motion. The converse direction has already been established in Theorem 11(i).

We prove the required isomorphism (Lemma 16) through the following three lemmas, the first of which is the key. For $\lambda > 0$ we let P_λ denote the orthogonal projection of H^2 onto $e^{i\lambda x}H^2$.

LEMMA 13. *Let f be a function on SO which does not belong to C_0. Then, for each $\lambda > 0$ there exists a sequence g_n of functions in $(I - P_\lambda)H^2$ such that $||g_n||_2 = 1$, g_n converges to zero weakly and such that $||fg_n||_2 \geqq \delta > 0$ for some positive δ.*

Proof. Consider first the function $g(x) = (e^{ibx} - e^{iax})/x(b - a)^{1/2}$, where $0 < a < b < \lambda$. Then $||g||_2 = 1$ and g belongs to $(I - P_\lambda)H^2$. This may be seen for example by noting that g is the Fourier Plancherel transform of

$$-i\sqrt{2\pi}(b - a)^{-1/2}\chi_{[a,b]}$$

where $\chi_{[a,b]}$ is the characteristic function of $[a, b]$. Let $\gamma = b - a$. Then we have

$$\frac{1}{2\pi}\int_y^{2y} |g|^2 = \frac{1}{2\pi\gamma}\int_y^{2y} \frac{2(1 - \cos\gamma x)}{x^2}\,dx.$$

Thus

$$\frac{1}{2\pi}\int_{\pi\gamma^{-1}}^{2\pi\gamma^{-1}} |g|^2 = \frac{1}{2\pi\gamma}\int_\pi^{2\pi} \frac{2(1 - \cos t)}{t^2}\,\gamma dt = K$$

where K is an absolute constant, which is strictly positive.

Suppose now that f belongs to SO and does not belong to C_0. Then there exists a positive constant L and disjoint intervals $[2\pi t_n, 4\pi t_n]$, $n = 1, 2, \ldots$, such that $|f| > L$ on these intervals. Let

$$g_n(x) = (e^{ix/t_n} - e^{ix/2t_n})(2t_n)^{1/2}x^{-1}.$$

Then it follows, from our initial comments, that the functions g_n are orthogonal unit vectors which lie in $(I - P_\gamma)H^2$ for large n. In particular the sequence g_n converges to zero weakly. Also

$$||fg_n||_2 \geqq \frac{1}{2\pi}\int_{2\pi t_n}^{4\pi t_n} |g|^2|g_n|^2 \geqq LK > 0$$

for sufficiently large n, and so the proof of the lemma is complete.

LEMMA 14. *Let $\lambda > \nu \geqq 0$ and let f belong to SO. Then*

$$||[(P_\lambda - P_\nu)T_f]|| = ||f|M_\infty(SO)||_\infty.$$

Proof. The mapping from SO into the commutative C^*-algebra $\{[(P_\lambda - P_\nu)T_f]; f \text{ in } SO\}$ given by

$$f \rightarrow [(P_\lambda - P_\nu)T_f] \quad (f \text{ in } SO)$$

is a homomorphism. Moreover if f is in C_0 then $(P_\lambda - P_\nu)T_f$ is compact. This may be seen by writing $(P_\lambda - P_\nu)T_f$ as $(T_\nu T_\nu^* - T_\lambda T_\lambda^*)T_f$, where $T_\mu = T_{e^{i\mu x}}$, and localising relative to the circle, that is, apply [**4**, Theorem 7.47]. Consequently the mapping

$$f|M_\infty(SO) \rightarrow [(P_\lambda - P_\nu)T_f] \quad (f \text{ in } SO)$$

is a well defined C^*-homomorphism of commutative C^* algebras, and it will be sufficient to show that this mapping is injective. But suppose

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$[(P_\lambda - P_\nu)T_f] = 0$. Since the automorphisms induced by the unitary (multiplication) operators $M_{e^{i\rho x}}$, ρ in $\mathbf{R}$, leave multiplication operators fixed but translate the operators P_λ, we see that $[(I - P_{\lambda-\nu})T_f] = 0$. But if Q is the projection from L^2 to H^2 we have $(I_{L^2} - Q)M_fQ$ compact, since f belongs to $H^\infty + C_0$. Thus we conclude that $M_f(I - P_{\lambda-\nu})|H^2$ is compact and Lemma 13 implies that f is in C_0.

Let $\mathscr{C}$ be the commutative C^*-algebra generated by the projections P_λ, $\lambda > 0$.

LEMMA 15. *Let T be an operator in $\mathscr{C}$ and let f belong to SO. Then*

$$||[TT_f]|| = ||[T]|| \, ||f|M_\infty(SO)||_\infty.$$

Proof. It will be sufficient to consider the case when T is in the algebra generated by the projections P_λ, $\lambda > 0$. For such a T it can be seen that there exists a complex number α of modulus one and $\lambda > \nu \geqq 0$ such that

$$(P_\lambda - P_\nu)T = \alpha||T||(P_\lambda - P_\nu).$$

Thus, using the previous lemma,

$$\begin{aligned} ||[TT_f]|| &\geqq ||\alpha[(P_\lambda - P_\nu)TT_f]|| \\ &= ||T|| \, ||[(P_\lambda - P_\nu)T_f]|| \\ &\geqq ||[T]|| \, ||f|M_\infty(SO)||_\infty. \end{aligned}$$

The inequality in the other direction is straightforward and so the lemma is proved.

LEMMA 16. *The natural mapping of $\mathscr{T}_{AP}$ onto $\mathscr{T}_{[SO,AP]}/\mathscr{I}_x$ is a star isometrical isomorphism.*

Proof. Let U_t, $t \in \mathbf{R}$, be the translation unitaries on H^2 defined by $(U_tf)(x) = f(x - t)$, x in $\mathbf{R}$. It is known ([**6**]) that for T in $\mathscr{T}_{AP}$ the mapping E on $\mathscr{T}_{AP}$ defined by

$$E(T) = \lim_{t\to\infty} t^{-1}\int_0^t U_sTU_s^*ds$$

is well defined and in fact E is a faithful expectation onto $\mathscr{C}$.

Suppose that T is a positive operator in the kernel of the mapping $T \to T + \mathscr{I}_x$ and fix $\epsilon > 0$. By Lemma 10 there exists a function φ in SO with $\varphi(x) = 1$, $0 \leqq \varphi \leqq 1$, such that $||[TT_\varphi]|| \leqq \epsilon$. Now if $\varphi_s = U_s\varphi$, the s-translate of φ, then $\varphi - \varphi_s$ belongs to C_0 and so

$$U_sT_\varphi U_s^* - T_\varphi = T_{\varphi_s - \varphi}$$

belongs to $\mathscr{I}_\infty$. Consequently

$$t^{-1}\int_0^t U_sTT_\varphi U_s^*ds - t^{-1}\int_0^t U_sTU_s^*dsT_\varphi = t^{-1}\int_0^t U_sTU_s^*(T_{\varphi_s} - T_\varphi)ds$$

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belongs to $\mathscr{I}_\infty$. Since the first term of the left hand side of this equation has essential norm no greater than ϵ, and since the second term converges with t to $E(T)T_\varphi$, we see that

$$d([E(T)T_\varphi], [\mathscr{I}_\infty]) \leqq \epsilon.$$

Now, by Lemma 10(i) there exists f in C_∞ with $0 \leqq f \leqq 1, f(-\infty) = f(+\infty) = 1$, and $\|[E(T)T_\varphi T_f]\| \leqq 2\epsilon$. But now, since $\|\varphi f|M_\infty(SO)\|_\infty = 1$ Lemma 15 indicates that $\|[E(T)]\| \leqq 2\epsilon$. Thus $[E(T)] = 0$. However $\mathscr{C}$ contains no compact operators ([**2**] [**11**]) and so $E(T) = 0$. Since E is faithful it follows that $T = 0$ and so the proof is complete.

Remark. Let A be an algebra of functions on the real line and let $\mathscr{C}_A$ denote the commutator ideal of the Toeplitz algebra $\mathscr{T}_A$. It can be shown that if the mapping $f \to T_f + \mathscr{C}_A$, f in A, is an algebra isomorphism, then so is the mapping $g \to T_g + \mathscr{C}_{[SO,A]}$, for g in $[SO, A]$. To see this note first that $\mathscr{C}_{[SO,A]}$ contains the compact operators, and note secondly that it will suffice to show that $T_{\varphi f \psi g} - T_{\varphi f}T_{\psi g}$ belongs to $\mathscr{C}_{[SO,A]}$ whenever φ, ψ are in SO and g, f are in A. This is because the mapping in question is always an isometry [**4**]. However

$$T_{\varphi f \psi g} - T_{\varphi f}T_{\psi g} = T_{\varphi\psi}(T_{fg} - T_f T_g) + \text{compact}$$

and so this semicommutator belongs to $\mathscr{C}_{[SO,A]}$ because $T_{fg} - T_f T_g$ does.

In particular, in view of [**13**] Theorem 2, it follows that $\mathscr{T}_{[SO,AP]}/\mathscr{C}_{[SO,AP]}$ is the isomorphic image of $[SO, AP]$ under the mapping $f \to T_f + \mathscr{C}_{[SO,AP]}$.

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