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SPATIAL BRANCHING PROCESSES AND SUBORDINATION

Dedicated to Professor Fukushima on his 60th birthday.

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ABSTRACT. We present a subordination theory for spatial branching processes. This theory is developed in three different settings, first for branching Markov processes, then for superprocesses and finally for the path-valued process called the *Brownian snake*. As a common feature of these three situations, subordination can be used to generate new branching mechanisms. As an application, we investigate the compact support property for superprocesses with a general branching mechanism.

1**. Introduction.** The goal of this work is to develop a subordination theory in the context of spatial branching processes. This theory applies in particular to the measure-valued branching processes called *superprocesses*. One of the most interesting features of subordination is that, starting from a superprocess with a given branching mechanism, it can be used to generate other superprocesses with different branching mechanisms.

We present our subordination theory in three different settings, first for branching Markov processes (where the branching phenomenon occurs only on a discrete set of times), then for superprocesses and finally for the path-valued process called the *Brownian snake*. These three cases are presented in Sections 2, 3, 4 respectively. There are obvious connections between the three situations. However, we believe that it is interesting to treat each case separately in detail. The case of branching Markov processes is elementary in the sense that the relevant objects can be defined and understood very easily. However, the key ideas of our subordination procedure are present in this discrete setting and the explicit formulas derived in that case are already nontrivial and interesting. The treatment of superprocesses is formally very similar to the case of branching Markov processes. This is not surprising since superprocesses can be viewed as limits of branching Markov processes. We obtain in particular a simple formula for the branching mechanism function of the subordinate superprocess, and, on a number of examples, we show how this formula can be used to get explicit calculations. However, the theory here is much less elementary as we need to use the machinery developed for studying superprocesses. In particular the exit measures studied by Dynkin [9], [11] play a fundamental role. Finally, the case of the Brownian snake corresponds to a special case of superprocesses (namely superprocesses with a finite-variance branching mechanism). One advantage of the Brownian snake is that it gives a more trajectorial understanding of the basic objects such as the exit measures. Our subordination theory for the Brownian snake also yields

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a path-valued process approach for superprocesses with a (rather) general branching mechanism, including the β-stable branching. Until now, such an approach was only available for finite-variance superprocesses [17].

Let us now explain the basic ideas of our subordination method for branching processes. We start from a càdlàg Markov process $\xi = (\xi_t, t \geq 0)$ with values in a Polish space E. We assume that ξ has a regular recurrent point r and denote by $L = (L_t, t \geq 0)$ a local time of ξ at r. Consider a branching Markov process with spatial motion ξ where all particles start at r. This means that at time $t = 0$ we have a finite number of particles located at r, that start moving independently with the law of the Markov process ξ, die at rate $\lambda > 0$ and give rise when they die to new particles (according to a certain reproduction law), which in turn move independently with the law of ξ, die at rate λ, *etc*. For each particle alive at t we can consider the total local time at r accumulated by this particle and its ancestors up to time t. For every $s \geq 0$, denote by X_s the number of particles which (at any time) have accumulated a local time s at the point r. Then, $(X_s, s \geq 0)$ is also a (continuous-time) branching process, corresponding to the evolution of a population where the individuals die at a new rate $\tilde{\lambda}$ and with a new reproduction law. Both $\tilde{\lambda}$ and the new reproduction law can be evaluated explicitly.

At this stage, we have not constructed the spatial motions of the individuals of this new branching process. This is however easy to do. If we want the new spatial motions to be given by another independent Markov process γ with values in E', we simply replace the process $(\xi_t, t \geq 0)$ by the pair $\big((\xi_t, \gamma_{L_t}), t \geq 0\big)$. We then consider instead of X_s the random measure Y_s which is defined as the sum, over all particles having accumulated a local time s, of the Dirac masses at the positions in E' of these particles. A detailed account of this construction is given in Section 2 (for technical reasons, we use the triple (ξ, L, γ_L) rather than the pair (ξ, γ_L)).

Let us now briefly explain the analogous construction for superprocesses (Section 3). We consider again the Markov process ξ and denote by P_x the law of ξ started at $x \in E$. If μ is a measure on E and g a nonnegative measurable function on E we denote by $\langle \mu, g \rangle$ the integral of g with respect to μ. We introduce a branching mechanism function of the type

(1) $$\psi(u) = au + bu^2 + \int_{(0,\infty)} (e^{-us} - 1 + us) n(ds)$$

where $a, b \geq 0$ and n is a measure on $(0, \infty)$ such that $\int (s \wedge s^2) n(ds) < \infty$. The superprocess with spatial motion ξ and branching mechanism ψ is the Markov process $(Z_s, s \geq 0)$ with values in the space $M_f(E)$ of all finite measures on E, whose transition kernel can be described as follows: If $\mathbb{P}_\mu$ denotes the law of Z started at $\mu \in M_f(E)$, then for any bounded nonnegative measurable function $g: E \longrightarrow \mathbb{R}_+$,

$$\mathbb{E}_\mu(\exp -\langle Z_t, g \rangle) = \exp -\langle \mu, v_t \rangle,$$

where the function $(v_t(x), t \geq 0, x \in E)$ is the unique nonnegative solution of the integral equation

(2) $$v_t(x) + E_x\Big(\int_0^t \psi\big(v_{t-s}(\xi_s)\big)\, ds\Big) = E_x\big(g(\xi_t)\big).$$

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The case $\psi(u) = bu^2$ corresponds to the finite-variance superprocess.

Loosely speaking, Z_s is uniformly distributed on a cloud of infinitesimal particles that move independently according to the law of ξ and are (continuously) subject to a branching mechanism governed by the function ψ. This interpretation suggests that it should be possible to adapt the construction explained above for branching Markov processes. However, it is not clear how to measure the "number of particles" that have accumulated a local time s at the regular point r. The right tool for this is the notion of exit measures [9], [11]. More precisely, we first replace the process ξ by the pair (ξ, L), taking values in $E \times [0, \infty)$, and we define X_s as the total mass of the exit measure from the open set $E \times [0, s)$. The process $(X_s, s \geq 0)$ is a continuous state branching process whose branching mechanism function $\tilde{\psi}$ can again be computed rather explicitly. By a trick similar to the one we used in the discrete case, we can also construct a superprocess with spatial motion γ and branching mechanism $\tilde{\psi}$ (X then corresponds to the total mass process of this superprocess). We have treated a number of examples that show that the function $\tilde{\psi}$ can effectively be computed. For instance, if ξ is a stable Lévy process on the real line with index $\alpha \in (1, 2]$ and $\psi(u) = cu^{1+\beta}$ for $\beta \in (0, 1]$, then $\tilde{\psi}(u) = c'u^{1+\beta(1-1/\alpha)}$.

In Section 4, we present our subordination procedure from the point of view of the Brownian snake. The usual Brownian snake with spatial motion ξ [17], [18] is a Markov process $W = (W_s, s \geq 0)$ in the space of E-valued stopped paths. The connection with superprocesses can be stated by saying that the process W generates the historical paths of a superprocess with spatial motion ξ and branching mechanism $\psi(u) = 2u^2$ (see [17] for more precise statements).

For definiteness, we specify the process ξ as follows. We let $S = (S_t, t \geq 0)$ be a subordinator in $\mathbb{R}_+$ and define ξ as the associated residual lifetime process: $\xi_s = \inf\{S_t - s, S_t > s\}$. We consider the regular point $r = 0$ and the corresponding local time is $L_s = \inf\{t, S_t > s\}$. Let γ be as previously an independent Markov process with values in E'. Our main result says that the Brownian snake with spatial motion (ξ, γ_L) is connected to a superprocess with spatial motion γ and with a new branching mechanism $\tilde{\psi}$ in much the same way as the usual Brownian snake is connected to the finite-variance superprocess. Moreover, the function $\tilde{\psi}$ is expressed explicitly in terms of the Lévy measure of S (see Theorem 8). In particular, if S is a stable subordinator with index $\alpha \in (0, 1]$, then $\tilde{\psi}(u) = cu^{1+\alpha}$.

An informal description of our construction can be given as follows. For every $s \geq 0$ we consider a path $\gamma^{(s)}$ of the Markov process γ stopped at a random time η_s, and simultaneously a path $S^{(s)}$ of the subordinator S stopped at the same random time η_s. The jumps of $S^{(s)}$ should be interpreted as point masses distributed along the path $\gamma^{(s)}$, and in particular $S^{(s)}_{\eta_s} =: \zeta_s$ is the "total mass" of the path $\gamma^{(s)}$. In contrast with the usual Brownian snake, it is the total mass process $(\zeta_s, s \geq 0)$, and not the "lifetime process" $(\eta_s, s \geq 0)$, that evolves according to the law of reflecting Brownian motion on $\mathbb{R}_+$. Thus, between times s and $s' > s$, the path $\gamma^{(s)}$ will first be "erased" from its tip in such a way that its total mass becomes $\inf_{[s,s']} \zeta_r$, and then it will be extended (with a creation of point masses on the new part of the path) in order to arrive at a total mass equal to $\zeta_{s'}$.

The paths $\gamma^{(s)}$ generated in this way are the historical paths of a superprocess with spatial motion γ and branching mechanism $\tilde{\psi}$. The instants of occurrence of point masses along the paths $\gamma^{(s)}$ correspond to discontinuity times for this superprocess.

We believe that this construction will be useful to investigate path properties of general superprocesses, in the same way as the usual Brownian snake has proved a powerful tool for studying super-Brownian motion (see *e.g.* [21]). As a typical application, we give in Section 4 sufficient conditions that ensure that the compact support property holds for superprocesses with a (rather) general branching mechanism. We refer to [4], [5] (Chapter 8) and [6] for previous results about the compact support property and the continuity properties of the support process.

Let us finally mention a related previous work of Kaj and Salminen [16], who consider for a one-dimensional branching Brownian motion started at the origin, the number X_x of particles that hit each level $x \geq 0$. They prove that the process $(X_x, x \geq 0)$ is a branching process, compute its offspring distribution and also investigate scaling limits of X. Via the famous Lévy theorem relating the supremum of linear Brownian motion to its local time at 0, the results of [16] correspond to a special case of the situation treated in Sections 2 and 3 (in this special case, the process ξ is reflecting Brownian motion, see subsection 3.2.2).

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2. Discrete branching.

2.1. *Notation.* Let E be a Polish space. We denote by $M_p(E)$ the space of finite point measures on E. As in Section 1, we consider a Borel right Markov process $\xi = (\xi_t, t \geq 0)$ taking values in E. We denote its law started at $\xi_0 = x$ by P_x. We will assume moreover that the sample paths of ξ are right-continuous and have left-limits (càdlàg).

Let Π be a *sub-critical* probability measure on $\mathbb{N}$, that is $\sum_n n\Pi(n) \leq 1$. The moment generating function

$$\hat{\Pi}(s) = \sum_{n\in\mathbb{N}} s^n \Pi(n), \qquad s \in (0, 1]$$

is then a Lipschitz function which satisfies $\hat{\Pi}(s) \geq s$.

For every parameter $\lambda > 0$, one can construct a branching Markov process letting the paths of ξ branch at rate λ with reproduction law Π. This process is viewed as a Markov process $Z = (Z_t, t \geq 0)$ taking values in $M_p(E)$; in particular the mass process $\langle Z, 1\rangle$ is a Galton-Watson process on $\mathbb{N}$. For every $\mu \in M_p(E)$, we denote by $\mathbb{P}_\mu$ the law of Z started at $Z_0 = \mu$. For every measurable function $f : E \longrightarrow (0, 1]$, put $g = -\log f$ and

$$\Phi_t[f](x) = \mathbb{E}_{\delta_x}(\exp\{-\langle Z_t, g\rangle\}), \qquad x \in E,$$

where δ_x stands for the Dirac point mass at x. The law of Z is determined by the branching property

$$\mathbb{E}_\mu(\exp\{-\langle Z_{t+t_n}, g\rangle\} \mid Z_{t_1}, \ldots, Z_{t_n}) = \exp\{\langle Z_{t_n}, \log \Phi_t[f]\rangle\} \tag{3}$$

(for every $0 \le t_1 \le \cdots \le t_n$ and $t \ge 0$) and by the equation

$$\Phi_t[f](x) = E_x\Big(f(\xi_t) + \lambda \int_0^t \big\{\hat{\Pi}\big(\Phi_{t-s}[f](\xi_s)\big) - \Phi_{t-s}[f](\xi_s)\big\}\, ds\Big) \tag{4}$$

which has a unique solution, thanks to Gronwall's lemma. Equation (4) can be deduced from the more intuitive identity

$$\Phi_t[f](x) = E_x\Big(e^{-\lambda t} f(\xi_t) + \lambda \int_0^t e^{-\lambda s}\hat{\Pi}\big(\Phi_{t-s}[f](\xi_s)\big)\, ds\Big) \tag{5}$$

which is obtained by considering the first branching time. A detailed argument for the derivation of (4) will be given in a more general context in the proof of Lemma 1. We sometimes call Z the branching Markov process associated with (Π, λ, ξ).

2.2. *Exit measure.* Our next goal is to associate with every closed set $F \subseteq E$ an *exit measure*, which, informally, is obtained by freezing each particle as it enters F. To give a rigorous definition, we first introduce the so-called historical process. For every $t \ge 0$, let $\mathcal{D}_t$ be the space of càdlàg paths $\omega\colon [0,t] \longrightarrow E$ and $\mathcal{D} = \bigcup_{t \ge 0} \mathcal{D}_t$. The set $\mathcal{D}_0$ is naturally identified with E. We will refer to $\mathcal{D}$ as the space of finite paths. Replacing ξ by the path valued process $(\xi_{\le t}, t \ge 0)$, where $\xi_{\le t} = (\xi_s, 0 \le s \le t)$, one can construct a branching Markov process $H = (H_t, t \ge 0)$ taking values in the space of point measures on finite paths. Assuming that H_0 is supported on $E = \mathcal{D}_0$, the measure H_t is supported on $\mathcal{D}_t$ for every $t \ge 0$. We denote by $p\colon \mathcal{D} \longrightarrow E$ the function that maps a finite path on its endpoint, *i.e.* $p(\omega) = \omega(t)$ for $\omega \in \mathcal{D}_t$. If $p(\mu)$ stands for the image of a measure μ under p, then the process $p(H) = \big(p(H_t), t \ge 0\big)$ is distributed as the branching Markov process Z. With a slight abuse of notation, we still denote by $\mathbb{P}_\mu$ the law of H started at $H_0 = \mu$.

For every closed set $F \subseteq E$, denote by $\mathcal{D}_F$ the subset of $\mathcal{D}$ consisting of finite paths for which the lifetime coincides with the first passage time in F

$$\mathcal{D}_F = \bigcup_{t \ge 0} \{\omega \in \mathcal{D}_t : \omega(s) \in F^c \text{ for } 0 \le s < t \text{ and } \omega(t) \in F\}.$$

The set of times $T_F = \{t \ge 0 : H_t(\mathcal{D}_F) > 0\}$ is a.s. finite. We then define the exit measure Z_F by

$$Z_F = p\Big(\sum_{t \in T_F} \mathbf{1}_{\mathcal{D}_F} \cdot H_t\Big),$$

where $\mathbf{1}_{\mathcal{D}_F} \cdot H_t$ stands for the restriction of the point measure H_t to $\mathcal{D}_F$.

For every measurable function $g \ge 0$ on E, we write for $f = e^{-g}$

$$\Phi_F[f](x) = \mathbb{E}_{\delta_x}(\exp\{-\langle Z_F, g\rangle\}) \quad \text{and} \quad U_F[g] = -\log(\Phi_F[f]).$$

Note that $U_F g = g$ on F. The following property of branching type is intuitively obvious, though the formal proof is quite tedious. A closely related result is stated as Proposition 2.1 in Chauvin [3]. Given a decreasing family of closed sets $F_k \subseteq F_{k-1} \cdots \subseteq F_1$, we have for every measurable function $g \ge 0$

$$\mathbb{E}_\mu(\exp\{-\langle Z_{F_k}, g\rangle\} \mid Z_{F_{k-1}}, \ldots, Z_{F_1}) = \exp\{-\langle Z_{F_{k-1}}, U_{F_k}[g]\rangle\}. \tag{6}$$

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In particular, (6) implies the simple identity

$$U_{F_1}(U_{F_2}) = U_{F_2}. \tag{7}$$

(write $\mathbb{E}_\mu(\exp -\langle Z_{F_2}, g\rangle) = \mathbb{E}_\mu\Big(\mathbb{E}_\mu(\exp -\langle Z_{F_2}, g\rangle \mid Z_{F_1})\Big)$).

Denote by $T_F = \inf\{t \geq 0, \xi_t \in F\}$ the first passage time of ξ in F. By considering the first branching time, one gets that for every measurable function $f\colon E \longrightarrow (0,1]$ and every $x \in E$

$$\Phi_F[f](x) = E_x\Big(e^{-\lambda T_F} f(\xi_{T_F}) + \int_0^{T_F} \lambda e^{-\lambda s}\hat{\Pi}\Big(\Phi_F[f](\xi_s)\Big)\, ds\Big). \tag{8}$$

LEMMA 1. *Assume that $P_x(T_F < \infty) = 1$ for every $x \in E$. Then we have*

$$\Phi_F[f](x) = E_x\Big(f(\xi_{T_F}) + \lambda \int_0^{T_F} \Big\{\hat{\Pi}\Big(\Phi_F[f](\xi_s)\Big) - \Phi_F[f](\xi_s)\Big\}\, ds\Big).$$

PROOF. Set $D = F^c$, and introduce the Poisson and resolvent kernels

$$H^F_\lambda f(x) = E_x\Big(\exp\{-\lambda T_F\} f(\xi_{T_F})\Big)$$
$$V^D_\lambda f(x) = E_x\Big(\int_0^{T_F} e^{-\lambda s} f(\xi_s)\, ds\Big).$$

We rewrite (8) as

$$\Phi_F[f] = H^F_\lambda f + \lambda V^D_\lambda\Big(\hat{\Pi}(\Phi_F[f])\Big). \tag{9}$$

Hence,

$$(I - \lambda V^D_\lambda)\Phi_F[f] = H^F_\lambda f + \lambda V^D_\lambda\Big(\hat{\Pi}(\Phi_F[f]) - \Phi_F[f]\Big).$$

Applying $(\lambda - \lambda')V^D_{\lambda'}$, we get by the resolvent equation and the Markov property

$$(\lambda V^D_\lambda - \lambda' V^D_{\lambda'})\Phi_F[f] = H^F_{\lambda'} f - H^F_\lambda f + \lambda V^D_{\lambda'}\Big(\hat{\Pi}(\Phi_F[f]) - \Phi_F[f]\Big) - \lambda V^D_\lambda\Big(\hat{\Pi}(\Phi_F[f]) - \Phi_F[f]\Big).$$

Then let $\lambda' \longrightarrow 0+$, note that $\lambda' V^D_{\lambda'} 1 \longrightarrow 0$ (since $T_F < \infty$ a.s.); we obtain

$$H^F_0 f + \lambda V^D_0\Big(\hat{\Pi}(\Phi_F[f]) - \Phi_F[f]\Big) = H^F_\lambda f + \lambda V^D_\lambda\Big(\hat{\Pi}(\Phi_F[f])\Big) = \Phi_F[f],$$

by (9). This establishes Lemma 1. ∎

We can apply Lemma 1 with $g = \lambda$, $f = e^{-\lambda}$. Since $\hat{\Pi}(s) \geq s$, we get $\Phi_F(e^{-\lambda}) \geq e^{-\lambda}$ and it follows that

$$\mathbb{E}_{\delta_x}(\langle Z_F, 1\rangle) = \lim_{\lambda \downarrow 0} \lambda^{-1}\mathbb{E}_{\delta_x}(1 - \exp -\langle Z_F, \lambda\rangle) \leq 1. \tag{10}$$

2.3. *Subordination.* We assume from now on that $r \in E$ is a regular recurrent point for ξ, and that it has zero potential; see also the remark at the end of this subsection when the latter assumption is relaxed. We denote by $T_r = \inf\{t > 0, \xi_t = r\}$ the first hitting time of r. Notice that $T_r = T_{\{r\}}$, P_r a.s. since r is regular. Let $L = (L_t, t \geq 0)$ be a local time at r (so that L is a continuous additive functional that increases only when $\xi = r$, and $L_\infty = \infty$ a.s.) and $\tau_s = \inf\{t : L_t > s\}$ the inverse local time. Finally, we denote by $P_r^\star$ the excursion measure of ξ away from r associated with L (as usual this excursion measure is normalized so that $E_r(\exp -\lambda\tau_1) = \exp(-E_r^\star(1 - \exp -\lambda T_r))$, and by m the occupation measure under $P_r^\star$, that is

$$m(A) = E_r^\star\left(\int_0^{T_r} \mathbf{1}_{\{\xi_s \in A\}}\, ds\right). \tag{11}$$

It is well-known that m is σ-finite and invariant, see *e.g.* [7], p. 122.

An important role will be played by the function φ^x defined as the moment generating function of $\langle Z_{\{r\}}, 1\rangle$ under $\mathbb{P}_{\delta_x}$:

$$\varphi^x(s) = \mathbb{E}_{\delta_x}(s^{\langle Z_{\{r\}},1\rangle}) = \Phi_{\{r\}}[s](x), \qquad s \in (0,1].$$

Lemma 1 with $f = s$ entails the identity

$$\varphi^x(s) = s + \lambda E_x\left(\int_0^{T_r}\left\{\hat\Pi\left(\varphi^{\xi_t}(s)\right) - \varphi^{\xi_t}(s)\right\} dt\right). \tag{12}$$

By (8) we have also

$$\varphi^x(s) = E_x\left(se^{-\lambda T_r} + \int_0^{T_r} \lambda e^{-\lambda t}\hat\Pi\left(\varphi^{\xi_t}(s)\right) dt\right). \tag{13}$$

Introduce an independent Borel right process with càdlàg paths, $\gamma = (\gamma_t, t \geq 0)$, taking values in a Polish space E'. We will write E' for expectations relative to the process γ. We now replace ξ by $\bar\xi = (\bar\xi_t, t \geq 0)$, $\bar\xi_t = \left(\xi_t, L(t), \gamma_{L(t)}\right)$, and denote by $\bar Z$ the corresponding branching Markov process. For every $u \geq 0$, the exit measure $\bar Z_{\{r\}\times[u,\infty)\times E'}$ is a.s. supported on $\{r\} \times \{u\} \times E'$, provided that $\bar Z_0$ is supported on $E \times [0,u] \times E'$. Therefore, under the latter assumption, we can define a random measure $\tilde Z_u$ on E' by

$$\delta_r \otimes \delta_u \otimes \tilde Z_u = \bar Z_{\{r\}\times[u,\infty)\times E'}.$$

The function φ^x introduced above can also be written in terms of $\tilde Z_0$: For any $y \in E'$,

$$\varphi^x(s) = \mathbb{E}_{\delta_{(x,0,y)}}(s^{\langle \tilde Z_0, 1\rangle}).$$

We can now state the main result of this section.

THEOREM 2. *Let $y \in E'$. Under $\mathbb{P}_{\delta_{(r,0,y)}}$, $\tilde Z = (\tilde Z_u, u \geq 0)$ is a branching Markov process on E' started from δ_y, associated with the Borel right process γ, with branching rate*

$$\tilde\lambda = E_r^\star(1 - e^{-\lambda T_r}) \tag{14}$$

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and reproduction law $\tilde{\Pi}$ *characterized by its generating function*

$$\hat{\tilde{\Pi}}(s) = \sum_{k\in\mathbb{N}} s^k \tilde{\Pi}(k) = \frac{\Lambda(s)}{\tilde{\lambda}} + s, \quad s \in (0,1] \tag{15}$$

where

$$\Lambda(s) = \lambda \int_E \Big(\hat{\Pi}\big(\varphi^x(s)\big) - \varphi^x(s)\Big) m(dx). \tag{16}$$

REMARK. The finiteness of $\tilde{\lambda}$ follows from excursion theory: As we already pointed out, $\tilde{\lambda}$ is the Laplace exponent, evaluated at λ, of the inverse local time $(\tau_s, s \geq 0)$. The fact that $\Lambda(s) < \infty$ for every $s \in (0,1]$ will be established in the course of the proof of Theorem 2.

LEMMA 3. *Let* $\tilde{\lambda}$ *and* Λ *be defined by formulas (14) and (16). Then,*

$$\tilde{\lambda} = \lim_{\varepsilon\downarrow 0+} \uparrow E_r^{\star}\big(1 - E_{\xi_\varepsilon}(\exp\{-\lambda T_r\}), T_r > \varepsilon\big)$$

and

$$\Lambda(s) = \lim_{\varepsilon\downarrow 0+} \uparrow E_r^{\star}\big(\varphi^{\xi_\varepsilon}(s) - s, T_r > \varepsilon\big).$$

PROOF OF LEMMA 3. We use (11), then the Markov property under the excursion measure and finally (12) to get

$$\begin{aligned}
\Lambda(s) &= \lambda E_r^{\star}\left(\int_0^{T_r} \Big(\hat{\Pi}\big(\varphi^{\xi_t}(s)\big) - \varphi^{\xi_t}(s)\Big)\, dt\right) \\
&= \lim_{\varepsilon\downarrow 0+} \uparrow \lambda E_r^{\star}\left(\int_\varepsilon^{T_r} \Big(\hat{\Pi}\big(\varphi^{\xi_t}(s)\big) - \varphi^{\xi_t}(s)\Big)\, dt, T_r > \varepsilon\right) \\
&= \lim_{\varepsilon\downarrow 0+} \uparrow E_r^{\star}\big(\varphi^{\xi_\varepsilon}(s) - s, T_r > \varepsilon\big).
\end{aligned}$$

The argument for $\tilde{\lambda}$ is similar. ∎

PROOF OF THEOREM 2. The intuitive idea of the result is as follows. Under $\mathbb{P}_{\delta_{(r,0,y)}}$, $\bar{Z}$ starts with one particle located at $(r,0,y)$. Let ζ be the first branching time, which is exponentially distributed with parameter λ. Then, the first branching time for $\tilde{Z}$ is the local time of the initial particle at time ζ, and is therefore exponentially distributed with parameter $\tilde{\lambda}$. Moreover, the branching distribution is the law of the number of descendants of the initial particle that eventually come back to $\{r\} \times [0,\infty) \times E'$. This suggests the expression

$$\hat{\tilde{\Pi}}(s) = E_r\left(\lambda \int_0^\infty dt e^{-\lambda t} \hat{\Pi}\big(\varphi^{\xi_t}(s)\big)\right). \tag{17}$$

One can easily check that the latter formula is equivalent to (15). First, by standard excursion theory, the right-hand side of (17) coincides with

$$\frac{1}{\tilde{\lambda}} E_r^{\star}\left(\int_0^{T_r} \lambda e^{-\lambda t} \hat{\Pi}\big(\varphi^{\xi_t}(s)\big)\, dt\right) = \frac{1}{\tilde{\lambda}} \lim_{\varepsilon\downarrow 0} \uparrow E_r^{\star}\left(\int_\varepsilon^{T_r} \lambda e^{-\lambda t} \hat{\Pi}\big(\varphi^{\xi_t}(s)\big)\, dt, T_r > \varepsilon\right).$$

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Then, by the Markov property under the excursion measure and (13), the last displayed quantity can be written as

$$\frac{1}{\tilde{\lambda}} \lim_{\varepsilon \downarrow 0} \uparrow E_r^\star\Big(\varphi^{\xi_\varepsilon}(s) - sE_{\xi_\varepsilon}(e^{-\lambda T_r}), T_r > \varepsilon\Big) = \frac{1}{\tilde{\lambda}}\big(\Lambda(s) + \tilde{\lambda}s\big),$$

by Lemma 3.

For the rigorous proof, we will use a different argument that can be generalized to the continuous branching case treated in the next section. Let $f: E' \longrightarrow (0, 1]$ be a measurable function and for $(x, u, y) \in \bar{E}$, set

$$\Phi_u^x[f](y) = \mathbb{E}_{\delta_{(x,0,y)}}(\exp\{\langle \tilde{Z}_u, \log f\rangle\}).$$

Recall that τ stands for the inverse local time. We then apply Lemma 1 with $F = \{r\} \times [u, \infty) \times E'$. Then $T_F = \tau_{u-} = \tau_u$ a.s. We can split $[0, \tau_u]$ into excursion intervals and apply the compensation formula to obtain

$$\begin{aligned}
\Phi_u^r[f](y) &= E_y'\big(f(\gamma_u)\big) + \lambda E_r \otimes E_y'\Big(\int_0^{\tau_u} \big\{\hat{\Pi}\big(\Phi^{\xi_s}_{u-L_s}[f](\gamma_{L_s})\big) - \Phi^{\xi_s}_{u-L(s)}[f](\gamma_{L(s)})\big\}\, ds\Big) \\
&= E_y'\Big(f(\gamma_u) + \lambda E_r\Big(\sum_{0 \le v < u} \int_{\tau_{v-}}^{\tau_v} \big\{\hat{\Pi}\big(\Phi^{\xi_s}_{u-v}[f](\gamma_v)\big) - \Phi^{\xi_s}_{u-v}[f](\gamma_v)\big\}\, ds\Big)\Big) \\
&= E_y'\Big(f(\gamma_u) + \lambda \int_0^u E_r^\star\Big(\int_0^{T_r} \big\{\hat{\Pi}\big(\Phi^{\xi_s}_{u-v}[f](\gamma_v)\big) - \Phi^{\xi_s}_{u-v}[f](\gamma_v)\big\}\, ds\Big)\, dv\Big) \\
&= E_y'\Big(f(\gamma_u) + \lambda \int_0^u \int_E \big\{\hat{\Pi}\big(\Phi^{x}_{u-v}[f](\gamma_v)\big) - \Phi^{x}_{u-v}[f](\gamma_v)\big\} m(dx)\, dv\Big).
\end{aligned}$$

Moreover, by (7) applied to $F_1 = \{r\} \times [0, \infty) \times E'$ and $F_2 = \{r\} \times [u, \infty) \times E'$, we have

$$\Phi_u^x[f](y) = \varphi^x\big(\Phi_u^r[f](y)\big), \tag{18}$$

since $L_t \equiv 0$ on $[0, T_r]$. Recall that Λ has been defined by (16). We thus get

$$\Phi_u^r[f](y) = E_y'\Big(f(\gamma_u) + \int_0^u \Lambda\big(\Phi^r_{u-v}[f](\gamma_v)\big)\, dv\Big) \tag{19}$$

Next, note that

$$\mathbb{P}_{\delta_x}(\langle \tilde{Z}_0, 1\rangle = 1) \ge E_x(\exp\{-\lambda T_r\}) \tag{20}$$

since $\langle \tilde{Z}_0, 1\rangle = 1$ when there has been no branching before time T_r. This entails

$$\varphi^x(s) - s \le \mathbb{P}_{\delta_x}(\langle 1, \tilde{Z}_0\rangle \ne 1) \le E_x(1 - \exp\{-\lambda T_r\}).$$

The finiteness of Λ then follows from Lemma 3 (and $\tilde{\lambda} < \infty$).

Moreover, by Lemma 3 again,

$$\frac{\Lambda(s)}{\tilde{\lambda}} + s = \lim_{\varepsilon \to 0+} \frac{E_r^\star\big(\varphi^{\xi_\varepsilon}(s) - sE_{\xi_\varepsilon}(e^{-\lambda T_r}), T_r > \varepsilon\big)}{E_r^\star\big(1 - E_{\xi_\varepsilon}(e^{-\lambda T_r}), T_r > \varepsilon\big)}.$$

The functions in the right-hand side are moment generating functions since the coefficients of their series expansion are nonnegative (use (20) for the coefficient of s) and their value for $s = 1$ is 1. Also their limit is trivially bounded from below by s. It easily follows that this limit is the moment generating function of a probability measure on the integers, $\tilde{\Pi}$. Note that $\tilde{\Pi}$ is sub-critical since $\hat{\tilde{\Pi}}(s) \geq s$. We can then rewrite (19) as

$$\Phi^r_u[f](y) = E'_y\Big(f(\gamma_u) + \tilde{\lambda} \int_0^u \big\{\hat{\tilde{\Pi}}\big(\Phi^r_{u-v}[f](\gamma_v)\big) - \Phi^r_{u-v}[f](\gamma_v)\big\}\, dv\Big). \tag{21}$$

We thus recover an equation of the type (4). It is then easy to complete the proof of the theorem. For $0 \leq t_0 < t_1 < \cdots < t_k$, we can apply (6) to the branching Markov process $\bar{Z}$ and the closed sets $F_i = \{r\} \times [t_i, \infty) \times E'$. For $g = -\log f$ we get

$$\mathbb{E}_{\delta_{(r,0,y)}}(\exp -\langle \tilde{Z}_{t_k}, g\rangle \mid \tilde{Z}_{t_1}, \ldots, \tilde{Z}_{t_{k-1}}) = \exp -\langle \tilde{Z}_{t_{k-1}}, \tilde{U}_{F_k}(g)\rangle,$$

where, using the additivity property of the local time,

$$\begin{aligned}\tilde{U}_{F_k}(g)(y) &= -\log \mathbb{E}_{\delta_{(r,t_{k-1},y)}}(\exp -\langle \tilde{Z}_{t_k}, g\rangle)\\ &= -\log \mathbb{E}_{\delta_{(r,0,y)}}(\exp -\langle \tilde{Z}_{t_k - t_{k-1}}, g\rangle)\\ &= -\log \Phi^r_{t_k - t_{k-1}}[f](y).\end{aligned}$$

This gives both the Markov property of $\tilde{Z}$ and (using (21)) the fact that the Laplace transform of its semigroup has the desired form. ■

REMARKS. 1. It is straightforward to extend the result of Theorem 2 to the case when the initial value of $\bar{Z}$ is $\eta \otimes \delta_0 \otimes \delta_y$, for any point measure η on E. The conclusion is the same, except that the initial value of $\tilde{Z}_0$ is now random, $\tilde{Z}_0 = \langle \tilde{Z}_0, 1\rangle \delta_y$, where the moment generating function of $\langle \tilde{Z}_0, 1\rangle$ is

$$\mathbb{E}_{\eta\otimes\delta_0\otimes\delta_y}(s^{\langle \tilde{Z}_0, 1\rangle}) = \exp\Big(\int \log\big(\varphi^x(s)\big)\eta(dx)\Big).$$

2. When r has a non-zero potential, we may assume for the sake of simplicity that the local time is given by

$$L_t = \int_0^t \mathbf{1}_{\{\xi_s = r\}}\, ds.$$

We can follow the same calculation as in the proof of Theorem 2 after splitting the time-interval $[0, \tau_{u-}]$ into the excursion intervals of ξ away from r and $\{s \leq \tau_{u-} : \xi_s = r\}$. We then find that, in the previous notation, $\tilde{Z}$ is a branching Markov process under $\mathbb{P}_{\delta_{(r,0,y)}}$, with branching rate $\lambda + \tilde{\lambda}$ and with reproduction law $(\lambda\Pi + \tilde{\lambda}\tilde{\Pi})/(\lambda + \tilde{\lambda})$. Observe that $\lambda + \tilde{\lambda}$ is the value of the Laplace exponent of the inverse local time evaluated at λ.

2.4. *Example.* We assume here that ξ is a *residual lifetime* process. Specifically, let Υ be a Radon measure on $(0, \infty)$ with $\int (1 \wedge x)\Upsilon(dx) < \infty$ and $S = (S_t, t \geq 0)$ a subordinator with no drift and Lévy measure Υ. Next, consider

$$\xi_t = \inf\{S_s - t : S_s > t\}, \quad t \geq 0.$$

The regular point is $r = 0$ and the potential measure of ξ killed when it hits 0 is

$$E_x\Big(\int_0^{T_0} f(\xi_t)\,dt\Big) = \int_0^x f(t)\,dt.$$

Suppose also that for some $\beta \in (0,1]$

$$\hat{\Pi}(s) = \frac{1}{1+\beta}(1-s)^{1+\beta} + s.$$

Fix $s \in (0,1]$ and write $g(x) = \varphi^x(s)$. Equation (12) is then an integrated Ricatti equation

$$g(x) = s + \frac{\lambda}{1+\beta}\int_0^x \big(1-g(t)\big)^{1+\beta}\,dt.$$

The solution is

$$g(x) = 1 - \Big(\frac{\beta\lambda}{1+\beta}x + (1-s)^{-\beta}\Big)^{-1/\beta}.$$

On the other hand, the occupation measure m under the excursion measure of ξ is

$$m(dt) = \bar{\Upsilon}(t)\,dt, \quad \text{where } \bar{\Upsilon}(t) = \Upsilon((t,\infty)).$$

We finally obtain by (16)

$$\Lambda(s) = \frac{\lambda}{1+\beta}\int_0^\infty \Big(\frac{\beta\lambda}{1+\beta}t + (1-s)^{-\beta}\Big)^{-(1+\beta)/\beta}\bar{\Upsilon}(t)\,dt$$

and

$$\tilde{\lambda} = \int \Upsilon(dt)(1-e^{-\lambda t}) = \lambda\int_0^\infty \bar{\Upsilon}(t)e^{-\lambda t}\,dt.$$

3. **Continuous branching.**

3.1. *Main result.* Let ψ be a nonnegative function on $[0,\infty)$ of the type (1). Notice that ψ is locally Lipschitz. One can find, in many different ways, a family $(\Pi_\varepsilon, \varepsilon > 0)$ of reproduction laws and a family $(\sigma_\varepsilon, \varepsilon > 0)$ of positive constants such that

$$\psi(u) = \lim_{\varepsilon\to 0+} \frac{\sigma_\varepsilon}{\varepsilon}\Big(\hat{\Pi}_\varepsilon(1-\varepsilon u) - (1-\varepsilon u)\Big), \tag{22}$$

uniformly on compact subsets of $[0,\infty)$. It is known (see [12], Theorem I.3.1) that if $Z^{(\varepsilon)}$ denotes the branching Markov process associated with $(\Pi_\varepsilon, \sigma_\varepsilon, \xi)$ and with initial value given by a Poisson distribution with intensity $\varepsilon^{-1}\mu$, then $\varepsilon Z^{(\varepsilon)}$ converges in law, in the sense of finite dimensional distributions, towards the superprocess $\mathcal{Z}$ started at μ, with spatial motion ξ and branching mechanism ψ, whose law has been characterized in the introduction.

Simultaneously with the superprocess $\mathcal{Z}$, we can construct, for every closed subset F of E, the exit measure $\mathcal{Z}_F$. The intuitive idea is the same as in the discrete setting, but the rigorous construction is much more involved (see [12]). For simplicity, we consider only closed sets F such that $P_x(T_F < \infty) = 1$ for every $x \in E$. Then, the exit measure $\mathcal{Z}_F$

is a random measure on E which satisfies the following properties. For every measurable function $g: E \longrightarrow [0, \infty)$,

$$\mathbb{E}_\mu(\exp\{-\langle Z_F, g\rangle\}) = \exp\{-\langle \mu, U_F[g]\rangle\}, \tag{23}$$

where

$$U_F[g](x) = -\log\Big(\mathbb{E}_{\delta_x}(\exp\{-\langle Z_F, g\rangle\})\Big), \quad x \in E$$

solves the equation

$$U_F[g](x) + E_x\Big(\int_0^{T_F} \psi\big(U_F[g](\xi_s)\big)\, ds\Big) = E_x\big(g(\xi_{T_F})\big). \tag{24}$$

The bound $U_F[g](x) \le E_x\big(g(\xi_{T_F})\big)$ easily implies

$$\mathbb{E}_\mu(\langle Z_F, g\rangle) \le E_\mu\big(g(\xi_{T_F})\big). \tag{25}$$

The analogue of (6) holds (see [12], Theorem I.1.3): If $F_k \subseteq F_{k-1} \cdots \subseteq F_1$ is a decreasing family of closed sets, then

$$\mathbb{E}_{\delta_x}(\exp\{-\langle Z_{F_k}, g\rangle\} \mid Z_{F_1}, \ldots, Z_{F_{k-1}}) = \exp\{-\langle Z_{F_{k-1}}, U_{F_k}[g]\rangle\}. \tag{26}$$

This property is known as the special Markov property for superprocesses.

These properties being granted, it is now fairly easy to follow the route described in Section 2. Suppose that ξ has a regular recurrent point r of zero potential (see the remark at the end of this section for the case when r has a positive potential). As previously, we denote by m the occupation measure under the excursion measure $P_r^\star$ of ξ away from r (the latter is specified by the normalization of the local time L). For $u > 0$, set

$$v^x(u) = U_{\{r\}}[u](x) = -\log \mathbb{E}_{\delta_x}(\exp\{-u\langle Z_{\{r\}}, 1\rangle\}).$$

By (24), v^x satisfies the identity

$$u = v^x(u) + E_x\Big(\int_0^{T_r} \psi\big(v^{\xi_s}(u)\big)\, ds\Big). \tag{27}$$

As in the previous section, we introduce an independent Markov process γ, and the process $\bar\xi = (\xi, L, \gamma_L)$. We denote by $\bar Z$ the superprocess with spatial motion $\bar\xi$ and branching mechanism ψ. Let us fix $x \in E$ and $y \in E'$. By (25), for every $s \ge 0$, the exit measure $\bar Z_{\{r\}\times[s,\infty)\times E'}$ gives no mass to $\{r\} \times (s, \infty) \times E'$, $\mathbb{P}_{\delta_{(x,0,y)}}$ a.s. Hence, $\mathbb{P}_{\delta_{(x,0,y)}}$ a.s., we can define a random measure $\tilde Z_s$ by the formula

$$\delta_r \otimes \delta_s \otimes \tilde Z_s = \bar Z_{\{r\}\times[s,\infty)\times E'}.$$

A similar argument shows that $\tilde Z_0 = \langle \tilde Z_0, 1\rangle \delta_y$, $\mathbb{P}_{\delta_{(x,0,y)}}$ a.s. Also note that the distribution of $\langle \tilde Z_0, 1\rangle$ under $\mathbb{P}_{\delta_{(x,0,y)}}$ coincides with the law of $\langle Z_{\{r\}}, 1\rangle$ under $\mathbb{P}_{\delta_x}$, by a simple "projection" argument. We have in particular $\tilde Z_0 = \delta_y$, $\mathbb{P}_{\delta_{(r,0,y)}}$ a.s.

The main result of this section is:

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THEOREM 4. *Under* $\mathbb{P}_{\delta_{(r,0,y)}}$, $\tilde{Z}$ *is a superprocess started at* δ_y, *with spatial motion* γ *and branching mechanism* $\tilde{\psi}$ *given by*

$$\tilde{\psi}(u) = \int_E \psi\big(v^x(u)\big)m(dx) = \lim_{\varepsilon\downarrow 0} \uparrow E_r^\star\big(u - v^{\xi_\varepsilon}(u), \varepsilon < T_r\big). \tag{28}$$

The function $\tilde{\psi}$ *is of the type*

$$\tilde{\psi}(u) = \tilde{a}u + \int_{(0,\infty)} (e^{-ut} - 1 + ut)\tilde{n}(dt) \tag{29}$$

where $\tilde{a} \geq 0$ *and* $\tilde{n}$ *is a measure on* $(0,\infty)$ *such that* $\int (t \wedge t^2)\tilde{n}(dt) < \infty$.

PROOF. We have already noticed that $\mathbb{P}_{\delta_{(r,0,y)}}(\tilde{Z}_0 = \delta_y) = 1$. Let $0 \leq t_1 < t_2 < \cdots < t_p$. Then (26) (applied to the superprocess $\bar{Z}$ and the closed sets $F_i = \{r\} \times [t_i, \infty) \times E'$) and a simple translation argument give, for every measurable function $g\colon E' \longrightarrow [0,\infty)$,

$$\mathbb{E}_{\delta_{(r,0,y)}}(\exp -\langle \tilde{Z}_{t_p}, g\rangle \mid \tilde{Z}_{t_1}, \ldots, \tilde{Z}_{t_{p-1}}) = \exp -\langle \tilde{Z}_{t_{p-1}}, \tilde{U}^r_{t_p - t_{p-1}}[g]\rangle,$$

where, for $x \in E$, $y \in E'$, $t \geq 0$,

$$\tilde{U}^x_t[g](y) = -\log \mathbb{E}_{\delta_{(x,0,y)}}(\exp\{-\langle \tilde{Z}_t, g\rangle\}).$$

It remains to check that the function $(t,y) \longrightarrow \tilde{U}^r_t[g](y)$ solves an integral equation of the type (2) where ψ is replaced by the function $\tilde{\psi}$ defined in the theorem.

By (24) (with $F = \{r\} \times [t,\infty) \times E'$) and the same argument as in the proof of Theorem 2, we have

$$\begin{aligned}
E'_y\big(g(\gamma_t)\big) &= \tilde{U}^r_t[g](y) + E_r \otimes E'_y\Big(\int_0^{\tau_t} \psi\big(\tilde{U}^{\xi_s}_{t-L_s}[g](\gamma_{L_s})\big)\, ds\Big)\\
&= \tilde{U}^r_t[g](y) + E'_y\bigg(E_r\Big(\sum_{0\leq u<t} \int_{\tau_{u-}}^{\tau_u} \psi\big(\tilde{U}^{\xi_s}_{t-u}[g](\gamma_u)\big)\, ds\Big)\bigg)\\
&= \tilde{U}^r_t[g](y) + E'_y\Big(\int_0^t E_r^\star\Big(\int_0^{T_r} \psi\big(\tilde{U}^{\xi_s}_{t-u}[g](\gamma_u)\big)\, ds\Big)\, du\Big)\\
&= \tilde{U}^r_t[g](y) + E'_y\Big(\int_0^t \int_E \psi\big(\tilde{U}^x_{t-u}[g](\gamma_u)\big)m(dx)\, du\Big).
\end{aligned}$$

Moreover, using the remarks preceding the statement of the theorem,

$$\begin{aligned}
\mathbb{E}_{\delta_{(x,0,y)}}(\exp -\langle \tilde{Z}_t, g\rangle) &= \mathbb{E}_{\delta_{(x,0,y)}}\big(\mathbb{E}_{\delta_{(x,0,y)}}(\exp -\langle \tilde{Z}_t, g\rangle \mid \bar{Z}_{\{r\}\times[0,\infty)\times E'})\big)\\
&= \mathbb{E}_{\delta_{(x,0,y)}}(\exp -\langle \tilde{Z}_0, \tilde{U}^r_t[g]\rangle)\\
&= \mathbb{E}_{\delta_{(x,0,y)}}\Big(\exp\big(-\langle \tilde{Z}_0, 1\rangle \tilde{U}^r_t[g](y)\big)\Big)\\
&= \exp -v^x\big(\tilde{U}^r_t[g](y)\big),
\end{aligned}$$

which gives the identity

$$\tilde{U}^x_t[g](y) = v^x\big(\tilde{U}^r_t[g](y)\big).$$

It follows that

$$E'_y\big(g(\gamma_t)\big) = \tilde{U}^r_t[g](y) + E'_y\Big(\int_0^t \tilde{\psi}\big(\tilde{U}^r_{t-u}[g](\gamma_u)\big)\,du\Big) \tag{30}$$

where

$$\tilde{\psi}(u) = \int_E \psi\big(v^x(u)\big)m(dx)$$

is as in the statement of the theorem. We have thus derived the desired integral equation for $\tilde{U}^r_t[g](y)$. Using (11), (27) and the Markov property under $E^\star_r$, we have also

$$\begin{aligned}\tilde{\psi}(u) = E^\star_r\Big(\int_0^{T_r} \psi\big(v^{\xi_s}(u)\big)\,ds\Big) &= \lim_{\varepsilon\downarrow 0} \uparrow E^\star_r\Big(\int_\varepsilon^{T_r} \psi\big(v^{\xi_s}(u)\big)\,ds, \varepsilon < T_r\Big)\\ &= \lim_{\varepsilon\downarrow 0} \uparrow E^\star_r\big(u - v^{\xi_\varepsilon}(u), \varepsilon < T_r\big).\end{aligned}$$

To complete the proof, we have to check that $\tilde{\psi}$ can be written in the form (29) (at the present stage, we do not even know that $\tilde{\psi}(u) < \infty$ for every $u \geq 0$). It follows readily from (23) that for every closed set F, the mass of the exit measure, $\langle Z_F, 1\rangle$, has an infinitely divisible distribution. Applying this to $F = \{r\}$, we see that

$$v^x(u) = ua_x + \int_{(0,\infty)} (1 - e^{-ut})\mu_x(dt)$$

for some $a_x \geq 0$ and some measure μ_x on $(0,\infty)$ with $\int(1 \wedge t)\mu_x(dt) < \infty$. On the other hand, we know from (25) and the definition of v^x that

$$(v^x)'(0) = \mathbb{E}_{\delta_x}(\langle Z_r, 1\rangle) \leq 1$$

and therefore

$$a_x + \int_{(0,\infty)} t\mu_x(dt) \leq 1 \quad \text{for all } x \in E.$$

Using the second expression for $\tilde{\psi}$ in (28), we have

$$\begin{aligned}\tilde{\psi}(u) &= \lim_{\varepsilon\downarrow 0} \uparrow E^\star_r\Big(u - ua_{\xi_\varepsilon} - \int_{(0,\infty)} (1 - e^{-ut})\mu_{\xi_\varepsilon}(dt), T_r > \varepsilon\Big)\\ &= \lim_{\varepsilon\downarrow 0} \uparrow \Big(u\tilde{a}_\varepsilon + \int_{(0,\infty)} (e^{-ut} - 1 + ut)\tilde{n}_\varepsilon(dt)\Big)\end{aligned}$$

where

$$\tilde{a}_\varepsilon = E^\star_r\Big(1 - a_{\xi_\varepsilon} - \int_{(0,\infty)} t\mu_{\xi_\varepsilon}(dt), T_r > \varepsilon\Big) \quad \text{and} \quad \tilde{n}_\varepsilon(dt) = E^\star_r\big(\mu_{\xi_\varepsilon}(dt), T_r > \varepsilon\big).$$

From the last expression for $\tilde{\psi}(u)$, we see that either $\tilde{\psi}(u) = \infty$ for every $u > 0$ or $\tilde{\psi}(u) < \infty$ for every $u > 0$. The first case cannot occur, because otherwise the equation (30) written with $g = \lambda > 0$ could have no solution. Thus $\tilde{\psi}(u) < \infty$ for every $u > 0$. By a standard argument (see *e.g.* Gnedenko and Kolmogorov [15], Section 19) $\tilde{\psi}(u)$ has necessarily an expression in the form (29). ∎

REMARKS. 1. More generally, if μ is a finite measure on E, then one can check that Theorem 4 remains valid under $\mathbb{P}_{\mu\otimes\delta_0\otimes\delta_y}$, but the initial value of $\tilde{Z}$ is now $z_0\delta_y$, where $z_0 \geq 0$ is a random variable with Laplace transform

$$-\log \mathbb{E}_{\mu\otimes\delta_0\otimes\delta_y}(\exp\{-uz_0\}) = \int_E v^x(u)\mu(dx).$$

2. When r has a non-zero potential, we assume as usual that the local time L is simply the time spent at r. A straightforward variation of the argument in the proof of Theorem 4 shows that under $\mathbb{P}_{\delta_{(r,0,y)}}$, $\tilde{Z}$ is a superprocess with spatial motion γ and branching mechanism $\psi + \tilde{\psi}$.

3.2. *Examples.* We now present detailed calculations in some special cases to obtain explicit expressions for the functions v^x and $\tilde{\psi}$ which appear in Theorem 4. We will always suppose that the measure-valued process Z is governed by a stable branching, viz

$$\psi(u) = ku^{\beta+1}$$

for some $\beta \in (0, 1]$ and $k > 0$.

3.2.1. Residual lifetime process. We assume here that ξ is as in subsection 2.4. Fix $u > 0$ and write $g(x) = v^x(u)$. Equation (27) is then an integrated Ricatti equation

$$g(x) + k\int_0^x g(t)^{\beta+1}\, dt = u.$$

The solution is

$$g(x) = (\beta kx + u^{-\beta})^{-1/\beta}.$$

Recall that the occupation measure m is $m(dt) = \bar{\Upsilon}(t)\, dt$, where $\bar{\Upsilon}(t) = \Upsilon((t,\infty))$. We finally obtain by (28)

$$\tilde{\psi}(u) = k\int_0^\infty \bar{\Upsilon}(t)(\beta kt + u^{-\beta})^{-(\beta+1)/\beta}\, dt.$$

REMARK. If we had taken the *age process*, $\inf\{t - S_s : S_s \leq t\}$, instead of the residual lifetime, then equation (27) would have given

$$u = g(x) + \frac{k}{\bar{\Upsilon}(x)}\int_x^\infty g(t)^{\beta+1}\bar{\Upsilon}(t)\, dt.$$

This does not seem easy to solve except for $\beta = 1$ and $\bar{\Upsilon}(t) = t^{-\rho}$ for some $\rho \in (0, 1)$ (in other words, ξ is a stable (ρ) age-process). Indeed, we deduce that in that case,

$$g'(x) - kg(x)^2 - \rho g(x)x^{-1} + u\rho x^{-1} = 0$$

and putting $g = -f'/(kf)$, we obtain

$$f'' - \rho f'/x = ku\rho f/x.$$

The latter equation can be solved in terms of modified Bessel functions.

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3.2.2. Reflecting Brownian motion. Assume now that ξ is a reflecting Brownian motion in $[0,\infty)$. Writing again $g(x) = v^x(u)$, we have that equation (27) reads

$$g(x) + 2k\int_0^\infty (t\wedge x)g(t)^{\beta+1}\,dt = u.$$

It follows that g is a non-increasing function and it is then clear that $g(\infty) = 0$. The integral equation gives $g'' = 2kg^{\beta+1}$, then $2g'g'' = 4kg'g^{\beta+1}$, and finally there exists some real number c such that

$$(g')^2 = \frac{4k}{\beta+2}g^{\beta+2} + c.$$

It follows that g is the inverse function of

$$f(x) = \int_x^u \Big(\frac{4k}{\beta+2}s^{\beta+2} + c\Big)^{-1/2} ds, \quad 0 < x \le u.$$

The function f must be defined for all $x \in (0,u]$ and satisfy $f(0+) = \infty$, which forces $c = 0$. Hence

$$f(x) = \frac{1}{\beta}\sqrt{\frac{\beta+2}{k}}(x^{-\beta/2} - u^{-\beta/2}), \quad g(x) = \Big(\beta\sqrt{\frac{k}{\beta+2}}x + u^{-\beta/2}\Big)^{-2/\beta}.$$

On the other hand, the occupation measure under the Brownian excursion law is simply the Lebesgue measure and equation (28) gives

$$\tilde\psi(u) = \int_0^\infty k\Big(\beta\sqrt{\frac{k}{\beta+2}}t + u^{-\beta/2}\Big)^{-2(\beta+1)/\beta} dt = \sqrt{\frac{k}{\beta+2}}u^{1+\beta/2}.$$

REMARK. If we had taken a Brownian motion on a finite interval, say $[0,a]$, with instantaneous reflection at the boundary points, we would have gotten the same equation, except that the boundary condition $g(\infty) = 0$ would have been replaced by $g'(a) = 0$. The solution would have satisfied

$$x = \int_{g(x)}^u \Big(\frac{4k}{\beta+2}s^{\beta+2} + c\Big)^{-1/2} ds$$

for some non-zero constant c which can be specified by the latter condition. For example, if $\beta = 1$, the solution would have involved Weierstrass functions. These calculations are closely related to Neveu [22].

3.2.3. Use of the scaling property. We will show here that $\tilde\psi$ can be specified up to a constant factor when the Markov process ξ satisfies a certain scaling property. Typically, suppose that ξ takes values in $[0,\infty)$, that for some $\nu > 0$

(31) the P_x-law of $(\lambda\xi_{t\lambda^{-\nu}}, t \ge 0)$ is $P_{\lambda x}$, $\quad\forall\lambda > 0$

and for some $\rho < 1$

(32) $$m(dt) = t^{-\rho}\,dt.$$

(in fact, it can be seen that (32) always holds under (31)). We claim that then

$$\tilde{\psi}(u) = cu^{1+\beta+\beta(\rho-1)/\nu} \tag{33}$$

for some constant number $c > 0$.

To establish (33), we first consider a discrete branching Markov process associated with $(\Pi_\varepsilon, \sigma_\varepsilon, \xi)$ as described in Section 2 and the beginning of subsection 3.1. We choose $\Pi_\varepsilon = \Pi$ independently of ε, with

$$\hat{\Pi}(s) = \frac{1}{1+\beta}(1-s)^{1+\beta} + s,$$

as in subsection 2.4, and $\sigma_\varepsilon = \varepsilon^{-\beta}$. Then, (22) holds with

$$\psi(u) = (1+\beta)^{-1} u^{1+\beta}.$$

Denote by $n^{(\varepsilon)}$ the mass of the exit measure at $\{0\}$ for this branching Markov process. Loosely speaking, the scaling property (31) implies that if we modify the space-scale by a factor $\lambda = 1/x$ and the time-scale by a factor $x^{-\nu}$, then the law of the spatial motion of each particle is unchanged, while the branching rate becomes $x^\nu \sigma_\varepsilon = \sigma_{\varepsilon x^{-\nu/\beta}}$. In other words, we have the identity

$$(n^{(\varepsilon)}, \mathbb{P}_{\delta_x}) \overset{(d)}{=} (n^{(\varepsilon x^{-\nu/\beta})}, \mathbb{P}_{\delta_1}).$$

Let $\kappa^{(\varepsilon,x)}$ be the cumulant of $n^{(\varepsilon)}$ under $\mathbb{P}_{\delta_x}$, so that

$$\mathbb{E}_{\delta_x}(\exp\{-un^{(\varepsilon)}\}) = \exp\{-\kappa^{(\varepsilon,x)}(u)\} = \exp\{-\kappa^{(\varepsilon x^{-\nu/\beta},1)}(u)\}, \quad u > 0.$$

Next, consider an independent Poisson variable $N(\varepsilon)$ with parameter ε^{-1}, so that

$$\begin{aligned} -\log \mathbb{E}_{N(\varepsilon)\delta_x}(\exp\{-u\varepsilon n^{(\varepsilon)}\}) &= \varepsilon^{-1}\big(1-\exp\{-\kappa^{(\varepsilon,x)}(u\varepsilon)\}\big) \\ &= \varepsilon^{-1}\big(1-\exp\{-\kappa^{(\varepsilon x^{-\nu/\beta},1)}(u\varepsilon)\}\big). \end{aligned}$$

According to Theorem I.3.1 of Dynkin [12], the distribution of $\varepsilon n^{(\varepsilon)}$ under $\mathbb{P}_{N(\varepsilon)\delta_x}$ converges as $\varepsilon \longrightarrow 0+$ towards that of the mass of the exit measure at $\{0\}$ of a superprocess started at δ_x, with spatial motion ξ and branching mechanism ψ. We thus have in the notation of the previous subsection

$$\begin{aligned} v^x(u) &= \lim_{\varepsilon \longrightarrow 0+} \varepsilon^{-1}\big(1-\exp\{-\kappa^{(\varepsilon x^{-\nu/\beta},1)}(u\varepsilon)\}\big) \\ &= \lim_{\eta \longrightarrow 0+} (x^{\nu/\beta}\eta)^{-1}\big(1-\exp\{-\kappa^{(\eta,1)}(ux^{\nu/\beta}\eta)\}\big) \\ &= x^{-\nu/\beta} v^1(ux^{\nu/\beta}). \end{aligned}$$

Equation (28) now gives

$$\begin{aligned} \tilde{\psi}(u) &= (1+\beta)^{-1} \int_0^\infty t^{-\rho}\big(v^t(u)\big)^{\beta+1}\, dt \\ &= (1+\beta)^{-1} \int_0^\infty t^{-\rho} t^{-\nu(\beta+1)/\beta}\big(v^1(ut^{\nu/\beta})\big)^{\beta+1}\, dt \\ &= (1+\beta)^{-1} u^{\beta+1+\beta(\rho-1)/\nu} \int_0^\infty s^{-\rho} s^{-\nu(\beta+1)/\beta}\big(v^1(s^{\nu/\beta})\big)^{\beta+1}\, ds \end{aligned}$$

which proves (33).

Let us discuss some special cases. First, if ξ is a Bessel process of dimension $d \in (0,2)$, then $\nu = 2$, $\rho = 1 - d$ and $\tilde{\psi}(u) = cu^{1+\beta(1-d/2)}$. For $d = 1$, ξ is a reflecting Brownian motion and this agrees with subsection 3.2.2. Next, if ξ is the age process or the residual lifetime process associated with a stable subordinator with exponent $\alpha \in (0,1)$, then $\nu = 1$ and $\rho = \alpha$. It follows that $\tilde{\psi}(u) = cu^{1+\alpha\beta}$, which agrees with subsection 3.2.1. Finally, it is immediate to adapt the argument leading to (33) to processes taking values in $(-\infty,\infty)$. If ξ is a stable Lévy process of index $\alpha \in (1,2]$, then $\nu = \alpha$, $\rho = 0$ and $\tilde{\psi}(u) = cu^{1+\beta(1-1/\alpha)}$. The Brownian case $\alpha = 2$ of course agrees with subsection 3.2.2.

4**. Subordination via the Brownian snake.** In this section, we show that the general subordination method for superprocesses that is developed in Section 3 can be interpreted in terms of the path-valued process called the Brownian snake. We treat only a special situation, corresponding to the residual lifetime process of subsection 2.4. This case is already interesting as it yields a path-valued process approach for superprocesses with a rather general branching mechanism, including the β-stable branching mechanism for $1 < \beta \leq 2$. The usual Brownian snake [17], [18], [13] applies only to the case $\beta = 2$. This section can be read independently of the previous ones, although ideas are similar. In subsection 4.1 below, we state some basic facts about the Brownian snake, which are mainly simple extensions of results in [17], [18].

4.1. *The Brownian snake with a discontinuous spatial motion.* Let ξ be as in the previous sections a càdlàg Borel right Markov process with values in a Polish space E. We denote by d_E a (complete) metric on E compatible with the topology on E. We may and will assume that the process ξ is defined on the Skorokhod space $\mathbb{D}([0,\infty),E)$. The mapping $x \longrightarrow P_x$ is then measurable from E into the space of all probability measures on $\mathbb{D}\big([0,\infty),E\big)$.

A *killed path* in E is a càdlàg function $\mathrm{w}\colon [0,\zeta) \longrightarrow E$, where $\zeta = \zeta(\mathrm{w}) \in (0,\infty)$ is called the *lifetime* of w. Note that the limit $\mathrm{w}(\zeta-)$ need not exist. It is also convenient to agree that every point x of E is a killed path with lifetime 0. We let $\mathcal{W}$ be the set of all killed paths and, if $\mathrm{w},\mathrm{w}' \in \mathcal{W}$, we define

$$d(\mathrm{w},\mathrm{w}') = d_E\big(\mathrm{w}(0),\mathrm{w}'(0)\big) + |\zeta - \zeta'| + \int_0^{\zeta\wedge\zeta'} \big(d_u(\mathrm{w}_{\leq u},\mathrm{w}'_{\leq u}) \wedge 1\big)\,du,$$

where $\mathrm{w}_{\leq u}$ stands for the restriction of w to $[0,u]$, and d_u denotes the distance on the Skorokhod space $\mathcal{D}_u = \mathbb{D}([0,u],E)$ (defined as in [2], p. 111 for instance). In particular, if $x,x' \in E$, $d(x,\mathrm{w}) = d_E\big(x,\mathrm{w}(0)\big) + \zeta(\mathrm{w})$, $d(x,x') = d_E(x,x')$. It is easy to check that d is a distance on $\mathcal{W}$ and that $(\mathcal{W},d)$ is a Polish space.

We can then define the Brownian snake with spatial motion ξ in much the same way as in [17] (where we dealt with a continuous spatial motion and considered stopped paths instead of killed paths). Let us fix $x \in E$ and denote by $\mathcal{W}_x$ the set of all killed paths with initial point x. Let $\mathrm{w} \in \mathcal{W}_x$ with lifetime $\zeta > 0$. If $0 \leq a < \zeta$, and $b \geq a$, we let $Q_{a,b}(\mathrm{w},d\mathrm{w}')$ be the unique probability measure on $\mathcal{W}_x$ such that

(i) $\zeta' = b$, $Q_{a,b}(\mathrm{w}, d\mathrm{w}')$ a.s.
(ii) $\mathrm{w}'(t) = \mathrm{w}(t)$, $\forall t \in [0, a]$, $Q_{a,b}(\mathrm{w}, d\mathrm{w}')$ a.s.
(iii) the law under $Q_{a,b}(\mathrm{w}, d\mathrm{w}')$ of $\big(\mathrm{w}'(a+t), 0 \le t < b-a\big)$ is the law of $(\xi_t, 0 \le t < b-a)$ under $P_{\mathrm{w}(a)}$.

In particular, $Q_{0,0}(\mathrm{w}, d\mathrm{w}') = \delta_x(d\mathrm{w}')$ and $Q_{0,b}(\mathrm{w}, d\mathrm{w}')$ is the law of $(\xi_t, 0 \le t < b)$ under P_x. The latter law will be denoted by $P_x^b(d\mathrm{w}')$. By convention, we also set $Q_{0,b}(x, d\mathrm{w}') = P_x^b(d\mathrm{w}')$.

Denote by $\theta_s^\zeta(dadb)$ the joint distribution of $(\inf_{[0,s]} \beta_r, \beta_s)$ when β is a one-dimensional reflecting Brownian motion with initial value $\beta_0 = \zeta \ge 0$:

$$\theta_s^\zeta(dadb) = \frac{2(\zeta + b - 2a)}{\sqrt{2\pi s^3}} \exp -\left(\frac{(\zeta + b - 2a)^2}{2s}\right) 1_{(0<a<\zeta\wedge b)}\, da\, db$$
$$+ (2/\pi s)^{1/2} \exp -\left(\frac{(\zeta + b)^2}{2t}\right) 1_{(0<b)}\, \delta_0(da)\, db.$$

PROPOSITION 5. *There exists a continuous strong Markov process in $\mathcal{W}_x$, denoted by $(W_s, s \ge 0)$, whose transition kernels are given by the formula*

$$Q_s(\mathrm{w}, d\mathrm{w}') = \int_{[0,\infty)^2} \theta_s^\zeta(da\, db) Q_{a,b}(\mathrm{w}, d\mathrm{w}').$$

If ζ_s denotes the lifetime of W_s, the process $(\zeta_s, s \ge 0)$ is a reflecting Brownian motion in $\mathbb{R}_+$.

Loosely speaking, the path W_s is erased from its tip when the lifetime ζ_s decreases and, on the other hand, it is extended (independently of the past) when ζ_s increases, using the law of the underlying spatial motion ξ for the extension. It is easy to check that a.s. for every $s < s'$, the killed paths W_s, $W_{s'}$ coincide for $t < m(s, s') := \inf_{[s,s']} \zeta_r$ (they also coincide at $t = m(s, s')$ provided that $m(s, s') < \zeta_s \wedge \zeta_{s'}$).

The proof of Proposition 5 is much similar to that of Theorem 2.1 in [17]. The process (W_s) is constructed from the Kolmogorov extension theorem. The existence of a continuous version is much easier here than in [17], where we used a different (stronger) metric. Indeed, the form of the metric d shows that, for $a \le \zeta$,

$$d(\mathrm{w}, \mathrm{w}') \le |\zeta - \zeta'| + |\zeta \wedge \zeta' - a|, \quad Q_{a,b}(\mathrm{w}, d\mathrm{w}') \text{ a.s.}$$

and therefore, if $(\beta_s, s \ge 0)$ is as previously,

$$\int Q_s(\mathrm{w}, d\mathrm{w}')\, d(\mathrm{w}, \mathrm{w}')^k \le c_k(E[|\beta_s - \beta_0|^k] + E[|\beta_s \wedge \beta_0 - \inf_{[0,s]} \beta_r|^k]) \le c_k' s^{k/2},$$

so that the classical Kolmogorov lemma yields the desired result. Finally, the strong Markov property is proved by the same argument as in [17].

We may and will assume that the process W is the canonical process on the space $C(\mathbb{R}_+, \mathcal{W}_x)$ of all continuous functions from $\mathbb{R}_+$ into $\mathcal{W}_x$. We also let $C_K(\mathbb{R}_+, \mathcal{W}_x)$ be the subset of $C(\mathbb{R}_+, \mathcal{W}_x)$ determined by the condition $W_s = x$ for s large enough (equivalently $\zeta_s = 0$ for s large). We denote by $\mathbf{P}_\mathrm{w}$ the law of (W_s) started at w.

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It is clear that x is a recurrent regular point for W. We denote by $\mathbf{N}_x$ the associated excursion measure. The law under $\mathbf{N}_x$ of $(\zeta_s, s \geq 0)$ is Itô's measure of positive excursions of linear Brownian motion. We assume that $\mathbf{N}_x$ is normalized so that

$$\mathbf{N}_x(\sup_{s\geq 0} \zeta_s > \varepsilon) = \frac{1}{2\varepsilon}.$$

We also set $\sigma = \inf\{s > 0, \zeta_s = 0\}$, which represents the duration of the excursion. Then, for any nonnegative measurable function G on $\mathcal{W}_x$,

$$\mathbf{N}_x \int_0^\sigma ds G(W_s) = \int_0^\infty dt E_x^t(G) \tag{34}$$

(see [18] Proposition 2.4).

Let D be an open subset of E such that $x \in D$ and let $F = D^c$. We may construct the exit local time of W from the set D along the lines of [18], Section 3. For $\mathrm{w} \in \mathcal{W}$, or $\mathrm{w} \in \mathbb{D}\big([0,\infty), E\big)$, we set $\tau(\mathrm{w}) = \inf\{t \geq 0, \mathrm{w}(t) \notin D\}$. In order to avoid trivialities, we assume that $P_x(\tau < \infty) > 0$. Then, as in Proposition 3.1 of [18], one easily sees that, if

$$U_s = \big(\zeta_s - \tau(W_s)\big)^+, \quad \eta_s = \inf\Big\{t, \int_0^t 1_{\{\tau(W_u)<\zeta_u\}}\, du > s\Big\},$$

the process $(U_{\eta_s}, s \geq 0)$ is under $\mathbf{P}_{\mathrm{w}}$ a reflecting Brownian motion in $\mathbb{R}_+$. From this it follows that the limit

$$L_s^D = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_0^s 1_{\{\tau(W_u)<\zeta_u<\tau(W_u)+\varepsilon\}}\, du$$

exists for every $s \geq 0$, $\mathbf{P}_{\mathrm{w}}$ a.s. and $\mathbf{N}_x$ a.e., and defines a continuous increasing process. By passing to the limit in (34) (see [18], Proposition 3.4 for details) we get that for any nonnegative measurable function G on $\mathcal{W}_x$,

$$\mathbf{N}_x \int_0^\sigma dL_s^D G(W_s) = E_x^D(G) \tag{35}$$

where E_x^D denotes the law of $(\xi_s, 0 \leq s < \tau)$ under $P_x(\cdot \cap \{\tau < \infty\})$.

Before introducing the exit measure in the Brownian snake setting, we make an additional assumption:

(H) For every $y \in D$, the process ξ is continuous at $s = \tau$, P_y a.s. on $\{\tau < \infty\}$.

This assumption is not really necessary, but it simplifies the theory and will hold in the applications we have in mind.

For every $\mathrm{w} \in \mathcal{W}$, set $\hat{\mathrm{w}} = \lim_{s\uparrow\zeta} \mathrm{w}(s) = \mathrm{w}(\zeta-)$ when the limit exists, and otherwise $\hat{\mathrm{w}} = \partial$, where ∂ denotes a cemetery point added to E as an isolate point. By convention, $\hat{x} = x$. By using **(H)** and (35) applied to the function $\Phi(\mathrm{w}) = 1_{E\setminus\partial D}(\hat{\mathrm{w}})$, we get that $\hat{W}_s \in \partial D$, dL_s^D a.e., $\mathbf{N}_x$ a.e. The exit measure from D, denoted as $\mathcal{Z}_F$ (recall that $F = D^c$, we use a notation different from [18], [19], but which is consistent with the previous sections), is the random measure on $\partial D = \partial F$ defined by

$$\langle \mathcal{Z}_F, g\rangle = \int_0^\sigma dL_s^D g(\hat{W}_s).$$

As an immediate consequence of (35) we have, for any bounded function G on ∂D,

$$\mathbf{N}_x(\langle \mathcal{Z}_F, \varphi\rangle) = E_x\big(\varphi(\xi_\tau)1_{\{\tau<\infty\}}\big). \tag{36}$$

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PROPOSITION 6. *Let φ be a bounded nonnegative measurable function on ∂D, and for every $x \in D$ set*

$$u(x) = \mathbf{N}_x(1 - \exp -\langle Z_F, \varphi\rangle).$$

Then,

$$u(x) = E_x(\varphi(\xi_\tau)1_{\{\tau<\infty\}}) - 2E_x\Big(\int_0^\tau dt u(\xi_t)^2\Big). \tag{37}$$

For the proof, see Theorem 4.2 of [18]. Our setting is more general than in [18], but the arguments are exactly the same. The connections between the Brownian snake and superprocesses (see below and [17]) show that (37) may be interpreted as a special case of (24).

The last ingredient that we need is the special Markov property for the Brownian snake (which is closely related to the formula (26) for superprocesses). We first define the excursions of W outside D. By the properties of the Brownian snake, the set

$$\{s, \tau(W_s) < \zeta_s\} = \{s, \tau(W_s) < \infty\}$$

is open $\mathbf{N}_x$ a.e. We denote by (a_i, b_i), $i \in I$ its connected components. For every $i \in I$, the paths $W_s, s \in (a_i, b_i)$ must coincide up to their exit time from D (see the proof of Proposition 3.1 in [18]). Denote by τ^i their common exit time and by x^i their common exit point. We define $W^i \in C_K(\mathbb{R}_+, \mathcal{W}_{x^i})$ by

$$W^i_s(t) = W_{(a_i+s)\wedge b_i}(\tau^i + t),$$

for $0 \le t < \zeta^i_s = \zeta_{(a_i+s)\wedge b_i} - \tau^i$.

Let $C_K(\mathbb{R}_+, \mathcal{W}) = \bigcup_{y\in E} C_K(\mathbb{R}_+, \mathcal{W}_y)$, which is equipped with the topology of uniform convergence with respect to the metric d. The measure $\sum \delta_{W^i}$ is a point measure on $C_K(\mathbb{R}_+, \mathcal{W})$, that accounts for the behavior of the paths W_s after their exit time from D. The goal of the special Markov property is to explicit the conditional distribution of this point measure knowing the σ-field that contains the information given by the paths W_s before they exit D.

To define the latter σ-field, we set for every $s \ge 0$

$$\kappa_s = \inf\Big\{t, \int_0^t du 1_{\{\tau(W_u)=\infty\}} > s\Big\}$$

and $W^D_s = W_{\kappa_s}$. Under $\mathbf{N}_x$, the process (W^D_s) is continuous a.e. It is obtained by removing the values of W over all intervals (a_i, b_i), and pasting together the remaining pieces. We let $\mathcal{E}^D$ be the σ-field generated by $(W^D_s, s \ge 0)$ and by the collection of all sets that are negligible for every measure $\mathbf{N}_y$, $y \in E$.

PROPOSITION 7. *The random measure Z_F is $\mathcal{E}^D$-measurable. If G is any nonnegative measurable function on $C_K(\mathbb{R}_+, \mathcal{W}_x)$,*

$$\mathbf{N}_x\Big(\exp - \sum_{i\in I} G(W^i) \mid \mathcal{E}^D\Big) = \exp - \int Z_F(dz)\mathbf{N}_z(1 - e^{-G}).$$

In other words, conditionally given $\mathcal{E}^D$, $\sum \delta_{W^i}(dW)$ is a Poisson measure on $C_K(\mathbb{R}_+, \mathcal{W})$ with intensity $\int Z_F(dz)\mathbf{N}_z(dW)$.

See [19] for the special case when ξ is Brownian motion in $\mathbb{R}^d$ and [20] for the general statement.

We finally explain the connections between the Brownian snake and superprocesses [17]. We assume for simplicity that ξ has no fixed discontinuities (for every $x \in E$, $t \geq 0$, $s \longrightarrow \xi_s$ is P_x a.s. continuous at $s = t$). By replacing the underlying process (ξ_t) by (t, ξ_t), we can easily extend the definition of the exit measure to space-time domains $D \subset [0, \infty) \times E$. For every $t > 0$, denote by $\mathcal{Z}_t$ the exit measure from the domain $[0, t) \times E$ (notice that assumption **(H)** holds for this special domain). In this special case, the exit local time L_s^D coincides with the usual local time of the process (ζ_s) at level t. As an immediate consequence of (35), we get that $\mathcal{Z}_t$ is a.e. supported on $\{t\} \times E$. Therefore we may and will identify $\mathcal{Z}_t$ with a random measure on E.

The laws under $\mathbf{N}_x$, $x \in E$ of the process $(\mathcal{Z}_t, t > 0)$ are the canonical measures of the superprocess with spatial motion ξ and branching mechanism $\psi(u) = 2u^2$ (see [5], Chapter 3 and [14], Section 4 for canonical measures of superprocesses). This means that, if μ is a finite measure on E and $\sum_{i \in I} \delta_{W^i}(dW)$ is a Poisson measure on $C_K(\mathbb{R}_+, \mathcal{W})$ with intensity $\int \mu(dx) \mathbf{N}_x(dW)$, the process

$$X_t = \sum_{i \in I} \mathcal{Z}_t(W^i), \quad (t > 0), \; X_0 = \mu$$

is a superprocess with spatial motion ξ and branching mechanism ψ, started at μ. In fact, the (time-homogeneous) Markov property follows from Proposition 7 and, on the other hand, if f is a bounded nonnegative measurable function on E,

$$E(\exp -\langle X_t, f \rangle) = \exp - \int \mu(dx) \mathbf{N}_x(1 - e^{-\langle \mathcal{Z}_t, f \rangle}) = \exp -\langle \mu, u_t \rangle,$$

where, by Proposition 6, the function $u_t(x) = \mathbf{N}_x(1 - \exp -\langle \mathcal{Z}_t, f \rangle)$ is the (unique) nonnegative solution of the equation

$$u_t(x) = E_x\big(f(\xi_t)\big) - 2E_x\Big(\int_0^t dh\, u_{t-h}(\xi_h)^2\Big). \tag{38}$$

4.2. *Subordination.* We now assume that ξ is the residual lifetime process of subsection 2.4, $\xi_t = \inf\{S_s - t, S_s > t\}$, where S is a subordinator. However, we allow the subordinator $S = (S_t, t \geq 0)$ to have a nonzero drift. Specifically, the Laplace transform of S_t is given by

$$E_0(\exp -\lambda S_t) = \exp -t\eta(\lambda),$$

where

$$\eta(\lambda) = b\lambda + \int_0^\infty \Upsilon(dh)(1 - e^{-\lambda h}),$$

where $b \geq 0$, $\int (1 \wedge h) \Upsilon(dh) < \infty$. The local time of ξ at 0 is $L_t = \inf\{s, S_s > t\}$. To avoid trivial cases, we also assume that $b > 0$ or $\Upsilon((0, \infty)) = \infty$, which implies that the process (L_t) is continuous.

As in the previous sections, let γ be an independent càdlàg Borel right Markov process with values in a Polish space E'. We assume here that γ has no fixed discontinuities.

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We can apply the construction of subsection 4.1 to the Markov process $\bar{\xi}_t = (\xi_t, L_t, \gamma_{L_t})$, taking values in $\bar{E} = \mathbb{R}_+ \times \mathbb{R} \times E'$. We write $\bar{E}$ for expectations relative to the process $\bar{\xi}$. To simplify the notation, we set $\Gamma_t = \gamma_{L_t}$.

Following subsection 4.1 we denote by $(\bar{W}_s)$ the Brownian snake with spatial motion $\bar{\xi}$. For $\mathrm{w} \in \bar{\mathcal{W}}$, we write $\mathrm{w}(t) = \big(\xi_t(\mathrm{w}), L_t(\mathrm{w}), \Gamma_t(\mathrm{w})\big)$. For every $h > 0$, let D_h be the domain $D_h = \mathbb{R}_+ \times [0, h) \times E'$, $F_h = D_h^c$, and let $\tau_h(\mathrm{w})$ be the exit time from D_h for a killed path w:

$$\tau_h(\mathrm{w}) = \inf\{t, L_t(\mathrm{w}) \geq h\}.$$

As previously, we also write τ_h for the exit time from D_h of the process $\bar{\xi}$.

Note that assumption **(H)** holds for the domain D_h, essentially because γ has no fixed discontinuities. We can thus define Z_{F_h} and, by formula (36), we have for $a \in D_h$

$$\mathbf{N}_a(\langle Z_{F_h}, g\rangle) = \bar{E}_a\big(g(\bar{\xi}_{\tau_h})\big).$$

This formula makes it clear that Z_{F_h} is a.s. supported on the set $\{0\} \times \{h\} \times E'$. Therefore there exists a random measure $\tilde{Z}_h$ such that $Z_{F_h} = \delta_0 \otimes \delta_h \otimes \tilde{Z}_h$, $\mathbf{N}_a$ a.e.

THEOREM 8. *The laws of the process $(\tilde{Z}_t, t > 0)$ under the measures $\mathbf{N}_{(0,0,y)}$, $y \in E'$ are the canonical measures of the superprocess with spatial motion γ and branching mechanism*

$$\tilde{\psi}(v) = 2bv^2 + \int_{(0,\infty)} \Upsilon(d\rho)\frac{2\rho v^2}{1 + 2\rho v}.$$

Equivalently, for $y \in E'$ and $0 < h < t$, for every bounded nonnegative measurable function f on E',

$$\mathbf{N}_{(0,0,y)}(1 - \exp -\langle \tilde{Z}_t, f\rangle) = v_t(y)$$

and

$$\mathbf{N}_{(0,0,y)}(1 - \exp -\langle \tilde{Z}_t, f\rangle \mid \tilde{Z}_u, 0 < u \leq h) = 1 - \exp -\langle \tilde{Z}_h, v_{t-h}\rangle, \tag{39}$$

where the function $\big(v_t(y), t \geq 0, y \in E'\big)$ is the unique nonnegative solution of the equation

$$v_t(y) = E'_y\big(f(\gamma_t)\big) - E'_y\Big(\int_0^t du\,\tilde{\psi}\big(v_{t-u}(\gamma_u)\big)\Big).$$

REMARK. Theorem 8 can be viewed as a special case of Theorem 4 above (take $\beta = 1$ in the explicit calculations of subsection 3.2.1). However, it is not so easy to identify the objects defined in terms of the Brownian snake with the corresponding quantities for superprocesses, and therefore we present a direct derivation of Theorem 8. Moreover the snake approach is crucial in the applications developed in subsection 4.3.

PROOF. For $a = (\alpha, l, y) \in D_h$, set

$$u_h(a) = \mathbf{N}_a(1 - \exp -\langle \tilde{Z}_h, f\rangle)$$

and $v_h(y) = u_h(0, 0, y)$. By Proposition 6, with a slight abuse of notation,

$$\begin{aligned} u_h(a) &= \bar{E}_a\big(f(\bar{\xi}_{\tau_h})\big) - 2\bar{E}_a\Big(\int_0^{\tau_h} dt\, u_h(\bar{\xi}_t)^2\Big) \\ &= E'_y\big(f(\gamma_h)\big) - 2\bar{E}_a\Big(\int_0^{\tau_h} dt\, u_h(\xi_t, L_t, \Gamma_t)^2\Big). \end{aligned} \tag{40}$$

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We are mainly interested in $v_h(y)$. A trivial translation argument gives $u_h(0,l,y) = u_{h-l}(0,0,y) = v_{h-l}(y)$. We will now check that $u_h(\alpha,l,y)$ can be expressed in terms of $u_h(0,l,y)$.

Let $D_h^* = (0,\infty)\times[0,h)\times E'$, $F_h^* = (D_h^*)^c$ and let τ_h^* denote the exit time from D_h^*. Notice that, for $a\in D_h^*$, $\tau_h^* < \tau_h$, $\bar{P}_a$ a.s. It follows from (35) that the measure $d_sL_s^{D_h}$ is $\mathbf{N}_a$ a.e. supported on paths w such that $\tau_h^*(\mathrm{w}) < \zeta(\mathrm{w})$. By decomposing $\tilde{Z}_h$ according to the different excursions outside D_h^* and using the special Markov property (Proposition 7), we get for $(\alpha,l,y)\in D_h^*$

$$\begin{aligned}\mathbf{N}_{(\alpha,l,y)}(1-\exp-\langle\tilde{Z}_h,f\rangle) &= \mathbf{N}_{(\alpha,l,y)}\Big(\mathbf{N}_{(\alpha,l,y)}(1-\exp-\langle\tilde{Z}_h,f\rangle \mid E^{D_h^*})\Big)\\ &= \mathbf{N}_{(\alpha,l,y)}(1-\exp-\langle Z_{F_h^*},u_h\rangle).\end{aligned}$$

On the other hand, by (36), $Z_{F_h^*} = \langle Z_{F_h^*},1\rangle\delta_{(0,l,y)}$, $\mathbf{N}_{(\alpha,l,y)}$ a.e.

We then make the following simple observation. If $D' = (F')^c$, $D'' = (F'')^c$ are two domains with respective exit times τ', τ'' and if $a\in D'\cap D''$ is such that $\tau' = \tau''$, $\bar{P}_a$ a.s., then $Z_{F'} = Z_{F''}$, $\mathbf{N}_a$ a.e. This follows from the approximation of the exit local time given in subsection 4.1, since we have $\tau'(W_u) = \tau''(W_u)$, du a.e., $\mathbf{N}_a$ a.e.

The residual lifetime process started at $\alpha > 0$ hits 0 at time α. Therefore, for $a = (\alpha,l,y)\in D_h^*$, the exit time from D_h^* coincides $\bar{P}_a$ a.s. with the exit time from the space-time domain $[0,\alpha)\times\bar{E}$. By the previous observation, $Z_{F_h^*} = Z_\alpha$, $\mathbf{N}_a$ a.e., where Z_α is as in the final remark of subsection 4.1. On the other hand, by a well-known formula for the Brownian local time under the Itô excursion measure, we have for $\lambda\ge 0$, $t>0$,

$$\mathbf{N}_a(1-\exp-\lambda\langle Z_t,1\rangle) = \frac{\lambda}{1+2\lambda t}.$$

By combining the previous results, we get

$$\begin{aligned}u_h(\alpha,l,y) = \mathbf{N}_{(\alpha,l,y)}(1-\exp-\langle\tilde{Z}_h,f\rangle) &= \mathbf{N}_{(\alpha,l,y)}(1-\exp-u_h(0,l,y)\langle Z_{F_h^*},f\rangle)\\ &= \frac{u_h(0,l,y)}{1+2\alpha u_h(0,l,y)}.\end{aligned}$$

We can now prove that the function $v_t(y)$ solves the integral equation of Theorem 8. We start from (40) and we evaluate

$$\bar{E}_{(0,0,y)}\Big(\int_0^{\tau_h} dt\, u_h(\bar{\xi}_t)^2\Big).$$

From our construction, $\bar{P}_{(0,0,y)}(\tau_h = S_h) = 1$ for every $h>0$. Therefore, for any function g,

$$\int_0^{\tau_h} dt\, g(t) = b\int_0^h ds\, g(S_s) + \sum_{s\le h, S_{s-}<S_s}\int_{S_{s-}}^{S_s} dt\, g(t),$$

$\bar{P}_{(0,0,y)}$ a.s. We then get

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$$
\begin{aligned}
&\bar{E}_{(0,0,y)}\Big(\int_0^{\tau_h} dt u_h(\bar{\xi}_t)^2\Big)\\
&\quad = b\bar{E}_{(0,0,y)}\Big(\int_0^h ds u_h(0,s,\gamma_s)^2\Big) + \bar{E}_{(0,0,y)}\Big(\sum_{s\le h, S_{s-}<S_s}\int_{S_{s-}}^{S_s} dt u_h(\xi_t, L_t, \gamma_{L_t})^2\Big)\\
&\quad = bE'_y\Big(\int_0^h ds v_{h-s}(\gamma_s)^2\Big) + \bar{E}_{(0,0,y)}\Big(\sum_{s\le h, S_{s-}<S_s}\int_{S_{s-}}^{S_s} dt u_h(S_s - t, s, \gamma_s)^2\Big)\\
&\quad = bE'_y\Big(\int_0^h ds v_{h-s}(\gamma_s)^2\Big) + \bar{E}_{(0,0,y)}\left(\sum_{s\le h, S_{s-}<S_s}\int_{S_{s-}}^{S_s} dt\Big(\frac{u_h(0,s,\gamma_s)}{1+2(S_s-t)u_h(0,s,\gamma_s)}\Big)^2\right)\\
&\quad = bE'_y\Big(\int_0^h ds v_{h-s}(\gamma_s)^2\Big) + E'_y\left(\int_0^h ds \int \Upsilon(d\rho)\int_0^\rho dt\Big(\frac{u_h(0,s,\gamma_s)}{1+2tu_h(0,s,\gamma_s)}\Big)^2\right)\\
&\quad = bE'_y\Big(\int_0^h ds v_{h-s}(\gamma_s)^2\Big) + E'_y\Big(\int_0^h ds \int \Upsilon(d\rho)\frac{\rho u_h(0,s,\gamma_s)^2}{1+2\rho u_h(0,s,\gamma_s)}\Big)\\
&\quad = \frac{1}{2}E'_y\Big(\int_0^h ds\tilde{\psi}\big(v_{h-s}(\gamma_s)\big)\Big).
\end{aligned}
$$

The integral equation for $v_t(y)$ follows using (40). The uniqueness of the nonnegative solution to this integral equation follows from Gronwall's lemma.

It remains to establish (39). We rely on the special Markov property (Proposition 7). Notice that $(\tilde{Z}_u, 0 \le u \le h)$ is measurable with respect to $\mathcal{E}^{D_h}$. Then, by considering the contributions to $\tilde{Z}_t$ coming from the different excursions outside D_h, we get

$$
\begin{aligned}
\mathbf{N}_{(0,0,y)}(1-\exp-\langle \tilde{Z}_t, f\rangle \mid \mathcal{E}^{D_h}) &= 1-\exp\Big\{-\int \tilde{Z}_h(dy')\mathbf{N}_{(0,h,y')}(1-\exp-\langle \tilde{Z}_t, f\rangle)\Big\}\\
&= 1-\exp-\langle \tilde{Z}_h, u_t(0,h,\cdot)\rangle\\
&= 1-\exp-\langle \tilde{Z}_h, v_{t-h}\rangle.
\end{aligned}
$$

■

REMARK. The function $\tilde{\psi}$ can be expressed in the usual form for branching mechanism functions. Notice that, for every $\rho > 0$,

$$\frac{2\rho v^2}{1+2\rho v} = \int_0^\infty \frac{d\alpha}{4\rho^2} e^{-\alpha/(2\rho)}(e^{-\alpha v} - 1 + \alpha v).$$

Therefore, the function $\tilde{\psi}$ of the theorem can be expressed as

$$\tilde{\psi}(v) = 2bv^2 + \int_0^\infty \Upsilon'(d\alpha)(e^{-\alpha v} - 1 + \alpha v),$$

where

$$\Upsilon'(d\alpha) = \Big(\int_0^\infty \frac{\Upsilon(d\rho)}{4\rho^2} e^{-\alpha/(2\rho)}\Big)\, d\alpha.$$

If $\nu(d\rho)$ denotes the image of the measure $\rho^{-1}\Upsilon(d\rho)$ under the mapping $\rho \longrightarrow (2\rho)^{-1}$, we have also $\Upsilon'(d\alpha) = (\frac{1}{2}\int \nu(d\rho) e^{-\alpha\rho})\, d\alpha$. We see that we get only a special class of

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measures $\Upsilon'(d\alpha)$, namely those measures which are absolutely continuous with respect to Lebesgue measure and whose density is the Laplace transform of a measure ν on $\mathbb{R}_+$ such that $\int \nu(d\rho)(\rho^{-1} \wedge \rho^{-2}) < \infty$. The β-stable branching mechanism, for $1 < \beta < 2$, is obtained by the choice $b = 0$ and $\Upsilon(d\rho) = c\rho^{-\beta}\, d\rho$.

4.3. *The compact support property.* Let $\tilde{\psi}$ be the branching mechanism function in Theorem 8 and let $(Z_t, t \geq 0)$ be a superprocess with spatial motion γ and branching mechanism $\tilde{\psi}$. We say that the compact support property holds for this superprocess if, for every $t > 0$, the topological support of Z_t is compact a.s., for any choice of the initial value $\mu \in M_f(E')$.

THEOREM 9. *Let $\tilde{\psi}$ be as in Theorem 8. Assume that*

(i) either $b > 0$, or $b = 0$ and there exist two positive constants c_1, $\rho < 1$ such that, for every $r \in [0, 1]$,

$$\int_0^\infty \Upsilon(dh)(h \wedge r) \geq c_1 r^{1-\rho};$$

(ii) there exist three positive constants c_2, k, p, with $p > 2$ if $b > 0$, $p\rho > 2$ if $b = 0$, such that, for every $y \in E'$, $t \in [0, 1]$,

$$E'_y\Big(\sup_{0\leq s\leq t} d_{E'}(y, \gamma_s)^k\Big) \leq c_2 t^p.$$

Then the compact support property holds for the $(\gamma, \tilde{\psi})$-superprocess.

REMARK. For the β-stable branching mechanism, condition (i) holds with $\rho = \beta - 1$. The compact support property will then hold provided that (ii) is verified with some $p > 2(\beta - 1)^{-1}$.

PROOF. We rely on Theorem 8, but write $Y_t = \tilde{Z}_t$ and $\mathbf{N}_y$ instead of $\mathbf{N}_{(0,0,y)}$ to simplify notation. By the canonical representation for superprocesses, Z_t has the same law as $\sum_{i\in I} Y_t(\omega_i)$, where $\sum_{i\in I} \delta_{\omega_i}(dW)$ is a Poisson measure with intensity $\int \mu(dy)\mathbf{N}_y(dW)$. We then observe that in the sum $\sum_{i\in I} Y_t(\omega_i)$ there is only a finite number of nonzero terms. This follows from Lemma 3.4 of [5] if we can check that, for some $\delta > 0$,

$$\liminf_{\lambda\to\infty} \lambda^{-1-\delta}\tilde{\psi}(\lambda) > 0. \tag{41}$$

When $b > 0$, this is obvious. Otherwise, we write

$$\tilde{\psi}(v) = 2v^2 \int_0^\infty \frac{dh}{(1+2hv)^2}\Upsilon\big([h,\infty)\big) \geq \frac{2}{9}v^2 \int_0^{1/v} dh\Upsilon\big([h,\infty)\big) \geq \frac{2}{9}c_1 v^{1+\rho},$$

by assumption (i).

It is therefore enough to check that the support of Y_t is compact $\mathbf{N}_y$ a.e., for every $y \in E'$. Recall that for $\mathrm{w} \in W$, we write $\mathrm{w}(t) = \big(\xi_t(\mathrm{w}), L_t(\mathrm{w}), \Gamma_t(\mathrm{w})\big)$ for $t < \zeta(\mathrm{w})$. We also set $\hat{\Gamma}(\mathrm{w}) = \lim_{t\uparrow\zeta(\mathrm{w})} \Gamma_t(\mathrm{w})$ if the limit exists, $\hat{\Gamma}(\mathrm{w}) = \partial$ otherwise. By convention, if $\mathrm{w} = (\alpha, l, y)$ is a trivial path, we take $\hat{\Gamma}(\mathrm{w}) = y$. By our construction of the exit measure, we have

$$(42) \qquad \langle Y_t, \varphi \rangle = \int_0^\sigma dL_s^{D_t} \varphi\big(\hat{\Gamma}(W_s)\big)$$

and we already know that $\hat{\Gamma}(W_s) \neq \partial$ for a.e. $s \in [0, \sigma]$, $\mathbf{N}_y$ a.e. We will prove that much more holds.

LEMMA 10. *Under the assumptions of Theorem 9, the mapping $s \longrightarrow \hat{\Gamma}(W_s)$ takes values in E' and is continuous $\mathbf{N}_y$ a.e.*

Theorem 9 immediately follows from Lemma 10. Simply observe that, by formula (42),

$$\operatorname{supp}(Y_t) \subset \{\hat{\Gamma}(W_s), 0 \le s \le \sigma\}$$

and the latter set is compact by Lemma 10.

PROOF OF LEMMA 10. By assumption (ii) and the Kolmogorov lemma, the process γ has continuous paths. From the construction of the Brownian snake, and the continuity of the local time L, it follows easily that $\mathbf{N}_y$ a.e. for every $s \in (0, \sigma)$, the mapping $t \longrightarrow \Gamma_t(W_s)$ is continuous on $[0, \zeta_s)$. We also know that for each fixed s, this mapping has a limit at $t = \zeta_s$, $\mathbf{N}_y$ a.e. on $\{s < \sigma\}$. Set

$$\Gamma(W_s) = \big(\Gamma_t(W_s), 0 \le t < \zeta_s\big), \quad \Gamma^*\big(W_s\big) = \big(\Gamma_t(W_s), 0 \le t \le \zeta_s\big)$$

where $\Gamma_{\zeta_s}(W_s) = \hat{\Gamma}(W_s)$. The mapping $s \longrightarrow \Gamma(W_s)$ takes values in the set of E'-valued killed paths, equipped with the metric d defined in subsection 4.1. On the other hand, for each fixed s, $\Gamma^*(W_s)$ is ($\mathbf{N}_y$ a.e. on $\{s < \sigma\}$) a stopped path in the sense of [17] (a finite path in the sense of subsection 2.1), that is a continuous mapping w from some compact interval $[0, \zeta]$ into E'. Following [17], the set of all E'-valued stopped paths is equipped with the distance

$$d^*(\mathrm{w}, \mathrm{w}') = |\zeta - \zeta'| + \sup_{t \ge 0} d_{E'}\big(\mathrm{w}(t \wedge \zeta), \mathrm{w}'(t \wedge \zeta')\big),$$

and is a Polish space for this distance.

Let us check that the process $\Gamma^*(W_s)$ has under $\mathbf{N}_y$ a version that is continuous for the metric d^*. For technical reasons, we first consider rational values of s. We may argue under the conditional distribution of $(W_s, s \in \mathbb{Q}_+)$ under $\mathbf{N}_y$ knowing the process $(\zeta_s, s \ge 0)$. We denote this conditional distribution by $\Theta_y^{(\zeta_s)}$. Under $\Theta_y^{(\zeta_s)}$, $(W_s, s \in \mathbb{Q}_+)$ is a time-inhomogeneous Markov process whose transition kernel between times s and s' is $Q_{m(s,s'),\zeta_{s'}}$, with $m(s, s') = \inf_{[s,s']} \zeta_h$ (notice that $m(s, s') < \zeta_s$ for every rational $0 < s < s' \le \sigma$, $\mathbf{N}_y$ a.e.). The previous description of $\Theta_y^{(\zeta_s)}$ immediately follows from the form of the transition kernels of W. We may furthermore assume that the mapping $s \longrightarrow \zeta_s$ is Hölder continuous with exponent $\frac{1}{2} - \varepsilon$ for every $\varepsilon > 0$. For rational $0 \le \alpha < \beta \le \sigma$, the paths $W_\alpha(t)$, $W_\beta(t)$ coincide for $t \le m := \inf_{[\alpha,\beta]} \zeta_s$, and then behave independently according to the law of the process ξ. Thus,

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$$
\begin{aligned}
d^*\big(\Gamma^*(W_\alpha),\Gamma^*(W_\beta)\big) &= |\zeta_\alpha-\zeta_\beta| + \sup_{m\le t} d_{E'}\big(\Gamma_{t\wedge\zeta_\alpha}(W_\alpha),\Gamma_{t\wedge\zeta_\beta}(W_\beta)\big)\\
&= |\zeta_\alpha-\zeta_\beta| + \sup_{m\le t\le\zeta_\alpha} d_{E'}\big(\Gamma_m(W_\alpha),\Gamma_t(W_\alpha)\big)\\
&\quad + \sup_{m\le t\le\zeta_\beta} d_{E'}\big(\Gamma_m(W_\beta),\Gamma_t(W_\beta)\big).
\end{aligned}
$$

Set $h=\zeta_\alpha-m$. From the behavior of the process $\bar\xi$ and the fact that $W_\alpha(t)$ is distributed under $\Theta_y^{(\zeta_s)}$ as a trajectory of $\bar\xi$ started at $(0,0,y)$ and stopped at time ζ_α, we see that

$$
\begin{aligned}
\Theta_y^{(\zeta_s)}\Big(\sup_{m\le t\le\zeta_\alpha} d_{E'}\big(\Gamma_m(W_\alpha),\Gamma_t(W_\alpha)\big)^k\Big) &\le \sup_{y'\in E'}\bar E_{(0,0,y')}\big(\sup_{0\le t\le L_h} d_{E'}(y',\gamma_t)^k\big)\\
&\le c_2\bar E_{(0,0,y)}\big((L_h)^p\big),
\end{aligned}
$$

by assumption (ii) and the independence of γ and L. Write E for $\bar E_{(0,0,y)}$ and let q be the smallest integer greater than p. Then,

$$
E\big((L_h)^p\big)\le E\big((L_h)^q\big)^{p/q}
$$

and by a well-known argument,

$$
E\big((L_h)^q\big)\le q!\,\big(E(L_h)\big)^q.
$$

Moreover,

$$
E(L_h)=\int_0^\infty ds P(L_h>s)=\int_0^\infty dsP(S_s<h)=U\big([0,h)\big),
$$

where U denotes the potential kernel of S. If $b>0$, it is trivial that $U\big([0,h)\big)\le h/b$. If $b=0$, a general property of subordinators (see *e.g.* [1], Proposition 3.1) gives for $h\le 1$

$$
U\big([0,h)\big)\le c\frac{h}{\int_0^h du\Upsilon\big([u,\infty)\big)}\le \frac{c}{c_1}h^\rho,
$$

by (i). Here, c is a constant independent of h.

By combining the previous bounds, we get, with $\rho=1$ if $b=0$,

$$
\Theta_y^{(\zeta_s)}\Big(\sup_{m\le t\le\zeta_\alpha} d_{E'}\big(\Gamma_m(W_\alpha),\Gamma_t(W_\alpha)\big)^k\Big)\le Ch^{p\rho}=C(\zeta_\alpha-m)^{p\rho}\le C'_\varepsilon(\beta-\alpha)^{p\rho(\frac12-\varepsilon)},
$$

provided that $\zeta_\alpha-m\le 1$. Here the constant C'_ε depends on $(\zeta_s,s\ge 0)$ and on ε but not on the choice of α, β in $[0,\sigma]$. An analogous bound holds for the symmetric term involving a supremum over $\{m\le t\le\zeta_\beta\}$. We finally obtain, for $\beta-\alpha\le\delta=\delta(\zeta_s,s\ge0)$,

$$\Theta_y^{(\zeta_s)}\Big(d^*\big(\Gamma^*(W_\alpha),\Gamma^*(W_\beta)\big)^k\Big)\le C''_\varepsilon(\beta-\alpha)^{p\rho(\frac12-\varepsilon)}.$$

We can choose ε so small that $p\rho(\frac12-\varepsilon)>1$ and then apply the Kolmogorov lemma. We get a process $(\tilde\Gamma(s),0\le s\le\sigma)$ which is continuous in the space of E'-valued stopped paths and coincides with $\big(\Gamma^*(W_s),0\le s\le\sigma\big)$ for rational values of s. It is then easy to see that $\big(\tilde\Gamma(s),0\le s\le\sigma\big)$ is a version of $\big(\Gamma^*(W_s),0\le s\le\sigma\big)$. If $\tilde\Gamma(s)=\big(\tilde\Gamma_t(s),0\le t\le\tilde\zeta_s\big)$, we have $\tilde\zeta_s=\zeta_s$ for every $s\in[0,\sigma]$, $\mathbf{N}_y$ a.e. and, because we already know that the killed paths $\Gamma(W_s)$ depend continuously on s, we have also

$$\tilde\Gamma_t(s)=\Gamma_t(W_s),\quad \forall t\in[0,\zeta_s),\ \forall s\in[0,\sigma],$$

$\mathbf{N}_y$ a.e. It follows that the limit

$$\hat\Gamma(W_s)=\lim_{t\uparrow\zeta_s}\Gamma_t(W_s)=\tilde\Gamma_{\zeta_s}(s)$$

exists for every $s\in[0,\sigma]$, $\mathbf{N}_y$ a.e., and defines a continuous function of s. This completes the proof of Lemma 10. ∎

REMARKS. (i) Under our assumptions, a general result of Fitzsimmons (see *e.g.* Theorem 2.1.3 of [5]) shows that the superprocess Z has a càdlàg version. Dealing with this version, the previous argument shows more precisely that, for every $\varepsilon>0$, the set

$$\bigcup_{t\ge\varepsilon}\operatorname{supp} Z_t$$

is a.s. relatively compact. In fact, by the right-continuity of the mapping $t\longrightarrow Z_t$, the closure of this set coincides with the closure of

$$\bigcup_{t\ge\varepsilon,t\in\mathbb{Q}}\operatorname{supp} Z_t.$$

Then, using the same argument as in the previous proof, we see that it is enough to check that

$$\bigcup_{t\ge\varepsilon,t\in\mathbb{Q}}\operatorname{supp} Y_t$$

is $\mathbf{N}_y$ a.e. relatively compact. However, this set is $\mathbf{N}_y$ a.e. contained in the compact set $\{\hat\Gamma(W_s),0\le s\le\sigma\}$.

(ii) Assume that γ is Brownian motion in $\mathbb{R}^d$, or more generally a nice diffusion process, so that assumption (ii) holds for every p with $k=2p$. The argument of the proof then shows that the mapping $s\longrightarrow\hat\Gamma(W_s)$ is Hölder continuous with exponent $\frac{\rho}{4}-\delta$ for

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every $\delta > 0$ (with $\rho = 1$ if $b > 0$). Hence,

$$\dim\{\hat{\Gamma}(W_s), 0 \le s \le \sigma\} \le \frac{4}{\rho}.$$

Let $\mathcal{R}$ denote the range of Z, defined as usual by

$$\mathcal{R} = \bigcup_{\varepsilon>0} \mathrm{cl}\Big(\bigcup_{t\ge\varepsilon} \mathrm{supp}\, Z_t\Big),$$

where cl(A) stands for the closure of A. We get from remark (i) that $\dim \mathcal{R} \le \frac{4}{\rho}$, a.s. This bound is not sharp. However, a forthcoming paper of Delmas [8] shows that the previous arguments can be used successfully to investigate the Hausdorff dimension properties of general superprocesses.

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