

## ON THE STRUCTURE AND TENSOR PRODUCTS OF JC-ALGEBRAS

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Norm closed (or weakly closed) Jordan algebras of self-adjoint operators on a Hilbert space were initially studied by Topping, Effros, and Størmer [15], [4], [12], [13]. These works are very “spatial”, in that the algebras are considered in one given representation. The introduction of their abstract counterparts, the *JB*- and *JBW*-algebras, has led to an increased interest in this subject. The author hopes this paper will support the view that a more “space-free” approach is fruitful, even if only the “concrete” algebras are under study. In accordance with this view, a “*JC*-algebra” in this paper will mean a normed Jordan algebra over the reals, which is isometrically isomorphic to a norm closed Jordan algebra of self-adjoint operators.

Some of the results in this paper are closely related to, or rewordings of, results in the above-mentioned papers. However, I feel that the present approach is sufficiently different to be of interest in itself. In particular, many of the technical difficulties associated with earlier approaches are avoided.

In Section 1 a spin algebra is defined as a global variant of a spin factor, and spin algebras are shown to have no dense representation of type higher than  $I_2$ . The spin algebras complement the “universally reversible” algebras, introduced in Section 2 (Theorem 2.5).

Lemma 2.1 has been proved in [9], but I believe my approach is even more direct. The universally reversible *JC*-algebras are shown to be those not having spin factor representations of too high dimensions. The study of those is reduced to the study of antiautomorphisms of order two on  $C^*$ -algebras.

Section 3 contains an example of a simple *JC*-algebra that has dense representations of both real, complex, and quaternionic types. This example is closely related to the fact, proved independently by Størmer [14] and Giordano [5], that there is, up to conjugation with an automorphism, only one antiautomorphism of order two on the hyperfinite  $II_1$ -factor.

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The universal  $C^*$ -algebra was first introduced as a technical tool in [2]. It is further studied in Section 4. We get an exact functor  $C^*$  from the category of  $JB$ -algebras to that of  $C^*$ -algebras. It is shown how to compute it for universally reversible  $JC$ -algebras, and, as an application, we generalize Størmer's result [11] on the decomposition of a Jordan homomorphism of  $C^*$ -algebras as a sum of a homomorphism and an anti-homomorphism.

Section 5 consists of an application of the earlier material to the problem of defining a tensor product of  $JC$ -algebras. A universal candidate is studied, and it is applied to show that a more naive approach would fail. The universal tensor product is computed in a few cases.

I got the idea for the example in Section 3 from a lecture in Marseille by Thierry Giordano on his uniqueness result for antiautomorphisms of the  $II_1$  factor. Bruno Iochum should also be thanked for his invitation and generous hospitality during my visit to Marseille. Most of the work for the present paper was conducted in Toronto during the year 1980/81. My warmest thanks go to Man-Duen Choi for inviting me, and for his generous financial support. It is a pleasure to thank Terry Gardner and his wife Connie for their warm hospitality during this year. My thanks also go to Fred Shultz, whom I visited for a week in May, 1981. It was through discussions with him that I finally got the results of Section 5 together.

Let us now turn to notation and preliminaries. Let  $A$  be a  $JB$ -algebra. A *dense representation* of  $A$  is a homomorphism of  $A$  onto a weakly dense subalgebra of a type  $IJBW$ -factor. The representation is said to be of *real, complex, or quaternionic type* if the factor is isomorphic to the algebra of self-adjoint operators on real, complex, or quaternionic Hilbert space. Up to equivalence, all dense representations arise in the following way: Consider a pure state  $\rho$  of  $A$ . Let  $c(\rho)$  be its central support in  $A^{**}$ ,  $A_\rho = c(\rho)A^{**}$ , and  $\varphi_\rho: A \rightarrow A_\rho$  is the map  $a \rightarrow c(\rho)a$ . Then  $A_\rho$  is a type  $I$  factor,  $\varphi_\rho$  is a dense representation, and  $\varphi_\rho^*$  maps the normal state space of  $A_\rho$  isomorphically onto the minimal split face  $F_\rho$  generated by  $\rho$ . See [2, Section 2]. A *concrete representation* of  $A$  is a homomorphism into the Jordan algebra  $\mathfrak{B}(H)_{sa}$ .

If  $a \in A$ , we define  $T_a(x) = a \circ x$ ,  $U_a(x) = \{axa\}$ . Then

$$U_a = 2T_a^2 - T_a^2$$

and  $U_a(x) = axa$  in  $\mathfrak{B}(H)_{sa}$ . Two elements  $a$  and  $b$  are said to *operator commute* if  $T_a$  and  $T_b$  commute.

If  $B$  is a subset of a  $C^*$ -algebra,  $[B]$  is the  $C^*$ -algebra generated by  $B$ . If  $\Phi$  is an antiautomorphism of order 2 on a  $C^*$ -algebra  $\mathfrak{A}$ ,  $\mathfrak{A}_{sa}^\Phi$  is the  $JC$ -algebra consisting of all elements  $a$  of  $\mathfrak{A}$  such that  $a = a^* = \Phi(a)$ .

**1. Spin algebras.** A *spin factor* is  $V_n = \mathbf{R}1 \oplus H_n$ , where  $H_n$  is a real Hilbert space of dimension  $n \cong 2$ .  $V_n$  is made into a *JB-algebra* by defining the product in such a way that 1 is the unit and  $\xi \circ \eta = (\xi|\eta)1$  if  $\xi, \eta \in H_n$ , and defining the norm

$$\|\lambda 1 \oplus \xi\| = |\lambda| + \|\xi\|.$$

Spin factors are simple *JC-algebras*, and reflexive as Banach spaces [16].

A *spin algebra* is a *JC-algebra* which has a faithful family of representations onto spin factors.

**PROPOSITION 1.1.** *A spin algebra has no dense representations other than onto spin factors and  $\mathbf{R}$ .*

This is an immediate consequence of Theorem 1.2 below. The primitive ideal space  $\text{Prim } A$  is defined as for  $C^*$ -algebras, only with dense representations replacing the irreducible ones [7]. For a dense representation  $\varphi: A \rightarrow M$ , we say  $\varphi$  has type  $I_k$  if  $M$  is a type  $I_k$  factor, and let  $\text{Prim}_n(A)$  be the set of kernels of type  $I_k$  representations, where  $k \leq n$ .

By Theorem 1.2, there is an ideal  $J$  of  $A$  such that a dense representation  $\varphi$  of  $A$  is onto  $\mathbf{R}$  or a spin factor if and only if  $\ker \varphi \supseteq J$ . If  $A$  is a spin algebra,  $J$  must be 0, and the conclusion of Proposition 1.1 follows.

**THEOREM 1.2.** *If  $A$  is a unital JB-algebra and  $n < \infty$ ,  $\text{Prim}_n A$  is closed in  $\text{Prim } A$ .*

*Proof.* For any pure state  $\rho$ , consider the dense representation  $\varphi_\rho: A \rightarrow A_\rho$ . Clearly,  $A_\rho$  is of type  $I_k$  ( $k \leq n$ ) if and only if for each  $b \in A_\rho$ , the powers  $1, b, \dots, b^n$  are linearly dependent. But  $\varphi_\rho^*$  maps the normal state space of  $A_\rho$  isomorphically onto  $F_\rho$ , and the pure normal states of  $A_\rho$  separate points. Therefore,  $\ker \varphi_\rho \in \text{Prim}_n A$  if and only if for each  $a \in A$  and  $\rho_0, \dots, \rho_n \in \partial_e F_\rho$  the  $(n + 1)$  by  $(n + 1)$  matrix  $(\langle a^i, \rho_j \rangle)_{i,j=0}^n$  is singular.

Choose  $\rho \in \partial_e S(A)$  with  $\ker \varphi_\rho \notin \text{Prim}_n A$ , and pick  $a \in A, \rho_0, \dots, \rho_n \in \partial_e F_\rho$  such that  $(\langle a^i, \rho_j \rangle)$  is non-singular. Choose neighbourhoods (in the relative weak \*-topology)  $V_j$  of  $\rho_j$  in  $\partial_e K$  such that whenever  $\sigma_j \in V_j, (\langle a^i, \sigma_j \rangle)$  is non-singular. Let  $W$  be the intersection of the sets  $\{\ker \varphi_\sigma: \sigma \in V_j\}$  for  $j = 0, \dots, n$ . Then  $W$  is a neighbourhood of  $\ker \varphi_\rho$  in  $\text{Prim } A$  [7, Theorem 4.1].

Assume that  $\varphi: A \rightarrow M$  is a dense representation, and

$$\ker \varphi \in W \cap \text{Prim}_n A.$$

Then there is  $\sigma_j \in V_j$  such that

$$\ker \varphi = \ker \varphi_{\sigma_j}.$$

Since  $M$  is a spin factor or finite dimensional,  $\varphi$  is onto  $M$ . Therefore,  $\varphi_{\sigma_j}$  is equivalent to  $\varphi$  for  $j = 0, \dots, n$ . In particular  $\sigma_j \in \partial_e F_{\sigma_0}$ . Since the matrix  $(\langle a^i, \sigma_j \rangle)$  is non-singular, then  $\ker \varphi \notin \text{Prim } {}_n A$ , a contradiction.

**2. Reversibility.** Recall from [12] that a  $J\mathcal{C}$ -algebra  $A$  contained in  $\mathfrak{B}(H)_{sa}$  is called *reversible* if

$$a_1, \dots, a_n \in A \Rightarrow a_1 \dots a_n + a_n \dots a_1 \in A.$$

We call  $A$  *universally reversible* if  $\pi(A)$  is reversible for each concrete representation  $\pi: A \rightarrow \mathfrak{B}(H)_{sa}$ .

LEMMA 2.1 [9]. *Of all spin factors, only  $V_2, V_3$  and  $V_5$  admit reversible representations, and only  $V_2$  and  $V_3$  are universally reversible.*

*Proof.* That  $V_3$  is universally reversible is proved in [2, Lemma 4.1]. The same proof holds for  $V_2$  as well.

Assume that  $V_n \subseteq \mathfrak{B}(H)_{sa}$  is reversible,  $n \geq 4$ . Choose an orthonormal set  $s_1, s_2, s_3, s_4 \in H_n$ . Equivalently,  $s_j$  is a symmetry in  $V_n$ , and  $s_i \circ s_j = 0$  if  $i \neq j$ . Let

$$s_5 = s_1 s_2 s_3 s_4 = \frac{1}{2}(s_1 s_2 s_3 s_4 + s_4 s_3 s_2 s_1).$$

Then  $s_5 \in V_n$  is a symmetry, anticommuting with  $s_1, s_2, s_3, s_4$ . So  $s_1, \dots, s_5$  is an orthonormal set in  $H_n$ , and this is impossible if  $n = 4$ . If  $n \geq 6$ , choose a new unit vector  $s_6$  in  $H_n$ , orthogonal to  $s_1, \dots, s_5$ . Then  $s_6$  is a symmetry which both anticommutes and commutes with  $s_5 = s_1 s_2 s_3 s_4$ ; again, this is a contradiction.

$V_5 \cong M_2(\mathbf{H})_{sa}$  has a reversible representation  $\pi$  in  $M_4(\mathbf{C})_{sa}$ . Then, for some orthonormal basis  $s_1, \dots, s_5$  of  $H_5$ ,

$$\pi(s_1)\pi(s_2)\pi(s_3)\pi(s_4) = \pi(s_5).$$

Define a representation  $\pi'$  by  $\pi'(s_i) = \pi(s_i)$  if  $1 \leq i \leq 4$  and  $\pi'(s_5) = -\pi(s_5)$ . Then  $\pi \oplus \pi'(V_5)$  is not reversible.

THEOREM 2.2. *A  $J\mathcal{C}$ -algebra is universally reversible if and only if it has no spin factor representations other than onto  $V_2$  and  $V_3$ .*

*Proof.* The “only if” part is immediate from Lemma 2.1. The proof of [2, Theorem 4.6] proves the “if” part, with the minor modification that in [2, Lemma 4.5]  $s_0 \bar{A}$  splits into two parts, one a “global  $V_2$ ” and one a “global  $V_3$ ”.

The study of universally reversible algebras can be reduced to the study of \*-antiautomorphisms of C\*-algebras:

**PROPOSITION 2.3.** *If  $A$  is a universally reversible JC-algebra then there is a C\*-algebra  $\mathfrak{A}$  and a \*-antiautomorphism  $\Phi$  of  $\mathfrak{A}$  of order two such that  $A$  is isomorphic to  $\{a \in \mathfrak{A} : a = a^* = \Phi(a)\}$ .*

*Proof.* If  $A \subseteq \mathfrak{B}(H)_{sa}$ , choose a transposition on  $\mathfrak{B}(H)$  corresponding to some orthonormal basis, and identify  $A$  with

$$\{a \oplus a' : a \in A\} \subseteq \mathfrak{B}(H) \oplus \mathfrak{B}(H).$$

Let  $\mathfrak{A}$  be the C\*-algebra generated by  $A$ , with the \*-antiautomorphism  $\Phi$  given by

$$\Phi(a \oplus b) = b' \oplus a'.$$

If  $z \in \mathfrak{A}$  then  $z$  belongs to the closed real linear span of elements of the form

$$z' = x_1 \dots x_n + iy_1 \dots y_n, \text{ where } x_i, y_i \in A.$$

If  $z = z^* = \Phi(z)$  then  $z = 1/4(z + z^* + \Phi(z) + \Phi(z)^*)$  can be approximated by linear combinations of elements of the form

$$\frac{1}{4}(z' + z'^* + \Phi(z') + \Phi(z')^*) = \frac{1}{2}(x_1 \dots x_n + x_n \dots x_1) \in A.$$

Therefore  $z \in A$ , and the non-trivial part of the proof is finished.

Finally, the study of general JC-algebras can be reduced to the study of spin algebras, universally reversible algebras, and certain extensions.

**LEMMA 2.4.** *Let  $I$  be a closed ideal in a JB-algebra  $A$ , and  $M$  a JBW-algebra. Any homomorphism  $I \rightarrow M$  extends to a homomorphism  $A \rightarrow M$ .*

*Proof.* Identify  $I^{**}$  with the weak closure in  $A^{**}$  of  $I$ . Then  $I^{**} = eA^{**}$ , for a central projection  $e$  in  $A^{**}$  [3, Theorem 3.3]. The homomorphism  $I \rightarrow M$  extends to a normal homomorphism  $I^{**} \rightarrow M$  [2, p. 270]. Compose with the map  $a \rightarrow ae$  of  $A$  into  $I^{**}$ .

**THEOREM 2.5.** *Any JC-algebra  $A$  has an ideal  $I$  which is universally reversible and such that  $A/I$  is a spin algebra.*

*Proof.* Let  $I$  be the intersection of all kernels of spin factor representations. Then  $A/I$  is a spin algebra. By Lemma 2.4 and Theorem 2.2,  $I$  is universally reversible.

**3. A simple  $J\mathcal{C}$ -algebra.** One might hope that the topology of  $\text{Prim } A$  will separate, say, the real representations from the complex ones. This section contains an example to effectively crush any such hope: There exists a simple  $J\mathcal{C}$ -algebra possessing representations of three different types.

First, however, we prove some auxiliary results. The symbol  $\perp$  below signifies positive annihilator in  $A^*$ , resp.  $A$ :

**LEMMA 3.1.** *Let  $A$  be a  $J\mathcal{B}$ -algebra and  $B$  a closed quadratic ideal in  $A$ . Then  $B^+ = B^{\perp\perp}$ .*

*Proof.* The weak closure  $\bar{B}$  of  $B$  in  $A^{**}$  is a quadratic ideal, and  $B = \bar{B} \cap A$ . By [3]  $\bar{B} = \{pA^{**}p\}$  for some projection  $p \in A^{**}$ . If  $b \in A^+$ ,  $b \notin B$  then

$$\{(1-p)b(1-p)\} \neq 0.$$

If  $\rho \in A^{*+}$  does not annihilate this element, then

$$\{(1-p)\rho(1-p)\} \in B^+$$

but does not annihilate  $b$ . Hence  $b \notin b^{\perp\perp}$ , and the proof is complete.

The next two results (for Jordan ideals) are contained in [4], but the proofs below are shorter and more direct.

**PROPOSITION 3.2.** *If  $A$  is a  $J\mathcal{C}$ -algebra,  $A \subseteq \mathfrak{B}(H)_{sa}$ , and  $B$  is a closed quadratic ideal of  $A$ , then  $B = A \cap [B]$ .*

*Proof.* If  $a \in A^+$  but  $a \notin B$  there is a positive linear functional annihilating  $B$  but not  $a$  (Lemma 3.1). Extend to a positive linear functional on  $\mathfrak{B}(H)$ . By the Cauchy-Schwarz inequality, the extension annihilates  $[B]$ , so  $a \notin [B]$ .

**PROPOSITION 3.3.** *Let  $\mathfrak{A}$  be a  $C^*$ -algebra,  $A \subseteq \mathfrak{A}_{sa}$  a  $J\mathcal{C}$ -algebra generating  $\mathfrak{A}$  and  $J$  an ideal of  $A$ . Then  $[J]$  is an ideal of  $\mathfrak{A}$ .*

*Proof.* We prove  $AJ \subseteq [J]$ . Indeed,  $J$  is generated by its squares, so it is enough to prove  $ab^2 \in [J]$  when  $a \in A$ ,  $b \in J$ . But

$$ab^2 = (a \circ b)b - b(a \circ b) + a \circ b^2$$

proves that. Now  $A[J] \subseteq [J]$  follows, and since  $A$  generates  $\mathfrak{A}$ ,  $\mathfrak{A}[J] \subseteq [J]$ . Similarly,  $[J]\mathfrak{A} \subseteq [J]$ .

**THEOREM 3.4.** *There exists a simple  $J\mathcal{C}$ -algebra which admits both real, complex, and quaternionic dense representations.*

*Proof.* Let

$$\mathfrak{A} = \bigotimes_{n=1}^{\infty} M_2(\mathbb{C}),$$

the CAR algebra. Let  $t_2: M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$  be the transposition and  $u_2: M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$  the “quaternionic flip”:

$$u_2 \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}.$$

Now consider the two antiautomorphisms  $t = t_2 \otimes t_2 \otimes \dots$  and  $u = u_2 \otimes t_2 \otimes t_2 \otimes \dots$  on  $\mathfrak{A}$ . We claim that  $t, u$  are conjugate, i.e., there is an automorphism  $\alpha$  of  $\mathfrak{A}$  such that  $u = \alpha^{-1}t\alpha$ . Indeed, if  $\sim$  denotes conjugacy, we have  $u_2 \otimes u_2 \sim t_2 \otimes t_2$  on  $M_4(\mathbb{C})$ , so

$$\begin{aligned} u &= u_2 \otimes (t_2 \otimes t_2) \otimes (t_2 \otimes t_2) \otimes \dots \\ &\sim u_2 \otimes (u_2 \otimes u_2) \otimes (u_2 \otimes u_2) \otimes \dots \\ &= (u_2 \otimes u_2) \otimes (u_2 \otimes u_2) \otimes \dots \\ &\sim (t_2 \otimes t_2) \otimes (t_2 \otimes t_2) \otimes \dots = t. \end{aligned}$$

It follows that there is an automorphism of  $\mathfrak{A}$  carrying  $\mathfrak{A}_{sa}^t$  onto  $\mathfrak{A}_{sa}^u$ . Below we shall construct a real and a complex representation of  $\mathfrak{A}_{sa}^t$ , and a quaternionic representation of  $\mathfrak{A}_{sa}^u$ .

We start with a real representation of  $\mathfrak{A}_{sa}^t$ . Let  $\rho_2$  be the pure state on  $M_2(\mathbb{C})$  given by  $\langle a, \rho_2 \rangle = a_{11}$ . Then  $\rho = \rho_2 \otimes \rho_2 \otimes \dots$  is a pure state on  $\mathfrak{A}$ . Let  $(e_1, e_2)$  be the natural basis of  $\mathbb{C}^2$  and denote by

$$\bigotimes_{n=1}^{\infty} (\mathbb{C}^2, e_1) = H$$

the Hilbert space direct limit of the sequence

$$H_n = \bigotimes_{k=1}^n \mathbb{C}^2$$

where  $H_n$  is identified with the subspace  $H_n \otimes e_1$  of  $H_{n+1}$ . Let  $J_2$  be coordinatewise conjugation in  $\mathbb{C}^2$ . Then  $J = J_2 \otimes J_2 \otimes \dots$  is well defined on  $H, J^2 = 1$ , the GNS-representation  $\pi_\rho$  of  $\mathfrak{A}$  can be thought of as acting in  $H$  (with representing vector  $e_1 \otimes e_1 \otimes \dots$ ) and

$$\pi_\rho(t(a)) = J\pi_\rho(a)J \tag{a \in \mathfrak{A}}.$$

It follows that the restriction of  $\pi_\rho$  to  $\mathfrak{A}_{sa}^t$  is a dense representation in the JBW-factor  $\{a \in \mathfrak{B}(H): a = a^* = JaJ\}$ , which can be identified with  $\mathfrak{B}(K)_{sa}$ , where  $K$  is the real Hilbert space  $\{\xi \in H: J = \xi\}$ .

Next, consider the mapping  $j_2: \mathbb{C}^2 \rightarrow \mathbb{C}^2$  defined by

$$j_2 \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \bar{\mu} \\ -\lambda \end{pmatrix},$$

and note that  $j = j_2 \otimes J_2 \otimes J_2 \otimes \dots$  is a conjugate linear unitary (an “antiunitary”) on  $H$ , with  $j^2 = -1$ . We find, for  $a \in \mathfrak{A}$ ,

$$\pi_\rho(u(a)) = -j\pi_\rho(a)^*j.$$

Therefore, the restriction of  $\pi_\rho$  to  $\mathfrak{A}_{sa}^u$  is a dense representation in the  $JW$ -factor  $\{a \in \mathfrak{B}(H): a = a^* \text{ and } aj = ja\}$ .  $H$  can be given a structure of quaternionic Hilbert space in which  $j$  is multiplication with the unit quaternion usually called  $j$ , so the above  $JW$ -factor consists of all quaternionic linear mappings (see also the proof of [2, Theorem 3.1]).

Finally, choose pure states  $\sigma_2 \neq \tau_2$  on  $M_2(\mathbb{C})$  such that  $\sigma_2 = \tau_2 \circ t_2$ . Let  $\sigma = \sigma_2 \otimes \sigma_2 \otimes \dots, \tau = \tau_2 \otimes \tau_2 \otimes \dots$ . Then  $\sigma, \tau$  are inequivalent pure states on  $\mathfrak{A}$ , and we get an antihomomorphism  $\beta$  of  $\mathfrak{B}(H_\sigma)$  onto  $\mathfrak{B}(H_\tau)$  such that

$$\pi_\tau t(a) = \beta\pi_\sigma(a) \tag{a \in \mathfrak{A}}.$$

Here  $\pi_\sigma$  and  $\pi_\tau$  are the  $GNS$ -representations in  $H_\sigma, H_\tau$  corresponding to  $\sigma$  and  $\tau$ . Let  $\pi = \pi_\sigma \oplus \pi_\tau$ . Since  $\pi_\sigma$  and  $\pi_\tau$  are inequivalent and irreducible,  $\pi(\mathfrak{A})$  is dense in  $\mathfrak{B}(H_\sigma) \oplus \mathfrak{B}(H_\tau)$ . Define the antiautomorphism  $\Phi$  on  $\mathfrak{B}(H_\sigma) \oplus \mathfrak{B}(H_\tau)$  by

$$\Phi(a \oplus b) = \beta^{-1}(b) \oplus \beta(a).$$

Then  $\Phi^2 = 1$ , and the restriction of  $\pi$  to  $\mathfrak{A}_{sa}^l$  is a dense representation in the  $JW$ -factor

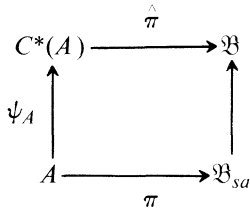
$$\{c \in \mathfrak{B}(H_\sigma) \oplus \mathfrak{B}(H_\tau): \Phi(c) = c = c^*\},$$

which is isomorphic with  $\mathfrak{B}(H_\sigma)_{sa}$ .

We have proved that  $\mathfrak{A}_{sa}^l$  has representations of all three types. That it is simple is an easy consequence of Propositions 3.2 and 3.3.

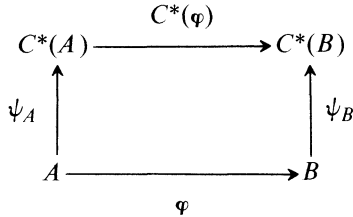
**4. The universal  $C^*$ -algebra.** Consider a  $JB$ -algebra  $A$ . To  $A$  we associate a  $C^*$ -algebra  $C^*(A)$ , which is the unique  $C^*$ -algebra with the following properties: There is a homomorphism  $\psi_A: A \rightarrow C^*(A)_{sa}$  such that  $\psi_A(A)$  generates  $C^*(A)$  as a  $C^*$ -algebra. Whenever  $\mathfrak{B}$  is a  $C^*$ -algebra and  $\pi: A \rightarrow \mathfrak{B}_{sa}$  is a homomorphism,  $\pi$  lifts to a  $*$ -homomorphism  $\hat{\pi}: C^*(A) \rightarrow \mathfrak{B}$  such that  $\pi = \hat{\pi} \circ \psi_A$ .





The existence of  $C^*(A)$  is proved in [2, Theorem 5.1].  $A$  need not have a unit (drop all references to the unit in the proof), but if it does, then  $\psi_A(1)$  is a unit of  $C^*(A)$ . It may be that  $C^*(A) = 0$ , for example if  $A = M_3^8$ . In general, the kernel of  $\psi_A$  is the exceptional ideal of [1].

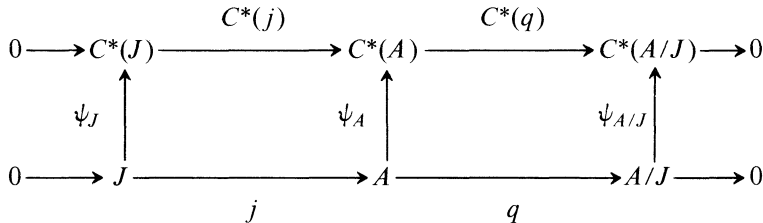
If  $A, B$  are  $JB$ -algebras and  $\varphi: A \rightarrow B$  is a homomorphism, we get a \*-homomorphism  $C^*(\varphi) = (\psi_B \circ \varphi)^\wedge: C^*(A) \rightarrow C^*(B)$ :



In this way,  $C^*$  becomes a functor from the category of  $JB$ -algebras to the category of  $C^*$ -algebras.

**THEOREM 4.1.** *The functor  $C^*$  is exact.*

*Proof.* This means that if  $J$  is a closed Jordan ideal in a  $JB$ -algebra  $A$ , then the top row in the commutative diagram below is exact:



Exactness at  $C^*(A/J)$  is trivial, since  $\psi_A(A)$  generates  $A$  and  $C^*(A/J)$  is generated by

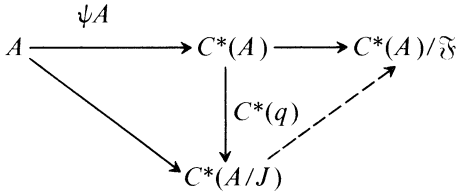
$$\psi_{A/J}(A/J) = \psi_{A/J}(q(A)) = C^*(q)\psi_A(A).$$

Next,  $C^*(q)C^*(j) = C^*(jq) = 0$ , so half the exactness at  $C^*(A)$  follows.

Let

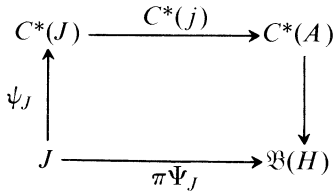
$$\mathfrak{I} = C^*(j)(C^*(J)).$$

Then  $\mathfrak{I}$  is generated by  $\psi_A(J)$ , which is an ideal in  $\psi_A(A)$ . By Proposition 3.3,  $\mathfrak{I}$  is an ideal in  $C^*(A)$ . But the composition  $A \rightarrow C^*(A) \rightarrow C^*(A)/\mathfrak{I}$  annihilates  $J$ , so it factors through  $C^*(A/J)$ .



Then, in this diagram, the circumference commutes, and so does the triangle on the left, by definition. Since  $\psi_A(A)$  generates  $C^*(A)$ , the triangle on the right commutes. Therefore, the kernel of  $C^*(q)$  is contained in the image  $\mathfrak{I}$  of  $C^*(j)$ , and exactness at  $C^*(A)$  is proved.

To prove exactness at  $C^*(J)$ , let  $\pi: C^*(J) \rightarrow \mathfrak{B}(H)$  be a faithful representation. Then  $\pi\psi_J$  is a homomorphism of  $J$  into  $\mathfrak{B}(H)_{sa}$ . By Lemma 2.4, this extends to a homomorphism of  $A$  into  $\mathfrak{B}(H)_{sa}$ . Factor the extension through  $C^*(A)$ :



But then the composition  $C^*(J) \rightarrow C^*(A) \rightarrow \mathfrak{B}(H)$  must be  $\pi$ , which is injective, so  $C^*(j)$  is injective.

By [2, Corollary 5.2]  $C^*(A)$  admits a unique antiautomorphism  $\Phi$  of order 2, which is the identity on  $\psi(A)$ . If  $A$  is a  $JC$ -algebra then  $\psi_A$  is injective; then we will identify  $A$  with  $\psi(A)$ .

LEMMA 4.2. *A  $JC$ -algebra  $A$  is universally reversible if and only if it is reversible in  $C^*(A)$  or, equivalently, if and only if  $A = C^*(A)_{sa}^\Phi$ .*

*Proof.* The first two conditions are trivially equivalent. So are the last two; see the proof of Proposition 2.3.

LEMMA 4.3. *Let  $A$  be a universally reversible JC-algebra. If  $\mathfrak{K}$  is a  $\Phi$ -invariant ideal of  $C^*(A)$ , then  $\mathfrak{K}$  is generated by  $\mathfrak{K} \cap A$ .*

*Proof.* Let  $(e_\mu)$  be a self-adjoint, bounded approximate unit for  $\mathfrak{K}$ . Then  $(\frac{1}{2}(e_\mu + \Phi(e_\mu)))$  is another such, so we may as well assume that  $e_\mu = \Phi(e_\mu)$ . Then  $e_\mu \in A$ , so  $\mathfrak{K} \cap A$  generates  $\mathfrak{K}$  (as an ideal, and hence as a  $C^*$ -algebra by Proposition 3.3).

We now characterize the universal  $C^*$ -algebra of a universally reversible JC-algebra.

THEOREM 4.4. *Assume  $A$  is a universally reversible JC-algebra, that  $\mathfrak{B}$  is a  $C^*$ -algebra, and that  $\theta:A \rightarrow \mathfrak{B}_{sa}$  is an injective homomorphism such that  $\theta(A)$  generates  $\mathfrak{B}$ . If  $\mathfrak{B}$  admits an antiautomorphism  $\varphi$  such that  $\varphi\theta = \theta$ , then  $\hat{\theta}$  is a  $*$ -isomorphism of  $C^*(A)$  onto  $\mathfrak{B}$ .*

*Proof.* Since  $\theta(A)$  generates  $\mathfrak{B}$ ,  $\hat{\theta}$  is onto. The composition  $\varphi\hat{\theta}\Phi$  is a  $*$ -homomorphism of  $C^*(A)$  to  $\mathfrak{B}$  extending  $\theta$ . Therefore, by the uniqueness of such extensions,  $\varphi\hat{\theta}\Phi = \hat{\theta}$ , or  $\varphi\hat{\theta} = \hat{\theta}\Phi$ . So the kernel of  $\hat{\theta}$  is a  $\Phi$ -invariant ideal, whose intersection with  $A$  is 0. By Lemma 4.3,  $\hat{\theta}$  is injective.

If  $\mathfrak{A}$  is a  $C^*$ -algebra, denote by  $[\mathfrak{A}, \mathfrak{A}]$  the commutator ideal of  $\mathfrak{A}$ , and by  $\mathfrak{A}^\circ$  the opposite  $C^*$ -algebra. There is a  $*$ -anti-isomorphism  $a \rightarrow a^\circ$  of  $\mathfrak{A}$  onto  $\mathfrak{A}^\circ$ .

COROLLARY 4.5. *If  $\mathfrak{A}$  is a  $C^*$ -algebra then  $C^*(\mathfrak{A}_{sa})$  can be identified with*

$$\mathfrak{B} = \{a \oplus b^\circ \in \mathfrak{A} \oplus \mathfrak{A}^\circ : a - b \in [\mathfrak{A}, \mathfrak{A}]\},$$

with  $\mathfrak{A}_{sa}$  identified with  $\{a \oplus a^\circ : a \in \mathfrak{A}_{sa}\}$ .

*Proof.* Clearly  $\mathfrak{B}$  is a norm closed  $*$ -subspace of  $\mathfrak{A} \oplus \mathfrak{A}^\circ$ . If  $a \oplus b^\circ$  and  $c \oplus d^\circ$  are elements of  $\mathfrak{B}$  then

$$(a \oplus b^\circ)(c \oplus d^\circ) = ac \oplus (db)^\circ, \quad \text{and}$$

$$ac - db = a(c - d) + (ad - da) + d(a - b) \in [\mathfrak{A}, \mathfrak{A}],$$

so  $\mathfrak{B}$  is a  $C^*$ -algebra.

Let  $\mathfrak{B}_\circ$  be the  $C^*$ -subalgebra of  $\mathfrak{B}$  generated by  $\{(a \oplus a^\circ) : a \in \mathfrak{A}_{sa}\}$ . If  $a, b \in \mathfrak{A}$  then

$$(a \oplus a^\circ)(b \circ b^\circ) - (ba) \oplus (ba)^\circ = (ab - ba) \oplus 0,$$

so  $\{x \in \mathfrak{A} : x \oplus 0 \in \mathfrak{B}_0\}$  contains all commutators. Since this set is an ideal of  $\mathfrak{A}$ , it contains  $[\mathfrak{A}, \mathfrak{A}]$ , and so  $[\mathfrak{A}, \mathfrak{A}] \oplus 0 \subseteq \mathfrak{B}_0$ . It follows that  $\mathfrak{B} \subseteq \mathfrak{B}_0$ , so  $\mathfrak{B}_0 = \mathfrak{B}$ .

In  $\mathfrak{B}$  we have the antiautomorphism  $a \oplus b^\circ \rightarrow b \oplus a^\circ$ , leaving  $\{a \oplus a^\circ : a \in A\}$  pointwise fixed. Theorem 4.4 completes the proof.

**COROLLARY 4.6.** ([11]) *If  $\mathfrak{A}$ ,  $\mathfrak{B}$  are  $C^*$ -algebras, and  $\mathfrak{A}$  has no one-dimensional representations, then any Jordan  $*$ -homomorphism of  $\mathfrak{A}$  into  $\mathfrak{B}$  is a sum of a  $*$ -homomorphism and a  $*$ -anti-homomorphism.*

*Proof.* In this case  $[\mathfrak{A}, \mathfrak{A}] = \mathfrak{A}$ , and so  $C^*(\mathfrak{A}_{sa}) = \mathfrak{A} \oplus \mathfrak{A}^\circ$ .

If  $\mathfrak{A}$  has one-dimensional representations, the conclusion of Corollary 4.6 may be false. Indeed, let

$$\mathfrak{A} = \{f \in C([-1, 1], M_2(\mathbf{C})) : f(0) \in \mathbf{C}1\},$$

and define  $\varphi : \mathfrak{A} \rightarrow \mathfrak{A}$  by  $\varphi f(x) = f(x)$  if  $x \geq 0$ ,  $\varphi f(x) = f(x)^t$  if  $x < 0$ . Then  $\varphi$  is not a sum of a  $*$ -homomorphism and a  $*$ -antihomomorphism.

We also remark that the conclusion of Corollary 4.6 holds if  $\mathfrak{A}$  is a von Neumann algebra, which may have one-dimensional representations. For then  $\mathfrak{A}$  is a direct sum of an abelian part, which offers no problems, and a non-abelian part, which has no one-dimensional representations. (Not even non-normal ones!).

We can use Theorem 4.4 to compute the universal  $C^*$ -algebras of some particular  $JC$ -algebras. For example,

$$\begin{aligned} C^*(M_n(\mathbf{R})_s) &= M_n(\mathbf{C}), \quad \text{and} \\ C^*(M_n(\mathbf{H})_{sa}) &= M_{2n}(\mathbf{C}) \quad \text{if } n \geq 3. \end{aligned}$$

The analogous results hold equally well in infinite dimension, whether we work with all bounded or only all compact operators.

It should also be noted here that  $C^*(V_n)$  may be computed explicitly for  $2 \leq n < \infty$ , and we get

$$C^*(V_{2n}) = M_{2^n}(\mathbf{C}), \quad C^*(V_{2n+1}) = M_{2^n}(\mathbf{C}) \oplus M_{2^n}(\mathbf{C}).$$

See [8, p. 276]. Note that  $C^*$  preserves direct limits. It then follows that the universal  $C^*$ -algebra of the infinite dimensional separable spin factor is the infinite tensor product  $\otimes M_2(\mathbf{C})$ , a well-known fact which motivates the name “ $CAR$ -algebra”.

**5. The universal tensor product of  $JC$ -algebras.** What should be expected of a “good” tensor product of two  $JC$ -algebras  $A$  and  $B$ ? Assuming that  $A$  and  $B$  have units (we shall keep this assumption

throughout), the tensor product  $C$  should contain copies of  $A$  and of  $B$  such that any element  $a$  of  $A$  and any element  $b$  of  $B$  operator commute.

We would like to identify  $C$  with the closure of  $A \otimes B$  in some norm. But we shall see below that this is rarely possible.

We define the *universal tensor product* of  $JC$ -algebras,  $A, B$  to be the  $JC$ -subalgebra of  $C^*(A) \otimes_{\max} C^*(B)$  generated by the subspace  $A \otimes B$ . We denote this  $JC$ -algebra  $A \tilde{\otimes} B$ . Before proceeding, we need a lemma on operator commutativity.

**LEMMA 5.1.** *Let  $A$  be a  $JC$ -subalgebra of a  $C^*$ -algebra  $\mathfrak{A}$ . Two elements in  $A$  operator commute in  $A$  if and only if they commute in  $\mathfrak{A}$ .*

*Proof.* The “if” part is evident. To prove the converse, assume  $a$  and  $b$  operator commute in  $A$ . We may assume  $A$  generates  $\mathfrak{A}$  in what follows. For any  $c \in A$  we have

$$4[T_a, T_b](c) = [[a, b], c].$$

Hence  $[a, b]$  commutes with every  $c \in A$ , and therefore  $[a, b]$  belongs to the center of  $\mathfrak{A}$ . For any irreducible representation  $\pi$  of  $\mathfrak{A}$ ,  $[\pi(a), \pi(b)]$  must be a scalar multiple of the identity. It is well known that then  $[\pi(a), \pi(b)] = 0$ , so  $[a, b] = 0$ .

$A \tilde{\otimes} B$  is characterized by a universal property, and its universal  $C^*$ -algebra is identified in the next result.

**PROPOSITION 5.2.** *If  $A, B$  and  $C$  are unital  $JC$ -algebras, and  $\varphi: A \rightarrow C$  and  $\psi: B \rightarrow C$  are unital homomorphisms with  $\varphi(a)$  operator commuting with  $\psi(b)$  for all  $a \in A, b \in B$ , there is a unique homomorphism  $\chi$  of  $A \tilde{\otimes} B$  into  $C$  such that  $\chi(a \otimes b) = \varphi(a) \circ \psi(b)$ . Moreover, the universal  $C^*$ -algebra of  $A \tilde{\otimes} B$  is  $C^*(A) \otimes_{\max} C^*(B)$ .*

*Proof.* We may assume  $C \subseteq \mathfrak{C}_{sa}$ , for a  $C^*$ -algebra  $\mathfrak{C}$ . Consider  $\hat{\varphi}: C^*(A) \rightarrow \mathfrak{C}$  and  $\hat{\psi}: C^*(B) \rightarrow \mathfrak{C}$ . By Lemma 5.1,  $\hat{\varphi}$  and  $\hat{\psi}$  have commuting ranges, so there exists a  $*$ -homomorphism  $\hat{\chi}$  of  $C^*(A) \otimes_{\max} C^*(B)$  into  $\mathfrak{C}$  such that

$$\hat{\chi}(a \otimes b) = \hat{\varphi}(a)\hat{\psi}(b).$$

Let  $\chi$  be the restriction of  $\hat{\chi}$  to  $A \tilde{\otimes} B$ . Clearly,  $\chi(A \tilde{\otimes} B) \subseteq C$ , and

$$\chi(a \otimes b) = \varphi(a) \circ \psi(b).$$

To prove the final statement, let  $\chi: A \tilde{\otimes} B \rightarrow \mathfrak{C}_a$  be a homomorphism, for any  $C^*$ -algebra  $\mathfrak{C}$ . Let  $\varphi(a) = \chi(a \otimes 1), \psi(b) = \chi(1 \otimes b)$ . Repeat the above

discussions to get an extension of  $\chi$  to a  $*$ -homomorphism  $\chi$  on  $C^*(A) \otimes_{\max} C^*(B)$ . Clearly,  $C^*(A) \otimes_{\max} C^*(B)$  is generated by  $A \otimes B$ , and therefore by  $A \tilde{\otimes} B$ . Then it must equal  $C^*(A \tilde{\otimes} B)$ .

The following result, when combined with Theorem 4.4, enables one to compute  $A \tilde{\otimes} B$  in many cases. Indeed, if  $\Phi_A$  is the antiautomorphism of  $C^*(A)$  leaving  $A$  pointwise fixed,  $\Phi_A \otimes \Phi_B$  is an antiautomorphism of  $C^*(A) \otimes_{\max} C^*(B)$  leaving  $A \tilde{\otimes} B$  pointwise fixed. Therefore, if  $A \tilde{\otimes} B$  is universally reversible, it is exactly the self-adjoint fixed points of  $\Phi_A \otimes \Phi_B$ , the tensor product of the canonical antiautomorphisms associated with  $A$  and  $B$ .

**PROPOSITION 5.3.**  *$A \tilde{\otimes} B$  is universally reversible unless one of  $A, B$  has a scalar representation and the other has a representation onto a spin factor  $V_n$ , where  $n \geq 4$ .*

*Proof.* Assume  $\chi: A \tilde{\otimes} B \rightarrow V_n$  is a representation onto  $V_n$ . Then  $\chi(A \otimes 1)$  and  $\chi(1 \otimes B)$  are operator commuting subalgebras of  $V_n$ , whose union generates  $V_n$ . Then one subalgebra must consist of the scalars only, and the other equals  $V_n$ . Indeed, if  $s \in V_n$  is a non-trivial symmetry (i.e., a unit vector of  $H_n$ ) then the only elements of  $V_n$  operator commuting with  $s$  are the linear combinations of  $s$  and 1. Theorem 2.3 completes the proof.

As an application of the above proposition and the remarks preceding it, we here show how to compute a few universal tensor products. For instance, if  $V_\infty$  is the infinite dimensional separable spin factor,  $V_\infty \tilde{\otimes} V_\infty$  is isomorphic to the simple  $JC$ -algebra considered in Section 3. For  $C^*(V_\infty)$  is just the  $CAR$  algebra, and  $\Phi_{V_\infty}$  is the transposition on the  $CAR$  algebra.

An antiautomorphism  $\alpha$  of  $\mathfrak{B}(H)$  of order 2 is induced by an antiunitary  $j$  with  $j^2 = \pm 1$ . If  $j^2 = +1$ ,  $\alpha$  is called real, while if  $j^2 = -1$ ,  $\alpha$  is called quaternionic. Now the behaviour of tensor product of antiautomorphisms on  $\mathfrak{B}(H)$  can be summarized in: real  $\otimes$  real = real, real  $\otimes$  quaternionic = quaternionic, quaternionic  $\otimes$  quaternionic = real. As a result, we get the following table for  $\tilde{\otimes}$ :

$\tilde{\otimes}$	$M_n(\mathbf{R})_s$	$M_n(\mathbf{H})_{sa}, n \geq 3$
$M_m(\mathbf{R})_s$	$M_{mn}(\mathbf{R})_s$	$M_{mn}(\mathbf{H})_{sa}$
$M_m(\mathbf{H})_{sa}, m \geq 3$		$M_{4mn}(\mathbf{R})_{sa}$

The computation of universal tensor products involving nonreversible spin factors requires more care: The canonical antiautomorphisms of their universal  $C^*$ -algebras must be analyzed. Their behaviour turns out to be

cyclic, depending on  $n \pmod 8$ . We just state here, without proof, that the universal tensor product of  $V_5 = M_2(\mathbf{H})_{sa}$  with itself is a direct sum of four copies of  $M_{16}(\mathbf{R})_s$ .

LEMMA 5.4. *If  $\mathfrak{A}$  is a  $C^*$ -algebra with no scalar representations and  $B$  is a  $JC$ -algebra then*

$$\mathfrak{A}_{sa} \otimes B \cong (\mathfrak{A} \otimes_{\max} C^*(B))_{sa}.$$

*Proof.* We have  $C^*(\mathfrak{A}_{sa}) = \mathfrak{A} \oplus \mathfrak{A}^\circ$ , with the antiautomorphism

$$\Phi_{\mathfrak{A}_{sa}}(a \oplus b^\circ) = (b \oplus a^\circ).$$

Hence

$$C^*(\mathfrak{A}_{sa} \tilde{\otimes} B) = \mathfrak{A} \otimes_{\max} C^*(B) \oplus \mathfrak{A}^\circ \otimes_{\max} C^*(B),$$

where the canonical antiautomorphism interchanges the two summands. Hence the self-adjoint fixed point algebra is isomorphic to  $(\mathfrak{A} \otimes_{\max} C^*(B))_{sa}$ .

We conclude with a result showing that a more naive approach to defining tensor products of  $JC$ -algebras is bound to fail.

THEOREM 5.5. *Assume  $A$  is a unital  $JC$ -algebra and that  $A \otimes M_2(\mathbf{C})_{sa}$  is a  $JC$ -algebra with some product satisfying*

- (i)  $(a \otimes 1)^2 = a^2 \otimes 1$
- (ii)  $(1 \otimes \alpha)^2 = 1 \otimes \alpha^2$
- (iii)  $(a \otimes 1) \circ (1 \otimes \alpha) = a \otimes \alpha$
- (iv)  $[T_{1 \otimes \alpha}, T_{a \otimes 1}] = 0$ .

*Then  $A$  is isomorphic to the self-adjoint part of a  $C^*$ -algebra.*

*Proof.* By Lemma 5.4,  $A \tilde{\otimes} M_2(\mathbf{C})_{sa}$  is the self-adjoint part of a  $C^*$ -algebra. By Proposition 5.2,  $A \otimes M_2(\mathbf{C})_{sa}$  is a quotient of  $A \tilde{\otimes} M_2(\mathbf{C})_{sa}$  and therefore also the self-adjoint part of a  $C^*$ -algebra. (Note that by Propositions 3.2 and 3.3, a Jordan ideal in the self-adjoint part of a  $C^*$ -algebra is the self-adjoint part of a two-sided ideal.)

By (ii), (iii) and (iv) we get

$$\begin{aligned} U_{1 \otimes \beta}(a \otimes \alpha) &= U_{1 \otimes \beta}T_{a \otimes 1}(1 \otimes \alpha) = T_{a \otimes 1}U_{1 \otimes \beta}(1 \otimes \alpha) \\ &= T_{a \otimes 1}(1 \otimes \{\beta\alpha\beta\}) = a \otimes \{\beta\alpha\beta\}. \end{aligned}$$

If  $\beta$  is a one-dimensional projection in  $M_2(\mathbf{C})$ , then  $\beta\alpha\beta$  is a scalar, so

$$U_{1\otimes\beta}(A \otimes M_2(\mathbf{C})_{sa}) = A \otimes \beta.$$

By (i) and (ii)  $a \rightarrow a \otimes \beta$  is a Jordan homomorphism, so

$$A \cong U_{1\otimes\beta}(A \otimes M_2(\mathbf{C})_{sa}).$$

But cutting down the self-adjoint part of a  $C^*$ -algebra with a projection  $(1 \otimes \beta)$  we get the self-adjoint part of a  $C^*$ -algebra, so the proof is complete.

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