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On the Stability of Homographic N-body Configurations

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Abstract. Stability of expanding homographic configurations formed by an even number of bodies of equal mass and placed at the vertices of regular polygons and polyhedrons is studied. It is shown that for every configuration considered, there is always an unstable perturbation that divides the system into pairs of bodies.

1. Introduction

An N-body configuration is called a central configuration if the gravitational force acting on the bodies is proportional to their distances from the center of mass. A. Wintner (1941) has shown that the configuration of gravitating bodies after simultaneous explosion (or during simultaneous collision) in the absence of rotation is the central configuration. The N-body system is called homographic if the configuration formed by the bodies in the inertial barycentric coordinate system changes in time staying similar to itself. Homographic configurations are examples of central configurations.

Nezhinskij (1972) has shown that central configurations are unstable. It is interesting to study the details of this instability for specific cases. Expanding homographic configurations formed by bodies of equal mass placed in the vertices of regular polygons and polyhedrons were chosen as convenient examples. The system expands as $t^{2/3}$. Perturbations growing faster than the general expansion of the system are considered to be unstable.

We have carried out an investigation of these perturbations in the linear approximation. For every configuration considered, there is always an unstable perturbation that divides the system into pairs of bodies. Computer simulations are used to study the subsequent non-linear stage of the process. Expanding and collapsing configurations formed by bodies of equal mass placed in the vertices of regular polygons are numerically studied. The expansion has the same law. Collapsing configurations with zero and non-zero initial velocities are considered. The numerical investigation was carried out using the modified code "Chain" developed by S. Mikkola and S. J. Aarseth (1993) and the author's code.

The perturbations violating the homographic structure are considered to be unstable (in the case of expanding configurations these perturbations grow faster than the general expansion of the system, i.e. faster than $t^{2/3}$). The computer simulations confirm the results of the linear stability analysis. In particular, an unstable perturbation that divides the system into pairs exists for each configuration under consideration.

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2. Basic equations

Let us consider an expanding homographic configuration in barycentric coordinates. All the radius vectors of the bodies have the same length: $|\mathbf{r_i}| = r$ ($\mathbf{r_i} = (x_i, y_i, z_i)$). Let the gravitational constant $G = 1$.

The equations of motion are:

$$(1) \qquad \begin{cases} \ddot{x}_i = \sum_{j\neq i} \frac{x_j - x_i}{r_{ij}^3} \\ \ddot{y}_i = \sum_{j\neq i} \frac{y_j - y_i}{r_{ij}^3} \\ \ddot{z}_i = \sum_{j\neq i} \frac{z_j - z_i}{r_{ij}^3} \end{cases}$$

where $r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2$. It is easy to obtain the expansion law:

$$(2) \qquad \mathbf{r_i} = k \cdot \mathbf{r_\circ} \cdot t^{2/3}.$$

Let us disturb the initial configuration: $x_i \to x_i + \delta x_i$, $\dot{x}_i \to \dot{x}_i + \delta \dot{x}_i$ and so on, where δx_i, $\delta \dot{x}_i$, ... are supposed to be small.

We obtain the following equations for the perturbations in linear approximation:

$$(3) \begin{cases} \delta\ddot{x}_i = \sum \left(\frac{\delta x_{ji}}{r_{ij}^3} - 3\frac{(x_j-x_i)^2 \delta x_{ji}}{r_{ij}^5} - 3\frac{(x_j-x_i)(y_j-y_i)\delta y_{ji}}{r_{ij}^5} - 3\frac{(x_j-x_i)(z_j-z_i)\delta z_{ji}}{r_{ij}^5} \right) \\ \delta\ddot{y}_i = \sum \left(\frac{\delta y_{ji}}{r_{ij}^3} - 3\frac{(x_j-x_i)(y_j-y_i)\delta x_{ji}}{r_{ij}^5} - 3\frac{(y_j-y_i)^2 \delta y_{ji}}{r_{ij}^5} - 3\frac{(z_j-z_i)(y_j-y_i)\delta z_{ji}}{r_{ij}^5} \right) \\ \delta\ddot{z}_i = \sum \left(\frac{\delta z_{ji}}{r_{ij}^3} - 3\frac{(x_j-x_i)(z_j-z_i)\delta x_{ji}}{r_{ij}^5} - 3\frac{(y_j-y_i)(z_j-z_i)\delta y_{ji}}{r_{ij}^5} - 3\frac{(z_j-z_i)^2 \delta z_{ji}}{r_{ij}^5} \right) \end{cases}$$

where $\delta x_{ji} = \delta x_j - \delta x_i$ and δy_{ji}, δz_{ji} have similar definitions.
The elementary solutions of system (3) are proportional to t^p:

$$(4) \qquad \begin{cases} \delta x_i = a_i t^p \\ \delta y_i = b_i t^p \\ \delta z_i = c_i t^p \end{cases}$$

After substituting Eqns (4) into (3) we obtain the homogeneous system of linear algebraic equations, which (after some bulky calculations) can be solved.

Let us call the perturbation relatively stable (or simply stable) if it grows slower than $t^{2/3}$. All other perturbations we shall call unstable. For any homographic configuration we'll have solutions corresponding to the displacement ($p = 0$), movement ($p = 1$), rotation ($p = 2/3$) of the system as a whole and changes in the speed of expansion, while the form of the configuration conserves ($p = -1/3$, $p = 4/3$). Some of these solutions are formally unstable.

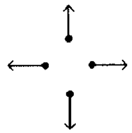

Figure 1.

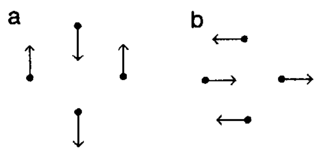

Figure 2. $p = 1.143$, $p = -0.143$

3. Results

Let us consider an expanding configuration formed by 4 bodies placed at the vertices of a square: (1,0), (0,1), (−1,0), (0,−1) and take into account only flat ($c_i = 0$ in Eqn (4)) perturbations (Figure 1). There are 8 non-trivial solutions in this case. Four of these are relatively stable. Another 4 perturbations are unstable (the maximal power of t is $p = 1.22...$). The results are shown in Table 1.

If we consider the 3-D perturbations, we also have (relatively) stable oscillations about X-Y plane ($p = 1/2 \pm 0.279\, i$).

The results are similar in the case of an expanding regular polygon (for simplicity we studied polygons with an even number of vertices). Let the initial coordinates of points be $x_i = \cos\frac{2\pi i}{n}$, $y_i = \sin\frac{2\pi i}{n}$, $i = 1, ..., n-1$.

In this case we also have unstable perturbations that divide the system into pairs (see Eqn 5).

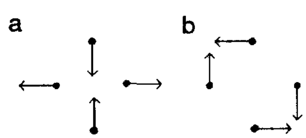

Figure 3. a) $p = 0.949$, $p = 0.050$, b) $p = 1.221$, $p = -0.221$

Table 1.

p	a_1	b_1	a_2	b_2	a_3	b_3	a_4	b_4	
1.143, −0.143	0	1	0	−1	0	1	0	−1	fig.2a
1.143, −0.143	1	0	−1	0	1	0	−1	0	fig.2b
0.949, 0.050	1	0	0	1	−1	0	0	−1	fig.3a
1.221, −0.221	0	−1	1	0	0	1	−1	0	fig.3b

$$(5)\quad \begin{cases} \delta x_{2s-1} = \left(\dfrac{\cos\frac{\pi(4s-1)}{n}}{\cos\frac{\pi}{n}} - \cos\dfrac{2\pi(2s-1)}{n}\right)\cdot t^p \\ \delta y_{2s-1} = \left(\dfrac{\sin\frac{\pi(4s-1)}{n}}{\cos\frac{\pi}{n}} - \sin\dfrac{2\pi(2s-1)}{n}\right)\cdot t^p \\ \delta x_{2s} = \left(\dfrac{\cos\frac{\pi(4s-1)}{n}}{\cos\frac{\pi}{n}} - \cos\frac{4\pi s}{n}\right)\cdot t^p \\ \delta y_{2s} = \left(\dfrac{\sin\frac{\pi(4s-1)}{n}}{\cos\frac{\pi}{n}} - \sin\frac{4\pi s}{n}\right)\cdot t^p \\ \text{where} \quad s = 1,2,3,...n/2. \end{cases}$$

The corresponding power of t is $p > 1$. For example, for $n = 8$, $p = 1.943...$ Similar results were obtained for regular polyhedrons (tetrahedron, cube, and octahedron): there are also unstable perturbations dividing the system into the pairs in these cases.

The above results correspond to the linear stage of the process. Numerical simulations are required to study the developed instability. Some simulations were carried out for some regular polygons ($n = 4, 8, 16, 20$) and polyhedrons (cube $n = 8$ and icosahedron $n = 20$). Computer simulations proved that the instability of the kind mentioned above (dividing the system into pairs, triples, quadruples and so on) really exist, though it grows a little slower than the linear theory predicts. These qualitative results may be useful in studying the clustering of galaxies and in explanation of the strong two-point correlation of clusters of galaxies (see Peebles 1980, Sutherland and Efstathiou 1991 and references therein).

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