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TILING BRANCHING MULTIPLICITY SPACES WITH GL_2 PATTERN BLOCKS

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Abstract

We study branching multiplicity spaces of complex classical groups in terms of GL_2 representations. In particular, we show how combinatorics of GL_2 representations are intertwined to make branching rules under the restriction of GL_n to GL_{n-2}. We also discuss analogous results for the symplectic and orthogonal groups.

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1. Introduction

1.1. Branching rules describe a way of decomposing an irreducible representation of a whole group into irreducible representations of a subgroup. With applications in physics, branching rules for classical groups have been extensively studied. See, for example, [6, 7, 9, 11].

In this paper, we study combinatorial aspects of branching rules for complex classical groups, under the restriction of GL_n to GL_{n-2}, Sp_{2n} to Sp_{2n-2}, and SO_m to SO_{m-2}, by investigating the GL_2 module structure of branching multiplicity spaces. Recently, Wallach, Yacobi and the present author studied Sp_{2n} to Sp_{2n-2} branching rules in terms of SL_2 representations [5, 10, 12]. Our results for the symplectic group are compatible with those in the above papers once we restrict GL_2 to SL_2.

1.2. A group homomorphism ϕ_α from the complex torus $(\mathbb{C}^*)^k$ to $\mathbb{C}^*$ defined by

$$\phi_\alpha(t_1, t_2, \ldots, t_k) = t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_k^{\alpha_k}$$

is called a *polynomial dominant weight* of the complex general linear group $GL_k = GL_k(\mathbb{C})$, if it satisfies

$$\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}^k \quad \text{and} \quad \alpha_1 \geq \cdots \geq \alpha_k \geq 0.$$

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We shall identify the polynomial dominant weight ϕ_α with the exponent α. We can also identify ϕ_α with Young diagram having α_i boxes in the ith row for all i. The sum $\alpha_1 + \cdots + \alpha_k$ will be denoted by $|\alpha|$.

Then, by theory of highest weight, polynomial dominant weights uniquely label complex irreducible polynomial representations of the general linear group, and we will let V_k^α denote the irreducible representation of GL_k labeled by Young diagram α, or equivalently, highest weight α. See, for example, [3, Section 9].

1.3. The irreducible representation V_n^λ of GL_n labeled by Young diagram λ is completely reducible as a GL_{n-2} representation. By Schur's lemma (for example, [1, Section 1.2]), for a pair of polynomial dominant weights λ and μ of GL_n and GL_{n-2} respectively, the branching multiplicity of V_{n-2}^μ in V_n^λ is equal to the dimension of the space

$$V^\lambda|_\mu = \mathrm{Hom}_{\mathrm{GL}_{n-2}}(V_{n-2}^\mu, V_n^\lambda) \tag{1.1}$$

of GL_{n-2} homomorphisms, and then, as a GL_{n-2} representation, V_n^λ decomposes into isotypic components as

$$V_n^\lambda = \bigoplus_\mu V_{n-2}^\mu \otimes \mathrm{Hom}_{\mathrm{GL}_{n-2}}(V_{n-2}^\mu, V_n^\lambda) \tag{1.2}$$

where the summation runs over the highest weights μ of V_{n-2}^μ appearing in V_n^λ. In this sense, we call the space (1.1) a GL_n *to* GL_{n-2} *branching multiplicity space*.

1.4. After a brief review on the representations of GL_2 in Section 2, we describe the GL_2 module structure of GL_n to GL_{n-2} branching multiplicity spaces in Section 3. We develop a combinatorial procedure of *tiling* branching multiplicity spaces with GL_2 *pattern blocks* in Section 4. This procedure will show, in particular, how combinatorics of GL_2 representations can be intertwined to make branching rules under the restriction of GL_n to GL_{n-2}. We will discuss analogous results for the branching of Sp_{2n} to Sp_{2n-2} and SO_m to SO_{m-2} in Section 5.

2. Irreducible representations of GL_2

In this section, we review algebraic and combinatorial models for GL_2 representations.

2.1. For a polynomial dominant weight $(x, z) \in \mathbb{Z}^2$ of GL_2, the irreducible representation with highest weight (x, z) can be realized as

$$V_2^{(x,z)} = \mathbb{C} \otimes \mathrm{Sym}^{x-z}(\mathbb{C}^2)$$

where $g \in \mathrm{GL}_2$ acts on the spaces $\mathbb{C}$ and $\mathbb{C}^2$ via scaling by the factor of $\det(g)^z$ and matrix multiplication, respectively. Here, $\mathrm{Sym}^d(\mathbb{C}^2)$ denotes the dth symmetric power of the space $\mathbb{C}^2$, and $\det(g)$ denotes the determinant of the matrix $g \in \mathrm{GL}_2$. See, for example, [1, Section 15.5].

2.2. The irreducible representations of GL_k can be described in terms of *Gelfand–Tsetlin patterns* [2]. For GL_2, Gelfand–Tsetlin patterns for $V_2^{(x,z)}$ are triangular arrays of the form

$$\begin{bmatrix} x & & z \\ & y & \end{bmatrix}$$

with $y \in \mathbb{Z}$ and $x \geq y \geq z$, which can label weight basis vectors $v \in V_2^{(x,z)}$,

$$\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \cdot v = ({t_1}^{y} {t_2}^{x+z-y}) v,$$

for all diagonal matrices $\mathrm{diag}(t_1, t_2)$ of GL_2. See, for example, [3, Section 8.1] or [8]. Then the character of the GL_2 representation $V_2^{(x,z)}$ is

$$\mathrm{ch}_{(x,z)}(t_1, t_2) = \sum_{y} {t_1}^{y} {t_2}^{x+z-y} \tag{2.1}$$

where the summation runs over all integers y such that $x \geq y \geq z$, or equivalently, over all Gelfand–Tsetlin patterns with top row (x, z).

2.3. We remark that if we restrict GL_2 to its subgroup SL_2, then $V_2^{(x,z)}$ is isomorphic to $\mathrm{Sym}^{x-z}(\mathbb{C}^2)$. By taking $t_1 = t$ and $t_2 = t^{-1}$ in (2.1), its character can be given as

$$\mathrm{ch}_{(d)}(t) = t^{-d} + t^{-d+2} + \cdots + t^{d-2} + t^{d}$$

where $d = x - z$. See, for example, [1, Section 11.1] or [3, Section 2.3].

3. Branching multiplicity spaces

In this section, we study the GL_2 module structure of GL_n to GL_{n-2} branching multiplicity spaces.

3.1. Let us recall branching rules for GL_k down to GL_{k-1}, under the embedding of GL_{k-1} in the upper left corner of GL_k. For polynomial dominant weights α and β of GL_k and GL_{k-1}, respectively, we write $\beta \sqsubseteq \alpha$ and say that β *interlaces* α, if

$$\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \cdots \geq \alpha_{k-1} \geq \beta_{k-1} \geq \alpha_k.$$

Lemma 3.1 ([3, Section 8.1], [8]). *Let α and β be polynomial dominant weights of GL_k and GL_{k-1}, respectively.*

(1) *The multiplicity of a GL_{k-1} irreducible representation V_{k-1}^{β} in V_k^{α}, as a GL_{k-1} representation, is at most one. It is precisely one when β interlaces α.*

(2) *As a $\mathrm{GL}_{k-1} \times \mathrm{GL}_1$ representation, V_k^{α} decomposes as*

$$V_k^{\alpha} = \bigoplus_{\beta \sqsubseteq \alpha} V_{k-1}^{\beta} \hat{\otimes} V_1^{(|\alpha|-|\beta|)}$$

where the summation runs over all β interlacing α.

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Next, let us consider polynomial dominant weights λ and μ of GL_n and GL_{n-2}, respectively. We say that μ *doubly interlaces* λ, if there exists a polynomial dominant weight κ of GL_{n-1} such that μ interlaces κ and κ interlaces λ, that is, $\mu \sqsubseteq \kappa \sqsubseteq \lambda$. By applying the above lemma twice, it is straightforward to see the following proposition.

Proposition 3.2.

(1) *The irreducible representation V^{μ}_{n-2} appears in V^{λ}_n as a GL_{n-2} representation if and only if μ doubly interlaces λ.*

(2) *The multiplicity of V^{μ}_{n-2} in V^{λ}_n is equal to the number of all possible κ satisfying $\mu \sqsubseteq \kappa \sqsubseteq \lambda$.*

(3) *As a $\mathrm{GL}_{n-2} \times \mathrm{GL}_1 \times \mathrm{GL}_1$ representation, V^{λ}_n decomposes as*

$$V^{\lambda}_n = \bigoplus_{\mu \sqsubseteq \kappa} \bigoplus_{\kappa \sqsubseteq \lambda} V^{\mu}_{n-2} \hat{\otimes} (V_1^{(|\kappa|-|\mu|)} \hat{\otimes} V_1^{(|\lambda|-|\kappa|)})$$

where the summation runs over all μ doubly interlacing λ and κ satisfying $\mu \sqsubseteq \kappa \sqsubseteq \lambda$.

By comparing (1.2) and Proposition 3.2, we can describe the branching multiplicity space

$$V^{\lambda}|_{\mu} = \mathrm{Hom}_{\mathrm{GL}_{n-2}}(V^{\mu}_{n-2}, V^{\lambda}_n)$$

in terms of integral sequences κ such that $\mu \sqsubseteq \kappa \sqsubseteq \lambda$, or arrays of the form

$$\begin{bmatrix} \lambda_1 & & \lambda_2 & & \lambda_3 & & \cdots & & \lambda_{n-1} & & \lambda_n \\ & \kappa_1 & & \kappa_2 & & \kappa_3 & & \cdots & & \kappa_{n-1} & \\ & & \mu_1 & & \mu_2 & & \cdots & & \mu_{n-2} & & \end{bmatrix}$$

where the entries are weakly decreasing along the diagonals from left to right, which we will call *interlacing patterns*.

3.2. Our next task is to show that every GL_n to GL_{n-2} branching multiplicity space can be factored into GL_2 representations. For polynomial dominant weights λ and μ of GL_n and GL_{n-2} respectively, let $\mathcal{IP}(\lambda, \mu)$ be the set of interlacing patterns whose top and bottom rows are λ and μ respectively. Also, for a sequence σ of weakly decreasing nonnegative integers

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{2n-3} \geq \sigma_{2n-2},$$

let $\mathcal{GT}(\sigma)$ be the set of all $(n-1)$-tuples of Gelfand–Tsetlin patterns for GL_2 whose top rows are $(\sigma_{2i-1}, \sigma_{2i})$ for $1 \leq i \leq n-1$.

Theorem 3.3. *Let λ and μ be polynomial dominant weights of GL_n and GL_{n-2}, and $\sigma = \sigma(\lambda, \mu)$ be the sequence $(x_1, z_1, \ldots, x_{n-1}, z_{n-1})$ obtained by rearranging the sequence*

$$(\lambda_1, \lambda_2, \ldots, \lambda_n, \mu_1, \mu_2, \ldots, \mu_{n-2})$$

in weakly decreasing order, that is, $x_1 \geq z_1 \geq \cdots \geq x_{n-1} \geq z_{n-1}$. *Then, the map from* $\mathcal{IP}(\lambda, \mu)$ *to* $\mathcal{GT}(\sigma)$ *sending*

$$\begin{bmatrix} \lambda_1 & & \lambda_2 & & \lambda_3 & & \cdots & & \lambda_{n-1} & & \lambda_n \\ & \kappa_1 & & \kappa_2 & & \kappa_3 & & \cdots & & \kappa_{n-1} & \\ & & \mu_1 & & \mu_2 & & \cdots & & \mu_{n-2} & & \end{bmatrix}$$

to

$$\left(\begin{bmatrix} x_1 & & z_1 \\ & \kappa_1 & \end{bmatrix}, \begin{bmatrix} x_2 & & z_2 \\ & \kappa_2 & \end{bmatrix}, \ldots, \begin{bmatrix} x_{n-1} & & z_{n-1} \\ & \kappa_{n-1} & \end{bmatrix} \right)$$

is a bijection.

We will prove the theorem in the context of pattern-tiling in Proposition 4.3. Our proof will show in particular how combinatorics of GL_2 representations are intertwined to make branching rules under the restriction of GL_n to GL_{n-2}. We also note that a direct proof can be given by using the observation that if μ doubly interlaces λ, then $x_1 = \lambda_1$, $z_{n-1} = \lambda_n$, and

$$z_j = \max(\lambda_{j+1}, \mu_j) \quad \text{and} \quad x_{j+1} = \min(\lambda_{j+1}, \mu_j) \tag{3.1}$$

for $1 \leq j \leq n-2$.

As an immediate consequence of Theorem 3.3, since there are exactly $x - z + 1$ possible Gelfand–Tsetlin patterns with top row (x, z), we have the following corollary.

COROLLARY 3.4. *For* μ *doubly interlacing* λ, *the multiplicity of* V_{n-2}^{μ} *in* V_n^{λ}, *or equivalently the dimension of the branching multiplicity space* $V^{\lambda}|_{\mu}$, *is*

$$\prod_{j=1}^{n-1} (x_j - z_j + 1)$$

where the x_j *and* z_j *are defined from the rearrangement* $(x_1, z_1, \ldots, x_{n-1}, z_{n-1})$ *of the sequence* $(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_{n-2})$ *in weakly decreasing order.*

We note that this formula can be derived from [12, Proposition 3.2]. See the remark after Theorem 3.5.

3.3. In the setting of Proposition 3.2, consider the diagonal block GL_2 complement to GL_{n-2} in GL_n:

$$\begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \in GL_n$$

where $g_1 \in GL_{n-2}$ and $g_2 \in GL_2$. This GL_2 commutes with GL_{n-2} acting on V_{n-2}^{μ} in (1.2), and therefore, the GL_n to GL_{n-2} branching multiplicity space carries the structure of a GL_2 module.

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THEOREM 3.5. *For μ doubly interlacing λ, the GL_n to GL_{n-2} branching multiplicity space $V^\lambda|_\mu$ is, as a GL_2 representation, isomorphic to the tensor product of GL_2 irreducible representations*

$$\mathrm{Hom}_{\mathrm{GL}_{n-2}}(V^\mu_{n-2}, V^\lambda_n) \cong \mathbb{C} \otimes V_2^{(x_1,z_1)} \otimes V_2^{(x_2,z_2)} \otimes \cdots \otimes V_2^{(x_{n-1},z_{n-1})}$$

where $\mathbb{C}$ is the one-dimensional representation given by $\det(g)^{-|\mu|}$ for $g \in \mathrm{GL}_2$; and x_j and z_j are defined from the rearrangement $(x_1, z_1, \ldots, x_{n-1}, z_{n-1})$ of the sequence $(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_{n-2})$ in weakly decreasing order.

PROOF. By taking $\mathrm{GL}_1 \times \mathrm{GL}_1$ in Proposition 3.2 as a maximal torus of GL_2, we can consider the following formula as the GL_2 character of the branching multiplicity space

$$ch(V^\lambda|_\mu) = \sum_\kappa t_1^{|\kappa|-|\mu|} t_2^{|\lambda|-|\kappa|}$$

where $(t_1, t_2) \in \mathrm{GL}_1 \times \mathrm{GL}_1$ and the summation runs over all κ such that $\mu \sqsubseteq \kappa \sqsubseteq \lambda$. Then

$$\begin{aligned}(t_1t_2)^{|\mu|} \cdot ch(V^\lambda|_\mu) &= \sum_\kappa {t_1}^{|\kappa|} {t_2}^{|\lambda|+|\mu|-|\kappa|} \\ &= \sum_\kappa {t_1}^{(\kappa_1+\cdots+\kappa_{n-1})} {t_2}^{(x_1+z_1+\cdots+x_{n-1}+z_{n-1})-(\kappa_1+\cdots+\kappa_{n-1})} \\ &= \prod_{j=1}^{n-1} \sum_{\kappa_j} {t_1}^{\kappa_j} {t_2}^{x_j+z_j-\kappa_j}\end{aligned}$$

and, by Theorem 3.3, $x_j \geq \kappa_j \geq z_j$ for each j. This shows that $ch(V^\lambda|_\mu)$ is the product of $(t_1t_2)^{-|\mu|}$, the character of the one-dimensional representation twisted by $\det(g)^{-|\mu|}$, and the characters of the $V_2^{(x_j,z_j)}$. This finishes our proof. □

The following SL_2 module structure of the branching multiplicity space was studied by Yacobi in his thesis (see [12, Proposition 3.2]):

$$\mathrm{Hom}_{\mathrm{GL}_{n-2}}(V^\mu_{n-2}, V^\lambda_n) \cong \mathrm{Sym}^{x_1-z_1}(\mathbb{C}^2) \otimes \cdots \otimes \mathrm{Sym}^{x_{n-1}-z_{n-1}}(\mathbb{C}^2).$$

Our theorem can be understood as a result obtained by lifting SL_2 to GL_2.

4. Tiling branching multiplicity spaces

In this section we develop a combinatorial procedure of tiling branching multiplicity spaces with Gelfand–Tsetlin patterns for GL_2, thereby proving Theorem 3.3.

4.1. First, in order to consider some directed paths in a graph, we place vertices on the coordinate plane as

$$P_n = \{(a, b) : b = 0, 1 \leq a \leq n\} \cup \{(a, b) : b = -1, 2 \leq a \leq n-1\}.$$

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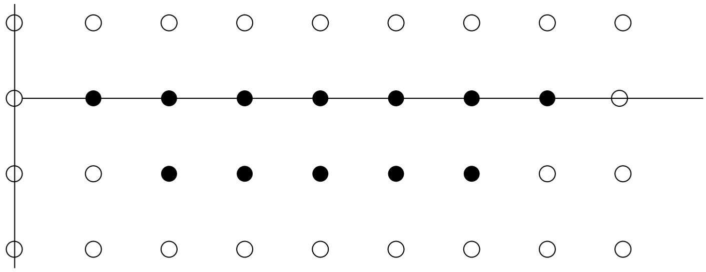
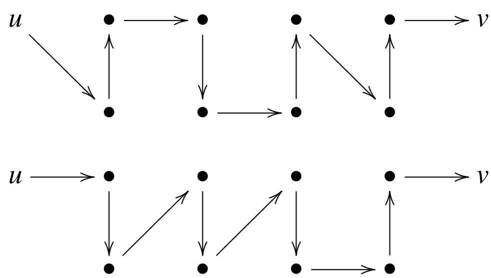
For example, P_7 is



Then we consider directed paths from $u = (1, 0)$ to $v = (n, 0)$ in $2n - 3$ steps visiting each point in P_n exactly once, when we are only allowed to move right($\rightarrow$) or up($\uparrow$) or down($\downarrow$) or up-right($\nearrow$) or down-right($\searrow$) at each step.

Example 4.1. These are two paths for P_6 out of 16 possible ones.



Each directed path can be presented by a sequence of allowed steps. For example, the two paths for P_6 in Example 4.1 can be presented as, respectively,

$$[\ \searrow\ \uparrow\ \rightarrow\ \downarrow\ \rightarrow\ \uparrow\ \searrow\ \uparrow\ \rightarrow\ ],$$
$$[\ \rightarrow\ \downarrow\ \nearrow\ \downarrow\ \nearrow\ \downarrow\ \rightarrow\ \uparrow\ \rightarrow\ ].$$

At each step of a path, it is clear whether we are on the line $y = 0$ or the line $y = -1$; and if we are on $y = 0$ then the next step should be down($\downarrow$), and if we are on $y = -1$ then the next step should be up($\uparrow$). Therefore, in presenting directed paths for P_n from $(1, 0)$ to $(n, 0)$, we may omit up($\uparrow$) and down($\downarrow$) arrows. Then, by denoting moving right($\rightarrow$) on the line $y = 0$ and on the line $y = -1$ by harpoon-up($\rightharpoonup$) and harpoon-down($\rightharpoondown$), respectively, we can present every path uniquely with the following four arrows:

$$\searrow,\quad \rightharpoonup,\quad \rightharpoondown,\quad \nearrow.$$

4.2. From this observation, we define *pattern blocks* attached to arrows and a *tiling* given by a directed path.

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Definition 4.2.

(1) For each i with $1 \le i \le n-1$, the ith pattern block corresponding to the down-right, harpoon-up, harpoon-down and up-right arrows is

$\searrow$	$\rightharpoonup$	$\rightharpoondown$	$\nearrow$
$\begin{matrix} x_i & & \\ & y_i & \\ & & z_i \end{matrix}$	$\begin{matrix} x_i & & z_i \\ & y_i & \\ & & \end{matrix}$	$\begin{matrix} & & \\ & y_i & \\ x_i & & z_i \end{matrix}$	$\begin{matrix} & & z_i \\ & y_i & \\ x_i & & \end{matrix}$

(2) For each directed path from $(1, 0)$ to $(n, 0)$ of P_n, its tiling is the concatenation of pattern blocks defined by the sequence of arrows presenting the path such that:

(a) y_i is at coordinate $(i + 0.5, -0.5)$;
(b) x_i and z_i above y_i are at coordinates $(i, 0)$ and $(i + 1, 0)$, respectively;
(c) x_i and z_i below y_i are at coordinates $(i, -1)$ and $(i + 1, -1)$, respectively

for $1 \le i \le n-1$.

With this definition, the two paths given in Example 4.1 can be given as

$$[\searrow \quad \rightharpoonup \quad \rightharpoondown \quad \searrow \quad \rightharpoonup] \quad \text{and} \quad [\rightharpoonup \quad \nearrow \quad \nearrow \quad \rightharpoondown \quad \rightharpoonup],$$

and the corresponding tilings are

$$\begin{bmatrix} x_1 & & x_2 & & z_2 & & x_4 & & x_5 & & z_5 \\ & y_1 & & y_2 & & y_3 & & y_4 & & y_5 & \\ & & z_1 & & x_3 & & z_3 & & z_4 & & \end{bmatrix}$$

and

$$\begin{bmatrix} x_1 & & z_1 & & z_2 & & z_3 & & x_5 & & z_5 \\ & y_1 & & y_2 & & y_3 & & y_4 & & y_5 & \\ & & x_2 & & x_3 & & x_4 & & z_4 & & \end{bmatrix}$$

respectively.

4.3. For each tiling, we identify two subsequences of $(x_1, z_1, \ldots, x_{n-1}, z_{n-1})$. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the subsequence on the line $y = 0$; and $\mu = (\mu_1, \ldots, \mu_{n-2})$ be the subsequence on the line $y = -1$. In the above example, λ and μ are, respectively,

$$\lambda = (x_1, x_2, z_2, x_4, x_5, z_5) \quad \text{and} \quad \mu = (z_1, x_3, z_3, z_4);$$
$$\lambda = (x_1, z_1, z_2, z_3, x_5, z_5) \quad \text{and} \quad \mu = (x_2, x_3, x_4, z_4).$$

We note that, with the order $x_1 \ge z_1 \ge x_2 \ge z_2 \ge \cdots$, the entries of the sequences λ and μ satisfy the identities (3.1).

The following proposition shows that the tiling procedure given in Definition 4.2 provides the correspondence stated in Theorem 3.3.

PROPOSITION 4.3.

(1) *For a given tiling, let us impose the order*

$$x_1 \geq z_1 \geq x_2 \geq z_2 \geq \cdots \geq x_{n-1} \geq z_{n-1},$$

on the entries x_i and z_i of pattern blocks, and let λ and μ be its subsequences placed on the lines $y = 0$ and $y = -1$, respectively. If y_i satisfies $x_i \geq y_i \geq z_i$ for each pattern block, then $\mu \sqsubseteq (y_1, \ldots, y_{n-1}) \sqsubseteq \lambda$, that is, for all r and s,

$$\lambda_r \geq y_r \geq \lambda_{r+1} \quad \text{and} \quad y_s \geq \mu_s \geq y_{s+1}.$$

(2) *Conversely, let an interlacing pattern*

$$\mu \sqsubseteq (y_1, \ldots, y_{n-1}) \sqsubseteq \lambda$$

be given. If we place its entries λ_i, μ_j and y_k on coordinates $(i, 0)$, $(j + 1, -1)$ and $(k + 0.5, -0.5)$ for all i, j and k, then we obtain a tiling defined by the directed path connecting the λ_i and μ_j in weakly decreasing order. That is, if $(x_1, z_1, \ldots, x_{n-1}, z_{n-1})$ is the rearrangement of the sequence $(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_{n-2})$ in weakly decreasing order, then x_i, y_i and z_i form a pattern block and satisfy

$$x_i \geq y_i \geq z_i$$

for $1 \leq i \leq n - 1$.

PROOF. It is enough to check out the inequalities for all possible pairs of consecutive pattern blocks in a tiling listed below. Note that these are also all possible partial interlacing patterns with two triples (x, y, z) and (x', y', z').

$$\begin{bmatrix} x & & x' & & \\ & y & & y' & \\ & & z & & z' \end{bmatrix} \begin{bmatrix} x & & x' & & z' \\ & y & & y' & \\ & & z & & \end{bmatrix} \begin{bmatrix} & & x' & & \\ & y & & y' & \\ x & & z & & z' \end{bmatrix}$$

$$\begin{bmatrix} & & x' & & z' \\ & y & & y' & \\ x & & z & & \end{bmatrix} \begin{bmatrix} x & & z & & \\ & y & & y' & \\ & & x' & & z' \end{bmatrix} \begin{bmatrix} x & & z & & z' \\ & y & & y' & \\ & & x' & & \end{bmatrix}$$

$$\begin{bmatrix} & & z & & \\ & y & & y' & \\ x & & x' & & z' \end{bmatrix} \begin{bmatrix} & & z & & z' \\ & y & & y' & \\ x & & x' & & \end{bmatrix}$$

In the first case, $(\lambda_1, \lambda_2) = (x, x')$ and $(\mu_1, \mu_2) = (z, z')$. With $x \geq z \geq x' \geq z'$, we have $x \geq y \geq z$ and $x' \geq y' \geq z'$ if and only if

$$x \geq y \geq x' \geq y' \quad \text{and} \quad y \geq z \geq y' \geq z'.$$

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In the second case, $(\lambda_1, \lambda_2, \lambda_3) = (x, x', z')$ and $\mu_1 = z$. With $x \geq z \geq x' \geq z'$, we have $x \geq y \geq z$ and $x' \geq y' \geq z'$ if and only if

$$x \geq y \geq x' \geq y' \geq z' \quad \text{and} \quad y \geq z \geq y'.$$

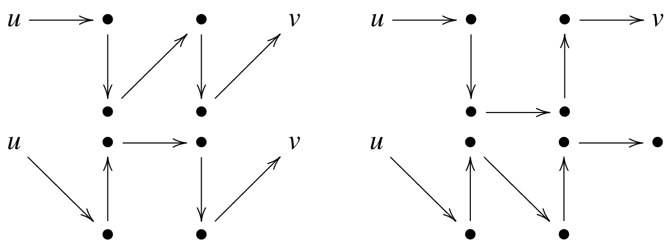
The rest of the cases can be shown similarly. □

4.4. We give an example illustrating tiling procedures, and therefore showing the GL_2 module structure of branching multiplicity spaces. Let us consider polynomial dominant weights $(x_i, z_i) \in \{(8, 5), (4, 2), (1, 0)\}$ of GL_2, and Gelfand–Tsetlin patterns

$$\left(\begin{bmatrix} 8 & & 5 \\ & y_1 & \end{bmatrix}, \begin{bmatrix} 4 & & 2 \\ & y_2 & \end{bmatrix}, \begin{bmatrix} 1 & & 0 \\ & y_3 & \end{bmatrix}\right)$$

where $y_i \in \mathbb{Z}$ varies for $x_i \geq y_i \geq z_i$ for all i.

In order to assemble these GL_2 pattern blocks to build GL_4 to GL_2 branching multiplicity spaces, we consider all the directed paths for P_4.



Using down-right, up-right, harpoon-up and harpoon-down arrows, they can be represented as

$$[\ \rightharpoonup\ \ \nearrow\ \ \nearrow\ ] \qquad [\ \rightharpoonup\ \ \rightharpoondown\ \ \rightharpoonup\ ]$$

$$[\ \searrow\ \ \rightharpoonup\ \ \nearrow\ ] \qquad [\ \searrow\ \ \searrow\ \ \rightharpoonup\ ].$$

Then, from Definition 4.2, we obtain the tilings

$$\begin{bmatrix} 8 & & 5 & & 2 & & 0 \\ & y_1 & & y_2 & & y_3 & \\ & & 4 & & 1 & & \end{bmatrix} \begin{bmatrix} 8 & & 5 & & 1 & & 0 \\ & y_1 & & y_2 & & y_3 & \\ & & 4 & & 2 & & \end{bmatrix}$$

$$\begin{bmatrix} 8 & & 4 & & 2 & & 0 \\ & y_1 & & y_2 & & y_3 & \\ & & 5 & & 1 & & \end{bmatrix} \begin{bmatrix} 8 & & 4 & & 1 & & 0 \\ & y_1 & & y_2 & & y_3 & \\ & & 5 & & 2 & & \end{bmatrix}$$

corresponding to the branching multiplicity spaces

$$\mathrm{Hom}_{GL_2}(V_2^{(4,1)}, V_4^{(8,5,2,0)}), \quad \mathrm{Hom}_{GL_2}(V_2^{(4,2)}, V_4^{(8,5,1,0)})$$
$$\mathrm{Hom}_{GL_2}(V_2^{(5,1)}, V_4^{(8,4,2,0)}), \quad \mathrm{Hom}_{GL_2}(V_2^{(5,2)}, V_2^{(8,4,1,0)})$$

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which are, by Theorem 3.5, as GL_2 representations, isomorphic to

$$\mathbb{C} \otimes V_2^{(8,5)} \otimes V_2^{(4,2)} \otimes V_2^{(1,0)}$$

where $g \in GL_2$ acts on $\mathbb{C}$ by $\det(g)^{-5}$, $\det(g)^{-6}$, $\det(g)^{-6}$ and $\det(g)^{-7}$, respectively.

We note that if some of the entries in the sequence $(x_1, z_1, \dots, x_{n-1}, z_{n-1})$ are equal, then different paths may give the same tiling, and therefore the same branching multiplicity space.

5. Branching multiplicity spaces of other classical groups

As in the case of the general linear group, we can study the GL_2 module structure of branching multiplicity spaces for the symplectic group. We can also obtain similar results for the orthogonal group within certain stable ranges. For more about stable range conditions in branching rules for classical groups, we refer readers to [4].

5.1. We denote the complex symplectic group of rank n and the complex special orthogonal group of rank $\lfloor m/2 \rfloor$ by Sp_{2n} and SO_m, respectively. The dominant weights of Sp_{2n} and SO_{2n+1} are of the form $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$; and the dominant weights of SO_{2n} are of the same form with $\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n|$.

We will state branching rules for individual cases (see, for example, [1, Section 25.3] or [3, Section 8.1]) with the convention of Gelfand–Tsetlin patterns, that is, the entries in each array are weakly decreasing along the diagonals from left to right.

5.2. Let W_{2n}^{λ} be the irreducible representation of Sp_{2n} with highest weight λ. Then for a dominant weight μ of Sp_{2n-2}, the multiplicity of W_{2n-2}^{μ} in W_{2n}^{λ} as a Sp_{2n-2} representation is equal to the number of Sp_{2n} dominant weights κ such that

$$\begin{bmatrix} \lambda_1 & & \lambda_2 & & \lambda_3 & & \cdots & & \lambda_n & \\ & \kappa_1 & & \kappa_2 & & \kappa_3 & & \cdots & & \kappa_n \\ & & \mu_1 & & \mu_2 & & \cdots & & \mu_{n-1} & \end{bmatrix}.$$

Note that we can identify this Sp_{2n} to Sp_{2n-2} branching rule with the GL_{n+1} to GL_{n-1} branching rule in Proposition 3.2. Therefore, as GL_2 representations,

$$\mathrm{Hom}_{Sp_{2n-2}}(W_{2n-2}^{\mu}, W_{2n}^{\lambda}) \cong \mathrm{Hom}_{GL_{n-1}}(V_{n-1}^{\mu}, V_{n+1}^{\lambda'})$$

where $\lambda' = (\lambda_1, \dots, \lambda_n, 0)$ (see [12, Theorem 3.1]). Then we can apply Theorem 3.3 to tile the Sp_{2n} to Sp_{2n-2} branching multiplicity space with GL_2 pattern blocks. From Theorem 3.5, we can express the branching multiplicity space as a tensor product of GL_2 representations. Also, by restricting GL_2 to its subgroup SL_2 and using the explanation in Section 2.3, we can obtain the SL_2 module structure of the branching multiplicity space.

We remark that Wallach and Yacobi studied Sp_{2n} to Sp_{2n-2} branching multiplicity spaces with $Sp_2 = SL_2$ and n copies of SL_2 in [10, 12], and Yacobi and the present author studied their algebraic and combinatorial properties in [5].

5.3. Let W^λ_{2n+1} be the irreducible representation of SO_{2n+1} with highest weight λ. Then for a dominant weight μ of SO_{2n-1}, the multiplicity of W^μ_{2n-1} in W^λ_{2n+1} as a SO_{2n-1} representation is equal to the number of dominant weights κ of SO_{2n} such that

$$\begin{bmatrix} \lambda_1 & & \lambda_2 & & \lambda_3 & & \cdots & & \lambda_n & \\ & \kappa_1 & & \kappa_2 & & \kappa_3 & & \cdots & & |\kappa_n| \\ & & \mu_1 & & \mu_2 & & \cdots & & \mu_{n-1} & \end{bmatrix}.$$

Note that if $\mu_{n-1} = 0$, then the interlacing condition makes $\kappa_n = 0$, and this branching rule becomes exactly the same as the GL_n to GL_{n-2} branching rule in Proposition 3.2. Therefore, if $\mu_{n-1} = 0$, as GL_2 representations,

$$\mathrm{Hom}_{SO_{2n-1}}(W^\mu_{2n-1}, W^\lambda_{2n+1}) \cong \mathrm{Hom}_{GL_{n-2}}(V^{\mu'}_{n-2}, V^\lambda_n)$$

where $\mu' = (\mu_1, \ldots, \mu_{n-2})$. Similarly, if $\lambda_n = 0$, as GL_2 representations,

$$\mathrm{Hom}_{SO_{2n-1}}(W^\mu_{2n-1}, W^\lambda_{2n+1}) \cong \mathrm{Hom}_{GL_{n-1}}(V^\mu_{n-1}, V^{\lambda'}_{n+1})$$

where $\lambda' = (\lambda_1, \ldots, \lambda_{n-1}, 0, 0)$. Then, we can apply Theorems 3.3 and 3.5 to tile the SO_{2n+1} to SO_{2n-1} branching multiplicity space with GL_2 pattern blocks and to factor it into GL_2 representations or SL_2 representations.

5.4. Let W^λ_{2n} be the irreducible representation of SO_{2n} with highest weight λ. Then for a dominant weight μ of SO_{2n-2}, the multiplicity of W^μ_{2n-2} in W^λ_{2n} as a SO_{2n-2} representation is equal to the number of SO_{2n-1} dominant weights κ such that

$$\begin{bmatrix} \lambda_1 & & \lambda_2 & & \lambda_3 & & \cdots & & \lambda_{n-1} & & |\lambda_n| \\ & \kappa_1 & & \kappa_2 & & \cdots & & \kappa_{n-2} & & \kappa_{n-1} & \\ & & \mu_1 & & \mu_2 & & \cdots & & \mu_{n-2} & & |\mu_{n-1}| \end{bmatrix}.$$

If $\mu_{n-2} = 0$, then the interlacing condition makes $\kappa_{n-1} = \lambda_n = \mu_{n-1} = 0$ and this branching rule becomes exactly the same as the GL_{n-1} to GL_{n-3} branching rule in Proposition 3.2. Therefore, if $\mu_{n-2} = 0$, then, as GL_2 representations,

$$\mathrm{Hom}_{SO_{2n-2}}(W^\mu_{2n-2}, W^\lambda_{2n}) \cong \mathrm{Hom}_{GL_{n-3}}(V^{\mu'}_{n-3}, V^{\lambda'}_{n-1})$$

where $\mu' = (\mu_1, \ldots, \mu_{n-3})$ and $\lambda' = (\lambda_1, \ldots, \lambda_{n-1})$. Similarly, if $\lambda_{n-1} = 0$, then $\kappa_{n-1} = \lambda_n = \mu_{n-1} = 0$ and as GL_2 representations,

$$\mathrm{Hom}_{SO_{2n-2}}(W^\mu_{2n-2}, W^\lambda_{2n}) \cong \mathrm{Hom}_{GL_{n-2}}(V^{\mu''}_{n-2}, V^{\lambda''}_n)$$

where $\mu'' = (\mu_1, \ldots, \mu_{n-2})$ and $\lambda'' = (\lambda_1, \ldots, \lambda_{n-2}, 0, 0)$. Then we can apply Theorems 3.3 and 3.5 to tile the SO_{2n} to SO_{2n-2} branching multiplicity space with GL_2 pattern blocks and to factor it into GL_2 representations or SL_2 representations.

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