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Convergence for moving averages

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Abstract. Assume T is an ergodic measure preserving point transformation from a probability space onto itself. Let $\{(n_k, l_k)\}_{k=1}^{\infty}$ be a sequence of pairs of positive integers, and define the sequence of averaging operators $A_k f(x) = (1/l_k)\sum_{j=0}^{l_k-1} f(T^{n_k+j}x)$. Necessary and sufficient conditions are given for this sequence of averages to converge almost everywhere. Weighted versions are also considered.

Introduction

Let (X, Σ, m) denote a non-atomic probability space, and T an invertible ergodic measure preserving point transformation of X onto itself. Let $\{(n_k, l_k)\}_{k=1}^{\infty}$ be a sequence of pairs of positive integers, and define the sequence of averaging operators

$$A_k f(x) = \frac{1}{l_k}\sum_{j=0}^{l_k-1} f(T^{n_k+j}x).$$

A number of authors have considered these averages with specific sequences $\{(n_k, l_k)\}_{k=1}^{\infty}$. They studied the question of almost everywhere convergence for the operators applied to functions in some L^p class. For example, in [**1**] it is shown that if $n_k = k$ and $l_k = \sqrt{k}$ then it is possible to find an f in L^∞ such that convergence fails. In fact they show convergence fails even for f the characteristic function of a measurable set. Later it was shown in [**10**] that if $n_k = p(k)$ where $p(x)$ is any polynomial with integer coefficients of degree at least one, and $l_k/n_k \to 0$, then convergence fails for some f in L^∞. It also follows from work in [**3**] that if $n_k = 4^k$ and $l_k = 2^k$ then convergence fails. On the other hand it can be shown that if $n_k = 2^{2^k}$ and $l_k = \sqrt{n_k}$ then almost everywhere convergence does occur for all f in L^1. Thus it becomes natural to investigate the question of convergence for general averages of the form $A_k f$. This is done in § 1. In § 2 these results are applied to give new

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proofs of the special cases mentioned above. In addition, the question of weighted averages can be considered. In [8] it was shown that the operators

$$2^{-n}\sum_{k=0}^{n}\binom{n}{k}f(T^kx)$$

fail to converge even for f the characteristic function of a measurable set. However in § 3 it will be shown that certain subsequences of these averages will converge almost everywhere for all f in L^1. Other weighted averages are also considered.

If A is a subset of integers, $|A|$ will denote the numbers of points in A. Throughout the paper, c will denote a constant, but not necessarily the same constant from one occurrence to the next. We will use the notation $n_k \nearrow \infty$ to mean $\{n_k\}_{k=1}^{\infty}$ is a nondecreasing unbounded sequence.

The assumption T ergodic can often be replaced by weaker assumptions. If the conclusion of a theorem is that a maximal inequality holds, or that, a.e. convergence holds, then by standard arguments, T ergodic can be replaced by T measure preserving. If the conclusion of the theorem is that a maximal inequality fails, then the assumption T ergodic can be replaced by T aperiodic. (The assumption T aperiodic is needed to make use of the Kakutani–Rokhlin construction.) If we want the stronger negative conclusion of 'strong sweeping out', we maintain the hypothesis T ergodic to insure the existence of a family of mixing transformations that commute with T.

1.

Let Ω be an infinite collection of lattice points with positive second coordinate. Define

$$\Omega_\alpha=\{(z,s)\,\big|\,|z-y|\le\alpha(s-r)\quad\text{for some } (y,r) \text{ in } \Omega,\ (z,s) \text{ a lattice point}\}.$$

Geometrically we visualize Ω_α as the union of all solid cones with aperture α and vertex in Ω. However for technical reasons, we define Ω_α to be only the lattice points in these cones.

The cross section of Ω_α at integer height $s>0$ is denoted by $\Omega_\alpha(s)$ and defined by

$$\Omega_\alpha(s)=\{k\,|\,(k,s)\in\Omega_\alpha\}.$$

Let T be an ergodic measure preserving point transformation from a probability space (X,Σ,m) to itself, and define the maximal function associated with the set Ω by

$$M_\Omega f(x)=\sup_{(k,n)\in\Omega}\frac{1}{n}\sum_{j=0}^{n-1}|f(T^{k+j}x)|.$$

THEOREM 1. (a) *Assume there exist constants* $A<\infty$ *and* $\alpha>0$ *such that* $|\Omega_\alpha(\lambda)|\le A\lambda$ *for all integer* $\lambda>0$; *then* M_Ω *is weak type* $(1,1)$ *and strong type* (p,p) *for* $1<p\le\infty$.

(b) *If* M_Ω *is weak type* (p,p) *for some finite* $p>0$ *then for every* $\alpha>0$ *there exists* $A_\alpha<\infty$ *such that for all integer* $\lambda>0$ *we have* $|\Omega_\alpha(\lambda)|\le A_\alpha\lambda$.

Theorem 1 is related to a generalization of Fatou's Theorem studied by Nagel and Stein [7]. The following proof is based on ideas in a subsequent paper by Sueiro [12], and the transfer principle of Calderón [4].

Proof of part (a). In the context of the maximal function problem, the Calderón transference principle allows us to transfer our considerations from (X, Σ, m, T) to the set of integers Z equipped with translation. Fix an x in X and a large integer N. For ease of notation, define

$$f(k)=f(T^k x)\chi_{[-N,N]}(k)$$

and extend the definition of the maximal function M_Ω to this setting in the obvious way. Let $\lambda>0$ be given. We now decompose the domain of f (which is now just the integers) into a collection of disjoint blocks $\{B_i\}$ where the two sided Hardy-Littlewood maximal function

$$f^*(p)=\sup_n \frac{1}{|n|+1}\sum_{j=0}^{n}|f(p+j)|>\lambda,$$

and the remainder where $f^*\leq\lambda$. By the usual Hardy-Littlewood maximal inequality we have

$$B_i=\{b_i, b_i+1,\ldots, b_i+r_i-1\}$$

and

$$|\cup B_i|\leq\frac{2}{\lambda}\|f\|_1=\frac{2}{\lambda}\sum_{j=-N}^{N}|f(j)|.$$

If $M_\Omega f(p)>2\lambda$ then by definition

$$\frac{1}{n}\sum_{j=0}^{n-1}|f(p+k+j)|>2\lambda \quad \text{for some } (k, n) \text{ in } \Omega.$$

This implies that the interval $(p+k, p+k+n-1)\subset B_i$ for some i. To see this, first note that the point $p+k$ is a place where f^* is greater than 2λ, hence greater than λ. We still need to show that $p+k+n-1\leq b_i+r_i-1$. If this were not true then there would be two possible cases.

Case 1. $p+k+n-1$ is not in one of the blocks where the maximal function is greater than λ. In this case we note that our maximal function f^* looks both to the left and the right. Looking at the average from $p+k+n-1$ back to $p+k$ we get less than or equal to λ, but this is the same average we assumed was greater than 2λ, a contradiction.

Case 2. $p+k+n-1$ is in some block other than B_i. In that case there must be at least one point q between $p+k$ and $p+k+n-1$ where $f^*(q)\leq\lambda$. Look from q left to $p+k$, the average will be less than or equal to λ, now look from q to the right to $p+k+n-1$, the average is less than or equal to λ. Thus we have

$$\sum_{j=p+k}^{q}|f(j)|\leq(q-p-k+1)\lambda,$$

and

$$\sum_{j=q}^{p+k+n-1}|f(j)|\leq(p+k+n-1-q+1)\lambda.$$

Adding these, we see that

$$\sum_{j=p+k}^{p+k+n-1}|f(j)|+|f(q)|\leq(n+1)\lambda.$$

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This implies

$$\sum_{j=p+k}^{p+k+n-1} |f(j)| \le (n+1)\lambda,$$

or

$$\frac{1}{n}\sum_{j=p+k}^{p+k+n-1} |f(j)| \le \frac{n+1}{n}\lambda \le 2\lambda,$$

a contradiction to the original assumption that the average was greater than 2λ. Thus we known that $n \le r_i$.

Let $c=[1+\alpha^{-1}]+1$. Then

$$\begin{aligned}|(b_i-p)-k| \le |b_i-(p+k)| &\le r_i \quad (\text{since } p+k \text{ is in } B_i)\\ &\le \alpha\left(r_i+\frac{1}{\alpha}r_i-r_i\right)\\ &\le \alpha(cr_i-r_i)\\ &\le \alpha(cr_i-n) \quad (\text{since } n\le r_i).\end{aligned}$$

Therefore by definition (b_i-p, cr_i) is contained in Ω_α, or $b_i-p\in\Omega_\alpha(cr_i)$. This implies that

$$p\in b_i-\Omega_\alpha(cr_i).$$

Using this we see that

$$\{p \mid M_\Omega f(p)>2\lambda\}\subset\bigcup_i\{b_i-\Omega_\alpha(cr_i)\}.$$

Taking measures (counting measure on the integers) we have

$$\begin{aligned}|\{p \mid M_\Omega f(p)>2\lambda\}| &\le \sum_i |b_i-\Omega_\alpha(cr_i)|\\ &\le \sum_i |\Omega_\alpha(cr_i)|\\ &\le \sum_i Acr_i\\ &\le Ac\sum_i |B_i| \le Ac|\cup B_i|\\ &\le Ac\frac{2}{\lambda}\sum_{j=-N}^{N}|f(j)|.\end{aligned}$$

From this and the transfer principle, (i.e. divide both sides by $2N$, return to the original notation, and use the fact that T is measure preserving), we deduce the weak type $(1, 1)$ conclusion of part a of the theorem. Note that the operator M_Ω is trivially bounded from L^∞ to L^∞. This and the weak type $(1, 1)$ estimate allow us to use the Marcinkiewicz interpolation theorem (see **[13]** page 111) to prove the operator M_Ω is strong type (p, p) for $1<p\le\infty$.

Proof of part (b). Assume that we are given $\alpha>0$ and an integer $\lambda>0$. First assume that $\Omega_\alpha(\lambda)$ is bounded. Form a very tall Kakutani–Rohklin tower of height N larger than

$$2(\alpha+1)\lambda+\sup\{|z| \mid z\in\Omega_\alpha(\lambda)\},$$

and with error less than $1/N$. Let (z, λ) be in Ω_α. Then by definition of Ω_α we know that $|z-k|\le\alpha(\lambda-n)$ for some $(k,n)\in\Omega$. Note that this implies $|z-k|\le\alpha\lambda$ and $\lambda\ge n$. Define a function f to be 1 on the top $2[(\alpha+1)\lambda]$ steps of the tower, and zero elsewhere. Let x be a point in the tower $[(\alpha+1)\lambda]$ steps below the top of the tower. Note that each step in the tower has measure at least 1/2N. We have

$$M_\Omega f(T^{-z}x)\ge\frac{1}{n}\sum_{j=0}^{n-1} f(T^{-z+k+j}x).$$

Since $|-z+k|\le\alpha\lambda$, the sum starts in the support of f. The sum also ends in the support of f because $|-z+k+n|\le|-z+k|+n\le\alpha\lambda+\lambda\le(\alpha+1)\lambda$. Thus the entire sum is within the support of f, and we conclude that

$$M_\Omega f(T^{-z}x)\ge 1.$$

This is true for all z such that $(z,\lambda)\in\Omega_\alpha$, i.e. for all z in $\Omega_\alpha(\lambda)$. The number of such z is no more than $2N$ times the measure of the set $\{M_\Omega f\ge 1\}$. Using this fact, and the assumption that the maximal function is weak type (p,p), we see that

$$\begin{aligned}\frac{|\Omega_\alpha(\lambda)|}{2N}&\le m(\{M_\Omega f\ge 1\})\\ &\le\frac{c}{1^p}\|f\|_p^p\\ &\le c\frac{[2(\alpha+1)\lambda]}{N}\\ &\le c\frac{2(\alpha+1)\lambda}{N}.\end{aligned}$$

Now multiply both sides by $2N$ to conclude the proof of part (b) in the case where $\Omega_\alpha(\lambda)$ is bounded. To see that this is in fact the general case, select B a large integer and replace $\Omega_\alpha(\lambda)$ by $\Omega_\alpha(\lambda)\cap[-B,B]$ in the above construction. We then conclude that $|\Omega_\alpha(\lambda)\cap[-B,B]|\le c2(\alpha+1)\lambda$. Note that the right hand side of the inequality is bounded independent of the choice of B. Thus the set $\Omega_\alpha(\lambda)$ must itself be bounded, and the proof is complete.

We can also consider a related symmetrical maximal function,

$$M^s_\Omega f(x)=\sup_{(k,n)\in\Omega}\frac{1}{2n+1}\sum_{j=-n}^{n}|f(T^{k+j}x)|.$$

Then Theorem 1 remains true with this maximal function. We state this as

THEOREM 2. (a) *Assume there exist constants $A<\infty$ and $\alpha>0$ such that $|\Omega_\alpha(\lambda)|\le A\lambda$ for all integer $\lambda>0$; then M^s_Ω is weak type $(1,1)$ and strong type (p,p) for $1<p\le\infty$.*

(b) *If M^s_Ω is weak type (p,p) for some finite $p>0$ then for every $\alpha>0$ there exists $A_\alpha<\infty$ such that for all integer $\lambda>0$ we have $|\Omega_\alpha(\lambda)|\le A_\alpha\lambda$.*

Proof. The proof is almost the same as the proof of Theorem 1. The following outline of the proof uses the notation from the proof of Theorem 1.

For part (a), we find a pair $(k,n)\in\Omega$ such that

$$\frac{1}{2n+1}\sum_{j=-n}^{n}|f(p+k+j|>2\lambda.$$

By the same argument as before, we find that $(p+k-n, p+k+n)\subset B_i$ for some i. In particular, this implies that $p+k$ is in B_i, and $n\le r_i$. From this we deduce that $p\in b_i-\Omega_\alpha(cr_i)$, and the result follows as before.

For part (b), note that $M_\Omega f(x)\le cM^s_\Omega f(x)$. To see this note that for each fixed $(k, n)\in\Omega$,

$$\sum_{j=0}^{n-1}|f(p+k+j|\le\sum_{j=-n}^{n}|f(p+k+j|$$

and hence

$$\frac{1}{n}\sum_{j=0}^{n-1}|f(p+k+j|\le\frac{2n+1}{n}\frac{1}{2n+1}\sum_{j=-n}^{n}|f(p+k+j|.$$

Consequently, if M^s_Ω is weak type (p, p) then so is the associated maximal function M_Ω, and hence by part (b) of Theorem 1, we see that there is a finite constant A_α such that $|\Omega_\alpha(\lambda)|\le A_\alpha\lambda$.

Corollary 1. *For each $p\ge 1$, the maximal functions $M_\Omega f$ and $M^s_\Omega f$ are weak type (p, p) if and only if the maximal functions $M_{\Omega_\alpha}f$ and $M^s_{\Omega_\alpha}f$ are weak type (p, p).*

Proof. Note that the union of cones with aperture α and base in Ω is exactly the same as the region formed by the union of cones with aperture α and base in Ω_α. Hence the condition in the theorem is either satisfied by both regions or neither region.

Let $\Omega^h=\{(k, n)|(k, n)\in\Omega \text{ and } n\ge h\}$. We say the

$$\lim_{\substack{n\to\infty\\(k,n)\in\Omega}}\frac{1}{n}\sum_{j=0}^{n-1}f(T^{k+j}x)$$

exists, (and equals $\tilde{f}$) if

$$\lim_{h\to\infty}\sup_{(k,n)\in\Omega^h}\left|\frac{1}{n}\sum_{j=0}^{n-1}f(T^{k+j}x)-\tilde{f}(x)\right|=0,$$

for a.e. x. With this notion of limit we can state the following corollary to Theorem 1.

Corollary 2. *If*

$$\lim_{\substack{n\to\infty\\(k,n)\in\Omega}}\frac{1}{n}\sum_{j=0}^{n-1}f(T^{k+j}x)$$

exists for all f in some L^p, $0<p<\infty$, then there exists an integer h such that for the set Ω^h we can find constants α and A_α such that for all integer $\lambda>0$, we have $|\Omega^h_\alpha(\lambda)|\le A_\alpha\lambda$.

Conversely if we can find an integer h and constants α and A_α such that for all integer $\lambda>0$, $|\Omega^h_\alpha(\lambda)|\le A_\alpha\lambda$ then the limit exists a.e. *for all f in L^1.*

Proof. Before proving the corollary we prove the following lemma which is really just a special case.

Lemma 1. *If $\Omega=\{(n_k, l_k)\,|\,l_k>0,\ k=1,2,\ldots\}$ with $l_k\nearrow\infty$, and*

$$\lim_{k\to\infty}\frac{1}{l_k}\sum_{j=0}^{l_k-1}f(T^{n_k+j}x)$$

exists a.e. *for all f in some L^p, $0<p<\infty$, then there exist constants α and A_α such that for all integer $\lambda>0$, we have $|\Omega_\alpha(\lambda)|\le A_\alpha\lambda$.*

Proof. Suppose convergence holds for all $f\in L^p$, for some $0<p<\infty$. If $1\le p<\infty$, apply Stein's theorem in L^p [**11**] (as extended by Sawyer [**9**]) to conclude that the maximal function M_Ω is weak type (p, p). If $0<p<1$, since $L^1\subset L^p$, apply Sawyer's theorem in L^1 to conclude that the maximal function is weak type $(1, 1)$. In either case, by Theorem 1, part (a), there exist constants $\alpha>0$ and $A_\alpha<\infty$ such that $|\Omega_\alpha(\lambda)|\le A_\alpha\lambda$.

LEMMA 2. *For each positive integer h define $\Omega(h)=\{(k, h)\,|\,(k, h)\in\Omega\}$. If*

$$\lim_{\substack{n\to\infty\\ (k,n)\in\Omega}} \frac{1}{n}\sum_{j=0}^{n-1} f(T^{k+j}x)$$

exists a.e. *for all $f\in L^p$ for some $0<p<\infty$ then $|\Omega(h)|$ is finite for all h greater than or equal to some integer h_0.*

Proof. If the limit exists then it must exist for each sequence of points from Ω with second coordinate tending to infinity. Assume Lemma 2 is false. Then we can find an increasing sequence of positive integers $h_1, h_2, \ldots$ such that $|\Omega(h_j)|=\infty$ for $j=1, 2, 3, \ldots$. We will now construct a 'bad' sequence $\{(n_k, l_k)\}_{k=1}^{\infty}$. Let $(m_1^1, h_1), (m_2^1, h_1), \ldots, (m_{h_1}^1, h_1)$ be any set of points in $\Omega(h_1)$. Let these be the first h_1 points in the sequence. Next let $(m_1^2, h_2), (m_2^2, h_2), \ldots, (m_{2h_2}^2, h_2)$ be any collection of $2h_2$ points in $\Omega(h_2)$. Let these be the next $2h_2$ points of the sequence. In general, at level h_i select ih_i points from $\Omega(h_i)$ and let these be the next ih_i points of the sequence. Define $\tilde\Omega$ to consist of the points in this sequence. The cone condition is obviously violated because at level $\lambda=h_k$ we see that $|\tilde\Omega_\alpha(h_k)|\ge kh_k$. Thus no finite constants α and A will work. By Lemma 1 we cannot have convergence along the sequence, and hence cannot have the more general convergence.

Proof of Corollary 2. By Lemma 2 we know that at a certain level h_0 and above, each level of Ω contains only a finite number of points. We will show that the linear growth condition is satisfied by the set $\tilde\Omega=\Omega^{h_0}$. Define an order on $\tilde\Omega$ as follows: $(n_i, l_i)<(n_j, l_j)$ if $l_i<l_j$ or $l_i=l_j$ and $n_i<n_j$. With this order we are really only considering a sequence with second coordinate tending to infinity. This puts us in the case of Lemma 1, from which we conclude that the growth condition is satisfied for some finite positive constants α and A.

To prove the converse, note that for functions in the dense class,

$$D=\{\varphi\,|\,\varphi(x)=g(x)-g(Tx)+c,\quad g\in L^\infty,\ \ c \text{ a constant}\}$$

convergence is trivial, because if $\varphi(x)=g(x)-g(Tx)+c$ then

$$\sup_{(k,l),(n,j)\in\Omega^h}\left|\frac{1}{l}\sum_{i=0}^{l-1}\varphi(T^{k+i}x)-\frac{1}{j}\sum_{i=0}^{j-1}\varphi(T^{n+i}x)\right|\le\frac{4\|g\|_\infty}{h},$$

and this clearly converges as h increases to infinity. Define

$$R(f)(x)=\lim_{h\to\infty}\ \sup_{(k,l),(n,j)\in\Omega^h}\left|\frac{1}{l}\sum_{i=0}^{l-1}f(T^{k+i}x)-\frac{1}{j}\sum_{i=0}^{j-1}f(T^{n+i}x)\right|.$$

Note that $R(f)(x) = R(f-\varphi)(x)$ for each $\varphi \in D$. Also note that $R(f)(x) \leq 2M_{\tilde{\Omega}}f(x)$. We would be done if we could show that $m\{x \mid R(f)(x) > \varepsilon\} = 0$ for each choice of $\varepsilon > 0$. This follows from the fact that for each $\varphi \in D$ we have

$$\begin{aligned} m\{x \mid R(f)(x) > \varepsilon\} &= m\{x \mid R(f-\varphi)(x) > \varepsilon\} \\ &\leq m\{x \mid 2M_{\tilde{\Omega}}(f-\varphi)(x) > \varepsilon\} \\ &\leq \frac{c}{\varepsilon/2} \|f-\varphi\|_1. \end{aligned}$$

We have used the growth condition and Theorem 1 to obtain the weak type inequality used in the last step. Since $\|f-\varphi\|_1$ can be taken as small as desired, the result follows.

Actually, more is true. By proving a variant of Sawyer's theorem, it will be possible to show that if convergence fails then it does so in a very strong way. We will show that if the growth condition fails, then given $\varepsilon > 0$, we can find a measurable set E such that $m(E) < \varepsilon$, but

$$\limsup \frac{1}{l_k} \sum_{j=0}^{l_k-1} \chi_E(T^{n_k+j}x) = 1 \quad \text{a.e.}$$

and

$$\liminf \frac{1}{l_k} \sum_{j=0}^{l_k-1} \chi_E(T^{n_k+j}x) = 0 \quad \text{a.e.}$$

In this situation we will say that we have the 'strong sweeping out property'.

A family of measure preserving transformations $\{S_\alpha\}$ is said to be mixing if for each pair of sets A and B in Σ, and $\rho > 1$, there exists S_α in the family such that $m(A \cap S_\alpha^{-1}(B)) < \rho \cdot m(A) \cdot m(B)$.

The following variant of Sawyer's Theorem will be needed to study the 'strong sweeping out property'.

THEOREM 3. *Let (X, Σ, m) denote a probability space. Assume that $\{T_k\}$ is a sequence of linear operators, $T_k : L^1 \to L^1$ with the properties*

(i) $T_k \geq 0$.

(ii) $T_k 1 = 1$.

(iii) *The T_k's commute with a family $\{S_\alpha\}$ which is a mixing family of measure-preserving transformations.*

For each n define $M_n f = \sup_{k \geq n} |T_k f|$. Assume that

(*) *For each $\varepsilon > 0$ and $n \in N$, there exists a sequence of sets $\{A_p\}$, such that if*

$$E_p = \{M_n \chi_{A_p} \geq 1-\varepsilon\} \quad \text{then } \sup_p \frac{m(E_p)}{m(A_p)} = +\infty.$$

Then the 'strong sweeping out property' holds: given $\varepsilon > 0$, we can find a set B, with $m(B) < \varepsilon$, such that

$$\limsup_k T_k 1_B = 1 \quad \text{a.e.}$$

$$\liminf_k T_k 1_B = 0 \quad \text{a.e.}$$

Proof. By Theorem 1.3 in del Junco and Rosenblatt [**6**], it is enough to show that:

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(**) For each $\varepsilon > 0$ and $n \in N$, we can find a set B, $m(B) < \varepsilon$ such that

$$M_n \chi_B \geq 1 - \varepsilon \quad \text{a.e.}$$

Now we argue as in Sawyer's proof. We may assume without loss of generality that $m(E_p)/m(A_p) \geq p^2$, otherwise we relabel the sequence. Let h_p be a natural number such that $1 \leq h_p m(E_p) \leq 2$ and hence

$$h_p m(A_p) \leq \frac{h_p m(E_p)}{p^2} \leq \frac{2}{p^2}.$$

Take $A_p^1, A_p^2, \ldots, A_p^{h_p}$ to be identical copies of A_p, i.e. $A_p^j = A_p$ for $j = 1, 2, \ldots, h_p$. Then $E_p^j = E_p$ for $j = 1, 2, \ldots, h_p$. By Sawyer's auxiliary lemma, (see M. de Guzmann's book [5], p. 20), there are $S_p^j \in \{S_\alpha\} p = 1, 2, \ldots, j = 1, 2, \ldots, h_p$ such that almost every x belongs to infinitely many $(S_p^j)^{-1}(E_p^j)$. (Here we use the fact that

$$\sum_{p=1}^{\infty} \sum_{j=1}^{h_p} m(E_p^j) = \sum_{p=1}^{\infty} h_p m(E_p) = \infty.)$$

Choose p_0 so that

$$\sum_{p=p_0}^{\infty} h_p m(A_p) \leq \sum_{p=p_0}^{\infty} \frac{2}{p^2} < \varepsilon$$

Define

$$\begin{aligned} F(x) &= \sup_{\substack{p \geq p_0 \\ 1 \leq j \leq h_p}} S_p^j \chi_{A_p^j}(x) \\ &= \chi_B(x). \end{aligned}$$

(As the sup of a countable family of indicator functions this is again an indicator function.) Then

$$\begin{aligned} \|F\|_1 &\leq \sum_{p \geq p_0} \sum_{j=1}^{h_p} \int_X S_p^j \chi_{A_p^j} \, dm \\ &= \sum_{p \geq p_0} \sum_{j=1}^{h_p} m(A_p^j) \\ &= \sum_{p \geq p_0} h_p m(A_p) \\ &< \varepsilon, \end{aligned}$$

and

$$\begin{aligned} M_n F(x) &\geq M_n(S_p^j \chi_{A_p^j})(x) \\ &= M_n(\chi_{A_p^j})(S_p^j x) \end{aligned}$$

(because our operators T_k commute with the family S_α: $T_k(Sg)(x) = S(T_k g)(x) = (T_k g)(Sx)$, whence $M_n(Sg)(x) = (M_n g)(Sx)$).

Now if $x \in (S_p^j)^{-1}(E_p^j)$, then $S_p^j x \in E_p^j$ and hence $M_n(\chi_{A_p^j})(S_p^j x) \geq 1 - \varepsilon$ which implies $M_n \chi_B(x) \geq 1 - \varepsilon$. Thus $M_n \chi_B(x) \geq 1 - \varepsilon$ for almost every $x \in X$, and the theorem is proved.

Remark. The assumptions on the sequence of operators $\{T_k\}$ used in the above theorem can be weakened and the conclusion can be strengthened. It is enough for these operators to satisfy the assumptions used by del Junco and Rosenblatt [**6**]. They assume that $\{T_k\}$ are monotone ($E \subset F \rightarrow T_k \chi_E \leq T_k \chi_F$) linear maps which are

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continuous in measure from Σ into the class of positive measurable functions and that $T_k\chi_X = 1$ for all k. The conclusion can be strengthened to show that an entire dense G_δ collection of subsets of Σ will work.

The following theorem applies Theorem 3 to the moving averages considered in Theorem 1. We state this as Theorem 4.

THEOREM 4. *Let $\Omega = \{(n_k, l_k) \mid l_k \nearrow \infty\}$. If the linear growth condition on $|\Omega_\alpha(\lambda)|$ fails, then we have the 'strong sweeping out property', i.e. given $\varepsilon > 0$ there exists a set E with $m(E) < \varepsilon$, such that*

$$\limsup \frac{1}{l_k} \sum_{j=0}^{l_k-1} \chi_E(T^{n_k+j}x) = 1 \quad \text{a.e.}$$

and

$$\liminf \frac{1}{l_k} \sum_{j=0}^{l_k-1} \chi_E(T^{n_k+j}x) = 0 \quad \text{a.e.}$$

Proof. Suppose that the set Ω is such that the linear growth condition on the cones fails. Take $\alpha = 1$. Then given any positive integer p, we can find a positive integer $\lambda = \lambda_p$ such that $p^2 5\lambda_p \leq |\Omega_1(\lambda_p)|$. Form a very tall Kakutani–Rohklin tower of height

$$N \gg 5\lambda_p + \sup\{z \mid z \in \Omega_1(\lambda_p)\}$$

and error less than $1/N$. Define A_p to consist of the top $4\lambda_p + 1$ steps of the tower. Then $m(A_p) \leq (4\lambda_p + 1)/N$. Let x be in the tower, exactly $2\lambda_p + 1$ steps from the top. Now for $z \in \Omega_1(\lambda_p)$ we have $|z - k| \leq (\lambda_p - n) \leq \lambda_p$ for some $(k, n) \in \Omega$, so $\lambda_p \geq n$. Also then $|z - (k+n)| \leq |z - k| + n \leq \lambda_p + \lambda_p = 2\lambda_p$, so that $|-z+k| \leq \lambda_p$ and $|-z+(k+n)| \leq 2\lambda_p$. This means that $T^{-z+k}x \in A_p, \ldots, T^{-z+(k+n)}x \in A_p$ and hence

$$M_\Omega \chi_{A_p}(T^{-z}x) \geq \frac{1}{n} \sum_{j=0}^{n-1} \chi_{A_p}(T^{-z+k+j}x) \geq 1.$$

(The sum on the right starts and ends in A_p.) Thus for each $z \in \Omega_1(\lambda_p)$, $T^{-z}x \in \{M_\Omega \chi_{A_p} \geq 1\}$. Since each step in the tower has measure at least $1/2N$, we have

$$\frac{|\Omega_1(\lambda_p)|}{2N} \leq m(M_\Omega \chi_{A_p} \geq 1 - \varepsilon) = m(E_p),$$

but

$$\frac{|\Omega_1(\lambda_p)|}{2N} \geq p^2 \frac{5\lambda_p}{2N} \geq \frac{p^2}{2} m(A_p)$$

and thus

$$\frac{m(E_p)}{m(A_p)} \geq \frac{p^2}{2}.$$

Note that if Ω' is the set Ω with the first few terms removed, then the above argument also applies to Ω'. Thus we showed that the condition (*) in Theorem 3 holds, finishing the proof.

2. *Applications*

In this section we consider a number of applications of the results obtained in § 1. We begin with a lemma which will prove useful in a number of the applications.

LEMMA 3. (Sliding Lemma.) *If $\Omega=\{(n_k, l_k)\}_{k=1}^{\infty}$, and $\tilde{\Omega}$ is obtained by increasing some of the l_k, then if $\tilde{\Omega}$ fails to satisfy the linear growth condition, so does Ω. Further if $g(k)=n_{k+1}-n_k$, and we shift the n_k in such a way that the new sequence $\tilde{n}_k$ has the property that $\tilde{n}_{k+1}-\tilde{n}_k \leq g(k)$ then if the sequence $\{(\tilde{n}_k, l_k)\}$ fails to satisfy the linear growth condition, so does the original sequence.*

Proof. Note that increasing l_k will decrease the cross section of the cone based at (n_k, l_k), and make it easier to satisfy the growth condition. Similarly, if we reduce the gaps between the cones, then the cross section at level λ will only decrease. If the reduced cross section is too large then the original cross section must have been too large also.

COROLLARY 3. *If $\{n_k\}$ and $\{l_k\}$ satisfy the growth conditions $n_{k+1}>n_k+l_k$ and for some fixed $j\geq 1$, $l_k>c\cdot n_{k-j}$, then $\lim_{k\to\infty}(1/l_k)\sum_{i=0}^{l_k-1} f(T^{n_k+i}x)$ exists* a.e. *for all $f\in L^1$.*

Remark. Note that a special case of this result is that if $n_k=2^{2^k}$ and $l_k=\sqrt{n_k}$ then we have convergence. This should be contrasted with the case considered in [**1**] where $n_k=k$, $l_k=\sqrt{n_k}$, and it is shown that convergence fails. Theorem 4 above shows that in fact convergence fails in a very dramatic way.

Proof. In this case the set Ω consists of points of the form (n_k, l_k). At level λ above the point (n_k, l_k) we see a contribution of $c(\lambda-l_k)$ where c depends on the angle chosen. Let m be the last integer from which there is a contribution. Assume first that $j=1$, and that the cone based at (n_{m-1}, l_{m-1}) is not contained in any earlier cone. Then the total contribution from all the cones is dominated by

$$\begin{aligned} n_{m-1}+\frac{c}{2}(\lambda-l_{m-1})+c(\lambda-l_m) &\leq \tfrac{3}{2}c\lambda+n_{m-1}\\ &\leq \tfrac{3}{2}c\lambda+c'l_m\\ &\leq \tfrac{3}{2}c\lambda+c'\lambda\\ &\leq A\lambda. \end{aligned}$$

(See figure 1.) If the cone based at (n_{m-1}, l_{m-1}) is contained in one or more earlier cones, let (n_k, l_k) be the vertex of the earlier cone which contains (n_{m-1}, l_{m-1}) and intersects the line $y=\lambda$ the farthest to the right. Then

$$\begin{aligned} |\Omega(\lambda)| &\leq n_k+\frac{c}{2}(\lambda-l_k)+c(\lambda-l_m)\\ &\leq n_{m-1}+\tfrac{3}{2}c\lambda\\ &\leq A\lambda, \end{aligned}$$

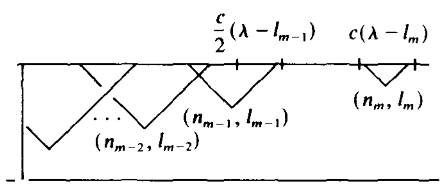

FIGURE 1.

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as above. If for example $j = 2$, and the cone at (n_{m-2}, l_{m-2}) is not contained in any earlier cones, then we have

$$|\Omega_\alpha(\lambda)| \leq n_{m-2} + \frac{c}{2}(\lambda - l_{m-2}) + (\lambda - l_{m-1}) + c(\lambda - l_m),$$
$$\leq A\lambda$$

where the growth condition was used to estimate n_{m-2} by a constant times l_m, and hence by a constant times λ. If the cone based at (n_{m-2}, l_{m-2}) is contained in one or more earlier cones, proceed as in the case $j = 1$. More generally, for any fixed j, we simply estimate the last $j+1$ terms by λ and use the growth condition to estimate the first $m-j$ terms.

COROLLARY 4. [3] *Let $n_k = r^k$ and $l_k = as^k$ with $1 < s < r$, a and s positive integers. Then for each choice of $p > 1$, $\lim_{k\to\infty} (1/l_k) \sum_{j=0}^{l_k-1} f(T^{n_k+j}x)$ fails to exist* a.e. *for some $f \in L^p$, and in fact the 'strong sweeping out property' holds.*

Proof. Here the set Ω consists of points of the form (r^n, as^n). The cone at (r^k, as^k) at height as^n has cross section $c(as^n - as^k) = cas^k(s^{n-k} - 1) \geq c's^n$. These cones will be disjoint at height as^n at those k such that $r^{k-1} + cas^n < r^k - cas^n$, i.e. for those k such that $2cas^n < r^k - r^{k-1}$, or $2cas^n < (r-1)r^{k-1}$. (See figure 2.) Let k_0 denote the smallest k such that $2cas^n < (r-1)r^{k-1}$. The number of cones which are disjoint at height as^n is at least $n - k_0$. Taking logs, we see that $n - k_0$ grows at least linearly with n. Thus the total contribution from these disjoint cross sections will be at least $nc''s^n$ for some constant c''. Since this is not dominated by As^n we cannot have a maximal inequality, and cannot have convergence.

COROLLARY 5. *Assume $r > 1$, and let $n_k = [r^k]$ and $l_k = o(n_k)$, then for each choice of $p \geq 1$,*

$$\lim_{k\to\infty} \frac{1}{l_k} \sum_{j=0}^{l_k-1} f(T^{n_k+j}x)$$

fails to exist a.e. *for some f in L^p and in fact we have the 'strong sweeping out property'.*

Proof. Note that we can assume without loss of generality that l_k are nondecreasing. If not, simply increase l_k's as necessary to achieve this. By Lemma 3 (the sliding lemma) if we do not have convergence for this new sequence, we could not have it for the original sequence. Note also that because of the rate of growth of the n_k and the non-decreasing nature of the l_k, if the cross section at height λ of two cones are disjoint then all future cross sections at height λ will also be disjoint. Let $\varepsilon > 0$ be given.

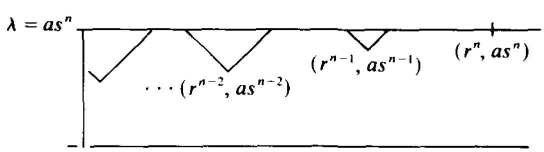

FIGURE 2.

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If there exists an infinite set S such that $n \in S$ implies that $l_n - l_{n-1} \geq \varepsilon l_n$ then we can proceed as follows: For $n \in S$ let $\lambda = l_n$, and let k_n be the largest k such that at level λ the cone at the point (n_k, l_k) intersects the cone at the point (n_{k-1}, l_{k-1}). Then we have

$$|\Omega_\alpha(\lambda)| \geq C\left(r^{k_n} + \sum_{j=k_n+1}^{n} (l_n - l_j)\right).$$

There are two cases.

Case 1. There is a constant c such that $n - k_n \leq c$ for all n in S. In this case we have $k_n \geq n - c$ and $|\Omega_\alpha(l_n)| \geq Cr^{k_n} \geq Cr^{n-c} = O(r^n)$ which implies we cannot have the weak type inequality.

Case 2. We have $n - k_n \to \infty$. For these $n \in S$ we have $|\Omega_\alpha(l_n)| \geq C(n-k_n)(l_n - l_{n-1}) \geq C(n-k_n)\varepsilon l_n$, but since $n - k_n$ goes to infinity we do not have the necessary size condition for a weak type inequality.

If the set S contains only a finite number of n, then we have $l_n - l_{n-1} < \varepsilon l_n$ for all n large enough. This can be rewritten $(1-\varepsilon)l_n < l_{n-1}$ or $l_n/l_{n-1} < 1/(1-\varepsilon)$. This leads to $l_n < A[1/(1-\varepsilon)]^n$ for some constant A. If ε is selected such that $1/(1-\varepsilon) < r$ then we are in the case of Corollary 4, and hence do not have a weak type result.

COROLLARY 6. **[10]** *Let $L > 0$ be given, and let $n_k = k^L$ (or the greatest integer in k^L if L is not an integer), and $l_k = o(n_k)$, then for each choice of $p \geq 1$,*

$$\lim_{k\to\infty} \frac{1}{l_k} \sum_{j=0}^{l_k-1} f(T^{n_k+j}x)$$

fails to exist a.e. *for some f in L^p and in fact we have the 'strong sweeping out property'.*

Proof. Look at the subsequence with index 2^k. We have $l_{2^k} = o(2^k)^L$, but $(2^k)^L = (2^L)^k = r^k$. Therefore looking only at this subsequence, we see that we are in the case of Corollary 5, and consequently cannot have convergence for this subsequence, and hence not for the original sequence.

COROLLARY 7. *Given any sequence $\{(n_k, l_k)\}_{k=1}^\infty$ such that $n_k \nearrow \infty$ and $l_k \nearrow \infty$, there is a subsequence $\{(n'_k, l'_k)\}_{k=1}^\infty$ such that convergence occurs along this subsequence.*

Proof. The sequence $\{(n'_k, l'_k)\}_{k=1}^\infty$ will be defined inductively. First let $(n'_1, l'_1) = (n_1, l_1)$. In general if (n'_k, l'_k) have been selected, select the pair (n'_{k+1}, l'_{k+1}) from the sequence $\{(n_k, l_k)\}_{k=1}^\infty$ such that $l'_{k+1} > n'_k$. The resulting subsequence satisfies the growth condition of Corollary 3 which is sufficient for convergence.

COROLLARY 8. *There is no universal strictly increasing subsequence $\{n_k\}_{k=1}^\infty$ such that for every sequence $\{(k, l_k)\}_{k=1}^\infty$, with $l_k \nearrow \infty$, convergence occurs along the subsequence $\{(n_k, l_{n_k})\}_{k=1}^\infty$.*

Proof. Given the proposed universally good sequence $\{n_k\}_{k=1}^\infty$, let $l_{n_k} = [\ln k]$. Define the remaining l_k so that the resulting sequence of l_k is non-decreasing. This sequence will be the required counter example. To see this, note that by the sliding lemma (Lemma 3) and the fact that $(k+1) - k \leq n_{k+1} - n_k$, $|\Omega_\alpha(\lambda)|$ is only decreased by 'sliding' the points (n_k, l_{n_k}) to the left, resulting in the sequence $\{(k, l_{n_k})\}_{k=1}^\infty$. Since the $l_{n_k} = [\ln k] = o(k)$, convergence fails by Corollary 6.

3. *Weighted averages*

There are many cases where the results of § 1 can be applied to problems involving weighted moving averages. Theorems 5 and 6 below show how this can be done. Further problems involving weights will be considered in a subsequent paper.

THEOREM 5. *Given any continuous function* φ *with the properties that*

(a) $\varphi(x) \geq 0$ *for all* x,

(b) $\int \varphi(x)\,dx = 1$,

(c) $\varphi(x)$ *is radial decreasing*, (i.e. $\varphi(|x|) = \varphi(x)$ *and* φ *is a decreasing function of* $|x|$),

(d) $\varphi(0) = A < \infty$.

define $\varphi_n(t) = n^{-1}\varphi(t/n)$. *Then the maximal function*

$$N_\Omega f(x) = \sup_{(k,n)\in\Omega} \sum_{j=-\infty}^{\infty} \varphi_n(j)|f(T^{k+j}x)|$$

is weak type $(1, 1)$ *if and only if the region* Ω *is a good region for the operator* M_Ω.

Proof. We can decompose the operator $N_\Omega f$ into the sum of pieces, each of which we can control by the maximal function $M^s_\Omega f$. By Theorem 2 we know that the operator M^s_Ω is weak type if and only if the linear growth condition on $\Omega(\lambda)$ is satisfied, and by Theorem 1 the same is true for the operator M_Ω.

Assume that f is non-negative. (If not replace f by $|f|$.) For all $(k, n) \in \Omega$ we have

$$\sum_{j=-\infty}^{\infty} \varphi_n(j)f(T^{k+j}x)$$

$$= \sum_{j=-n}^{n} \varphi_n(j)f(T^{k+j}x) + \sum_{m=0}^{\infty} \sum_{2^m n<|j|\leq 2^{m+1}n} \varphi_n(j)f(T^{k+j}x)$$

$$\leq \frac{2n+1}{n}\frac{A}{2n+1}\sum_{j=-n}^{n} f(T^{k+j}x) + \sum_{m=0}^{\infty} n2^{m+1}\varphi_n(n2^m)\frac{1}{n2^{m+1}}\sum_{n2^m<|j|\leq n2^{m+1}} f(T^{k+j}x)$$

$$\leq c\frac{1}{2n+1}\sum_{j=-n}^{n} f(T^{k+j}x) + c\sum_{m=0}^{\infty} n2^{m+2}\varphi_n(n2^m)\frac{1}{n2^{m+2}+1}\sum_{j=-n2^{m+1}}^{n2^{m+1}} f(T^{k+j}x)$$

$$\leq cM^s_\Omega f(x) + c\sum_{m=0}^{\infty} 2^{m+2}n\varphi_n(2^m n)M^s_{\Omega_\alpha}f(x)$$

$$\leq CM^s_{\Omega_\alpha}f(x).$$

To see that the last sum is finite we argue as follows:

$$\int_0^\infty \varphi(y)\,dy = \int_0^\infty \varphi_n(y)\,dy$$

$$\geq \int_0^n \varphi_n(y)\,dy + \sum_{m=0}^{\infty}\int_{n2^m}^{n2^{m+1}} \varphi_n(y)\,dy$$

$$\geq \varphi_n(n)\cdot n + \sum_{m=0}^{\infty} \varphi_n(n2^{m+1})\cdot(n2^m)$$

$$= \varphi_n(n)\cdot n + \frac{1}{2^3}\sum_{k=1}^{\infty} \varphi_n(n2^k)\cdot(n2^{k+2})$$

$$\geq \frac{1}{2^3} \varphi_n(n) \cdot n2^2 + \frac{1}{2^3} \sum_{k=1}^{\infty} \varphi_n(n2^k) \cdot (n2^{k+2})$$

$$\geq \frac{1}{2^3} \sum_{k=0}^{\infty} (n2^{k+2}) \cdot \varphi_n(n2^k).$$

The proof is completed by recalling that by Corollary 1, $M^s_{\bar{\Omega}_\alpha}$ satisfies the same weak type estimates as $cM^s_{\bar{\Omega}}$.

To prove the converse, note that

$$\sum_{j=-\infty}^{\infty} \varphi_n(j) f(T^{k+j}x) \geq \sum_{j=-n}^{n} \varphi_n(n) f(T^{k+j}x)$$

$$\geq c \frac{1}{2n+1} \sum_{j=-n}^{n} f(T^{k+j}x)$$

since $\varphi_n(n) = n^{-1}\varphi(n/n) = n^{-1}\varphi(1)$. *Hence* $N_\Omega f(x) \geq M^s_{\bar{\Omega}} f(x)$.

Remark 1. A similar theorem can be proved if we assume for example that φ is a non-negative function which is supported on the non-negative reals, decreasing for postive x, $\int \varphi(x)\,dx = 1$, and $\varphi(0) = A < \infty$. The only change in the proof is to use the one-sided maximal function rather than the symmetric two sided one used above.

Remark 2. Let $t(n)$ be a function from the positive integers to the integers. Then the region Ω is a good region for the maximal function

$$N'_\Omega f(x) = \sup_{(k,n)\in\Omega} \sum_{j-\infty}^{\infty} \varphi_{t(n)}(j) |f(T^{k+j}x)|,$$

i.e. the maximal function is weak type (p, p), if and only if the region $\bar{\Omega} = \{(k, t(n)) \mid (k, n) \in \Omega\}$ is a region which yields a weak type (p, p) inequality for the maximal function $M^s_{\bar{\Omega}_\alpha} f(x)$. To see this simply repeat the proof of Theorem 5, replacing n by $t(n)$. See the proof of Theorem 6 for an example of this idea.

Remark 3. More general weight functions can also be shown to work. For example if we have a sequence of probability measures p_n on the integers, which are non-decreasing on their support, and such that $\sup_k p_n(k)$ converges to zero as $n \to \infty$, then there is a subsequence p_{n_m} such that for all $f \in L^1$, $p_{n_m} f(x) = \sum_{k=-\infty}^{\infty} p_{n_m}(k) f(T^k x)$ converges for a.e. x. This will be considered further in a forthcoming paper.

Define the maximal function

$$N^b_\Omega f(x) = \sup_{(k,n)\in\Omega} \frac{1}{2^{2n}} \sum_{j=-n}^{n} \binom{2n}{n+j} |f(T^{k+j}x)|.$$

Let $\bar{\Omega} = \{(k, [\sqrt{n}]) \mid (k, n) \in \Omega\}$, where $[x]$ denotes the greatest integer function. Fix α and define $\bar{\Omega}_\alpha$ to be the collection of lattice points in the union of cones with vertex in $\bar{\Omega}$ and aperture α. We can now state the following theorem.

THEOREM 6. *For each* $p \geq 1$, *the operator* $N^b_\Omega f$ *is weak type* (p, p) *if and only if the operator* $M^s_{\bar{\Omega}} f$ *is weak type* (p, p). *In other words the region* Ω *is a good region for the operator* N^b_Ω *if and only if the region* $\bar{\Omega}$ *is a good region for the operator* $M^s_{\bar{\Omega}}$.

Proof. Note that by Corollary 1 looking, at the set $\bar{\Omega}_\alpha$ gives the same result as looking at $\bar{\Omega}$ from the point of view of the associated maximal functions being weak type (p, p). Therefore it suffices to compare the operator $N_\Omega^b f$ to the operator $M_{\bar{\Omega}_\alpha}^s f$.

Using Stirling's formula, it is easy to see that $(1/2^{2n})\binom{2n}{n}$ is well approximated by $1/\sqrt{\pi}\sqrt{n}$. Further,

$$\binom{2n}{n+k}=\binom{2n}{n}\frac{n(n-1)\cdots(n-k+1)}{(n+1)(n+2)\cdots(n+k)}.$$

Dividing top and bottom by n^k, and using the fact that

$$\frac{1-j/n}{1+j/n}\le e^{-2j/n},$$

it follows that

$$\binom{2n}{n+k}\le\binom{2n}{n}e^{-2(1/n)}e^{-2(2/n)}\cdots e^{-2(k-1)/n}\le\binom{2n}{n}e^{-k^2/n}.$$

(In fact the inequality can be reversed if $|k/n|<\frac{1}{2}$. This follows by the same argument and the fact that

$$\tfrac{1}{2}e^{-2j/n}\le\frac{1-j/n}{1+j/n}\quad\text{if } j/n\le\tfrac{1}{2}.)$$

Consequently, to study the maximal function $N_\Omega^b f$ we will use the fact that

$$\frac{1}{2^{2n}}\binom{2n}{n-k}=\frac{1}{2^{2n}}\binom{2n}{n+k}$$

can be dominated by

$$c\cdot\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{n}}e^{-k^2/n}=c\cdot\varphi_{\sqrt{2n}}(k)\quad\text{where } \varphi(x)=\frac{1}{\sqrt{\pi}}e^{-2x^2}$$

and $\varphi_t(x)=t^{-1}\varphi(x/t)$. In the following assume that f is non-negative. Thus we will actually study

$$N_\Omega^* f(x)=\sup_{(k,n)\in\Omega}\sum_{j=-n}^{n}\varphi_{\sqrt{2n}}(j)f(T^{k+j}x).$$

The operator can be decomposed into a sum of pieces, each of which will then be related to the operator $M_{\bar{\Omega}_\alpha}f$. We have

$$\begin{aligned}
&\sum_{j=-n}^{n}\varphi_{\sqrt{2n}}(j)f(T^{k+j}x)\\
&\quad\le\sum_{j=-[\sqrt{n}]}^{[\sqrt{n}]}\varphi_{\sqrt{2n}}(j)f(T^{k+j}x)+\sum_{m=0}^{\infty}\sum_{2^m[\sqrt{n}]<|j|\le 2^{m+1}[\sqrt{n}]}\varphi_{\sqrt{2n}}(j)f(T^{k+j}x)\\
&\quad\le c\frac{1}{2[\sqrt{n}]+1}\sum_{j=-[\sqrt{n}]}^{[\sqrt{n}]}f(T^{k+j}x)\\
&\qquad+\sum_{m=0}^{\infty}2^{m+1}[\sqrt{n}]\varphi_{\sqrt{2n}}(2^m[\sqrt{n}])\frac{1}{2^{m+1}[\sqrt{n}]}\sum_{2^m[\sqrt{n}]<|j|\le 2^{m+1}[\sqrt{n}]}f(T^{k+j}x)\\
&\quad\le c\frac{1}{2[\sqrt{n}]+1}\sum_{j=-[\sqrt{n}]}^{[\sqrt{n}]}f(T^{k+j}x)
\end{aligned}$$

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$$+c\sum_{m=0}^{\infty}2^{m+2}[\sqrt{n}]\varphi_{\sqrt{2n}}(2^m[\sqrt{n}])\frac{1}{2^{m+2}[\sqrt{n}]+1}\sum_{j=-2^{m+1}[\sqrt{n}]}^{2^{m+1}[\sqrt{n}]}f(T^{k+j}x)$$

$$\le cM^{s}_{\bar{\Omega}_\alpha}f(x)+c\sum_{m=0}^{\infty}2^{m+2}[\sqrt{n}]\varphi_{\sqrt{2n}}(2^m[\sqrt{n}])M^{s}_{\bar{\Omega}_\alpha}f(x)$$

$$\le cM^{s}_{\bar{\Omega}_\alpha}f(x).$$

We have used the fact that if a point $(k,[\sqrt{n}])$ is in $\bar{\Omega}_\alpha$ then so is (k,j) for any $j\ge[\sqrt{n}]$. This is true because if a point is in a cone, so is every point above that point, and $\bar{\Omega}_\alpha$ was defined to be a union of cones. We also need the fact that $\sum_{m=0}^{\infty}2^{m+2}[\sqrt{n}]\varphi_{\sqrt{2n}}(2^m[\sqrt{n}])$ is finite independent of n. To see this we argue as follows:

$$\int_0^\infty\varphi_{\sqrt{2n}}(y)\,dy\ge\int_0^{[\sqrt{n}]}\varphi_{\sqrt{2n}}(y)\,dy+\sum_{m=0}^{\infty}\int_{2^m[\sqrt{n}]}^{2^{m+1}[\sqrt{n}]}\varphi_{\sqrt{2n}}(y)\,dy$$

$$\ge[\sqrt{n}]\varphi_{\sqrt{2n}}([\sqrt{n}])+\sum_{m=0}^{\infty}2^m[\sqrt{n}]\varphi_{\sqrt{2n}}(2^{m+1}[\sqrt{n}])$$

$$\ge[\sqrt{n}]\varphi_{\sqrt{2n}}([\sqrt{n}])+\frac{1}{2^3}\sum_{k=1}^{\infty}2^{k+2}[\sqrt{n}]\varphi_{\sqrt{2n}}(2^k[\sqrt{n}])$$

$$\ge\frac{1}{2^3}\sum_{k=0}^{\infty}2^{k+2}[\sqrt{n}]\varphi_{\sqrt{2n}}(2^k[\sqrt{n}]),$$

and for each choice of t,

$$\int_0^\infty\varphi_t(y)\,dy=\int_0^\infty\frac{1}{t}\varphi(y/t)\,dy=\int_0^\infty\varphi(y)\,dy=\int_0^\infty\frac{1}{\sqrt{\pi}}e^{-2y^2}\,dy=\frac{1}{\sqrt{2}}.$$

Consequently if $M^{s}_{\bar{\Omega}_\alpha}f$ is weak type (p,p) then so is the operator $N^{b}_{\Omega}f$.

To prove the converse note that

$$\frac{1}{2^{2n}}\sum_{j=-n}^{n}\binom{2n}{n+j}|f(T^{k+j}x)|\ge c\sum_{j=-n/2}^{n/2}\varphi_{\sqrt{2n}}(j)f(T^{k+j}x)$$

$$\ge c\sum_{j=-[\sqrt{n}]}^{[\sqrt{n}]}\varphi_{\sqrt{2n}}(j)f(T^{k+j}x)$$

$$\ge c\frac{1}{\sqrt{n}}\sum_{j=-[\sqrt{n}]}^{[\sqrt{n}]}f(T^{k+j}x)$$

$$\ge c\frac{1}{2[\sqrt{n}]+1}\sum_{j=-[\sqrt{n}]}^{[\sqrt{n}]}f(T^{k+j}x),$$

and $(k,n)\in\Omega$ implies $(k,[\sqrt{n}])\in\bar{\Omega}_\alpha$. (Note that in the first inequality we have used the reverse inequality between the binomial coefficients and the function φ.) By the above, we have $N^{b}_{\Omega}f(x)\ge cM^{s}_{\bar{\Omega}_\alpha}f(x)$. Thus if $\bar{\Omega}$ is not a good region for the maximal function $M^s_{\bar{\Omega}}$, then $\bar{\Omega}_\alpha$ is not a good region for the maximal function $M^s_{\bar{\Omega}_\alpha}$ and hence Ω is not a good region for the maximal function N^b_Ω.

COROLLARY 9. *The operator*

$$\sup_{n>0} \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j} |f(T^j x)$$

fails to be weak type $(1, 1)$. *Consequently the associated limit operator*

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j} f(T^j x)$$

fails to exist a.e. *for some* $f \in L^p$, *and in fact we have the 'strong sweeping out property'.*

Proof. It suffices to look only at the even integers and to show that the maximal operator

$$\sup_{n>0} \frac{1}{2^{2n}} \sum_{j=0}^{2n} \binom{2n}{j} |f(T^j x)| = \sup_{n>0} \frac{1}{2^{2n}} \sum_{k=-n}^{n} \binom{2n}{n+k} |f(T^{n+k} x)|$$

fails to be weak type (1.1). But this is just $N^b_\Omega f$ for $\Omega = \{(n, n) \mid n \in N\}$ and hence $\bar{\Omega} = \{(n, [\sqrt{n}]) \mid n \in N\}$. Because the second coordinate grows more slowly than the first, by Corollary 6, $\bar{\Omega}$ is not a good region for $M^s_{\bar{\Omega}}$. Consequently by Theorem 6, Ω is not a good region for the maximal function N^b_Ω.

To see that we have the 'strong sweeping out property', note that by the Central Limit Theorem, given $\varepsilon > 0$ there exists a constant $b = b(\varepsilon)$ such that for all n large enough,

$$\frac{1}{2^{2n}} \sum_{k=-[b\sqrt{n}]}^{[b\sqrt{n}]} \binom{2n}{n+k} > 1 - \varepsilon.$$

For the set

$$\Omega = \{(n - [b\sqrt{n}], 2[b\sqrt{n}]) \mid n > 0\}$$

it is not difficult to see that the growth condition on the cones is not satisfied. Thus by Theorem 4, the ordinary averages along this sequence have the strong sweeping out property. Hence given $\varepsilon' > 0$, there exist sets E with arbitrarily small measure such that for almost every x there are infinitely many n with the property that

$$\frac{1}{2[b\sqrt{n}]} \sum_{k=-[b\sqrt{n}]}^{[b\sqrt{n}]-1} \chi_E(T^{n+k} x) = \frac{1}{2[b\sqrt{n}]} \sum_{j=0}^{2[b\sqrt{n}]-1} \chi_E(T^{n-[b\sqrt{n}]+j} x) > 1 - \varepsilon'.$$

This implies that if $k \in [n - [b\sqrt{n}], n + [b\sqrt{n}])$, then $\chi_E(T^k x) = 1$ except for at most $2[b\sqrt{n}]\varepsilon'$ terms. From this and the fact that the largest weight in the binomial average is dominated by $1/\sqrt{\pi}\sqrt{n}$, we can see that the binomial average is at least

$$1 - \varepsilon - \frac{2[b\sqrt{n}]}{\sqrt{\pi}\sqrt{n}} \varepsilon'.$$

Since ε and ε' are arbitrary, we are done.

COROLLARY 10. *Let* $S = \{2^{2^n} \mid n \geq 0\}$, *then the operator*

$$\sup_{n\in S} \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j} |f(T^j x)|$$

is weak type $(1, 1)$.

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Proof. In this case $\Omega=\{(2^{2^n},2^{2^n})\,|\,n\geq 0\}$, $\bar{\Omega}=\{(2^{2^n},\sqrt{2^{2^n}})\,|\,n\geq 0\}$. Note that $2^{2^{n-1}}=2^{2^n(1/2)}=[2^{2^n}]^{1/2}$. Hence by Theorem 6, and the fact that $\bar{\Omega}$ satisfies the growth condition of Corollary 3, the maximal inequality holds.

COROLLARY 11. *Let $S=\{2^{2^n}\,|\,n\geq 0\}$, then for all $f\in L^1$ the operator*

$$\lim_{n\in S}\frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j}f(T^jx)$$

exists for a.e. x, *and the limit is the integral of f.*

Proof. Consider the usual dense class of functions,

$$\{f\,|\,f(x)=g(x)-g(Tx)+c,\ g\in L^\infty\}.$$

For functions in this class convergence is true even without passing to a subsequence. Let $f(x)=g(x)-g(Tx)$, and assume $g\in L^\infty$. For this f, we must show we have convergence to zero. This follows because

$$\begin{aligned}\frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j}f(T^jx)&=\frac{1}{2^n}\sum_{j=0}^{[n/2]}\binom{n}{j}\{g(T^jx)-g(T^{j+1}x)\}\\&\quad+\frac{1}{2^n}\sum_{j=[n/2]+1}^{n}\binom{n}{j}\{g(T^jx)-g(T^{j+1}x)\}\\&=\frac{1}{2^n}\sum_{j=1}^{[n/2]}\left\{\binom{n}{j}-\binom{n}{j-1}\right\}g(T^jx)+\frac{1}{2^n}g(x)\\&\quad-\frac{1}{2^n}\binom{n}{[n/2]}g(T^{[n/2]+1}x)\\&\quad+\text{similar terms for the second sum.}\end{aligned}$$

Consequently,

$$\left\|\frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j}f(T^jx)\right\|_\infty\leq\|g\|_\infty\frac{1}{2^n}\sum_{j=1}^{[n/2]}\left\{\binom{n}{j}-\binom{n}{j-1}\right\}+\|g\|_\infty\frac{1}{2^n}+\|g\|_\infty\frac{1}{2^n}\binom{n}{[n/2]}$$

$$+\text{similar terms for the second sum.}$$

Because the sum $\sum_{j=1}^{[n/2]}\{\binom{n}{j}-\binom{n}{j-1}\}$ telescopes, we have

$$\left\|\frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j}f(T^jx)\right\|_\infty\leq\|g\|_\infty\frac{1}{2^n}\binom{n}{[n/2]}+\|g\|_\infty\frac{1}{2^n}+\|g\|_\infty\frac{1}{2^n}\binom{n}{[n/2]}$$

$$+\text{similar terms for the second sum.}$$

This converges to zero as $n\to\infty$, completing the proof of convergence for a dense class of functions. The corollary then follows by recalling the fact that we have a maximal inequality and using Banach's principle.

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