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METRIC PROJECTIONS AND THE DIFFERENTIABILITY OF DISTANCE FUNCTIONS

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Let M be a closed subset of a Banach space E such that the norms of both E and E^* are Fréchet differentiable. It is shown that the distance function $d(\cdot, M)$ is Fréchet differentiable at a point x of $E \sim M$ if and only if the metric projection onto M exists and is continuous at X. If the norm of E is, moreover, uniformly Gateaux differentiable, then the metric projection is continuous at x provided the distance function is Gateaux differentiable with norm-one derivative. As a corollary, the set M is convex provided the distance function is differentiable at each point of $E \sim M$. Examples are presented to show that some of our hypotheses are needed.

1. Introduction

For a nonempty subset M of a real Banach space E, let

$$\phi(x) = \inf\{\|x-y\| : y \in M\}$$

be the distance function associated to M and let

$$P(x) = \{y \in M : \|x-y\| = \phi(x)\}$$

be the set of *nearest points* in M to x, for each $x \in E$. We call a sequence (y_n) from M a *minimizing sequence* for x provided $\|x-y_n\| \to \phi(x)$ as $n \to \infty$.

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If M is also bounded we define, for each $x \in E$,

$$\psi(x) = \sup\{\|x-y\| : y \in M\}$$

and

$$F(x) = \{y \in M : \|x-y\| = \psi(x)\} ,$$

the set of *farthest points* in M from x, and we call a sequence (y_n) from M a *maximizing sequence* for x provided $\|x-y_n\| \to \psi(x)$ as $n \to \infty$. The maps P and F are called the *metric projection* and *antiprojection* for M respectively.

A real-valued function f on E is said to be *Gateaux differentiable* at a point x of E if there is an element $df(x)$ of E^* (the dual of E) such that, for each y in E,

$$\lim_{t\to 0} t^{-1}\big(f(x+ty)-f(x)\big) = \langle df(x), y\rangle ,$$

and we call $df(x)$ the *Gateaux derivative* of f at x.

We say that f is *Fréchet differentiable* at a point x if there is an element $f'(x)$ of E^* such that

$$\lim_{\|y\|\to 0} \|y\|^{-1}\big(f(x+y)-f(x)-\langle f'(x), y\rangle\big) = 0 ,$$

and we call $f'(x)$ the *Fréchet derivative* of f at x. Clearly, if f is Fréchet differentiable at a point x then it is also Gateaux differentiable at x and $f'(x) = df(x)$.

Denote by $B(E)$ the closed unit ball at the Banach space E and let $S(E) = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. We say that a nonzero element x^* of E^* *strongly exposes* $B(E)$ *at* $x \in S(E)$ provided a sequence (y_n) from $B(E)$ converges to x whenever $(\langle x^*, y_n\rangle)$ converges to $\|x^*\|$. A Banach space E is said to be *strictly convex* if $S(E)$ contains no line segments and *locally uniformly convex* provided a sequence (x_n) from $B(E)$ converges to a point x of $S(E)$ whenever $\|x+x_n\| \to 2$.

Let D denote the (set-valued) *norm-one duality map* on E, defined for each $x \in E$ by

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$$D(x) = \{x^* \in B(E^*) : \langle x^*, x\rangle = \|x\|\} .$$

The Banach space E is *smooth* provided $D(x)$ is a singleton for all non-zero x in E, in which case, for each nonzero x and y,

$$(1) \qquad t^{-1}(\|x+ty\|-\|x\|) - \langle D(x), y\rangle \to 0 \text{ as } t \to 0 .$$

This is clearly equivalent to the assertion that the norm of E is Gateaux differentiable at each nonzero point x, with Gateaux derivative $D(x)$. It is easily seen that the norm of E is Fréchet differentiable (at each nonzero point x) if (1) holds uniformly for $y \in S(E)$. We say that the norm of E is *uniformly Gateaux differentiable* if (1) holds uniformly for $x \in S(E)$, for each y in E.

Šmulian [18] showed that the norm of E^* is Fréchet differentiable at $x^* \in E^*$ if and only if x^* strongly exposes $B(E)$, and that if E^* has Fréchet differentiable norm then E is reflexive. Lovaglia [12] showed that if E is reflexive and locally uniformly convex then E^* has Fréchet differentiable norm. In [17], Šmulian proved that the norm of E is uniformly Gateaux differentiable if and only if E^* is *weak* uniformly convex*, that is, whenever (x_n^*) and (y_n^*) are sequences from $B(E^*)$ such that $\|x_n^*+y_n^*\| \to 2$ we have $x_n^* - y_n^* \to 0$ weak*. Using this characterization, it can be shown that every separable Banach space has an equivalent norm which is uniformly Gateaux differentiable (see [24], p. 429), and Trojanski [20] showed that there are nonseparable reflexive Banach spaces with no equivalent uniformly Gateaux differentiable norm. Sullivan [19] has investigated some consequences of uniform Gateaux differentiability.

A multivalued mapping T from a Banach space X to a Banach space Y is said to be *continuous* at $x \in X$ provided T is single-valued at x and $y_n \to Tx$ whenever $x_n \to x$ and $y_n \in Tx_n$.

In [3], Asplund showed that if M is a closed subset of Hilbert space and x has a nearest point in M, then the metric projection onto M is continuous at the point x if and only if ϕ is Fréchet differentiable at x; moreover, P is norm-weak continuous at x if and only if ϕ is Gateaux differentiable at x. His proofs used properties that are unique to Hilbert space.

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We will prove similar results in more general Banach spaces, and we will not need to assume the existence of a nearest point to x . If ϕ is only Gateaux differentiable at x we will need to assume that $\|d\phi(x)\| = 1$, but this will actually yield (norm-norm) continuity of P at x . The case where ϕ is differentiable for all $x \notin M$ is of special interest: we give conditions under which this yields the convexity of the set M .

Throughout this chapter, M will be assumed to be a non-empty subset of the Banach space E .

2. Consequences of the differentiability of ϕ and ψ

The first lemma we need is obvious.

LEMMA 2.1. *For a nonempty subset* M *of a Banach space* E *and any points* y *and* z *of* E *, we have* $|\phi(y)-\phi(z)| \le \|y-z\|$ *and if* M *is bounded then* $|\psi(y)-\psi(z)| \le \|y-z\|$.

LEMMA 2.2 (Vlasov [21], Blatter [6]). *Let* M *be a subset of a Banach space* E .

(a) If $x \in E \sim \overline{M}$ *is a point of Gateaux differentiability of* ϕ *and* $y \in P(x)$ *then* $\langle d\phi(x), x-y \rangle = \|x-y\|$ *and* $\|d\phi(x)\| = 1$.

(b) If M *is bounded,* $x \in E$ *is a point of Gateaux differentiability of* ψ *and* $y \in F(x)$ *then* $\langle d\psi(x), x-y \rangle = \|x-y\|$ *and* $\|d\psi(x)\| = 1$.

Proof. *(a)* Clearly $\|x-y\| = \phi(x) > 0$. For $0 < t < 1$,

$$\begin{aligned}\phi(x) - t\|x-y\| &= (1-t)\|x-y\| = \|x+t(y-x)-y\| \\ &\ge \phi\big(x+t(y-x)\big) \quad \text{since } y \in M , \\ &\ge \phi(x) - t\|x-y\|\end{aligned}$$

by Lemma 2.1. So equality holds throughout and

$$\begin{aligned}\langle d\phi(x), y-x \rangle &= \lim_{t\to 0} t^{-1}\{\phi[x+t(y-x)]-\phi(x)\} \\ &= -\|y-x\| ;\end{aligned}$$

hence $\langle d\phi(x), x-y \rangle = \|x-y\|$. But Lemma 2.1 implies that $\|d\phi(x)\| \le 1$, so this also shows that $\|d\phi(x)\| = 1$.

(b) If M is a single point this is clear. Otherwise $x \ne y$ and

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$\|x-y\| = \psi(x)$. For $0 < t < 1$,

$$\begin{aligned}\psi(x) + t\|x-y\| &= (1+t)\|x-y\| = \|x+t(x-y)-y\| \\ &\le \psi\big(x+t(x-y)\big) \quad \text{since } y \in M , \\ &\le \psi(x) + t\|x-y\| .\end{aligned}$$

As above, this implies that $\langle d\psi(x), x-y\rangle = \|x-y\|$ and $\|d\psi(x)\| = 1$.

Now we can give a proof of a result of Zhivkov.

THEOREM 2.3 [23]. *Suppose* M *is a subset of a strictly convex Banach space* E .

(a) If x *is a point of Gateaux differentiability of* ϕ *then there is at most one nearest point in* M *to* x .

(b) If M *is bounded and* x *is a point of Gateaux differentiability of* ψ *then there is at most one farthest point in* M *to* x .

Proof. *(a)* If $x \in \overline{M}$, this is obvious. Otherwise Lemma 2.2 shows that for all elements y and z of $P(x)$,

$$\langle d\phi(x), x-y\rangle = \|d\phi(x)\|\cdot\|x-y\| = \|d\phi(x)\|\cdot\|x-z\| = \langle d\phi(x), x-z\rangle .$$

Since E is strictly convex, $d\phi(x)$ can attain its norm at only one point of $S(E)$, which implies that $P(x)$ has at most one element.

(b) This is proved similarly.

Lemma 2.2 tells us that if ϕ is Gateaux differentiable at x and $\|d\phi(x)\| < 1$ then $P(x)$ is empty. (We will give an example later to show that this situation can occur even in Hilbert space.) When $\|d\phi(x)\| = 1$, we can prove the existence of nearest points but we need some strong assumptions.

THEOREM 2.4. *Let* M *be a closed subset of a Banach space* E *with uniformly Gateaux differentiable norm.*

(a) Suppose that $x \in E \sim M$ *is a point of Gateaux differentiability of* ϕ *with* $\|d\phi(x)\| = 1$. *Assume*

(i) that $d\phi(x)$ *strongly exposes* $B(E)$ *at some point* z *and*

(ii) that z *strongly exposes* $B(E^*)$ *at* $d\phi(x)$.

Then every minimizing sequence for x *converges to* $x - \phi(x)z$, *and the latter is the unique nearest point in* M *to* x .

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(b) Suppose that M *is bounded and that* x *is a point of Gateaux differentiability of* ψ *with* $\|d\psi(x)\| = 1$. *Assume*

(i) that $d\psi(x)$ *strongly exposes* $B(E)$ *at some point* z *and*

(ii) that z *strongly exposes* $B(E^*)$ *at* $d\psi(x)$.

Then every maximizing sequence for x *converges to* $x - \psi(x)z$, *and the latter is the unique farthest point in* M *from* x .

Proof. *(a)* Suppose (y_n) is a minimizing sequence for x . For all $t > 0$,

$$\phi(x+tz) - \phi(x) \le \inf_n \|x+tz-y_n\| - \lim_{n\to\infty} \|x-y_n\|$$
$$\le \liminf_{n\to\infty} \left(\|x+tz-y_n\|-\|x-y_n\|\right) .$$

By assumption *(i)*,

$$1 = \langle d\phi(x), z\rangle = \lim_{t\to 0} t^{-1}\left(\phi(x+tz)-\phi(x)\right) ,$$

so

$$1 \le \liminf_{t\to 0+} \liminf_{n\to\infty} t^{-1}\left(\|x+tz-y_n\|-\|x-y_n\|\right) .$$

We claim that, as $t \to 0$,

$$t^{-1}\left(\|x+tz-y_n\|-\|x-y_n\|\right) - \langle D\left(x-y_n\right), z\rangle \to 0 \tag{2}$$

uniformly for $n \in \mathbb{N}$. To see this, let $\alpha_n = \|x-y_n\|$ and note that by the uniform Gateaux differentiability of the norm,

$$t^{-1}\left\{\left\|\alpha_n^{-1}\left(x-y_n\right)+tz\right\|-\left\|\alpha_n^{-1}\left(x-y_n\right)\right\|\right\} - \left\langle D\left[\alpha_n^{-1}\left(x-y_n\right)\right], z\right\rangle \to 0$$

uniformly in n as $t \to 0$. From the fact that $D(ru) = D(u)$ for $u \in E$ and $r > 0$, together with homogeneity of the norm, we see that

$$\left(\alpha_n t\right)^{-1}\left(\|x-y_n+\alpha_n tz\|-\|x-y_n\|\right) - \langle D\left(x-y_n\right), z\rangle \to 0$$

uniformly in n as $t \to 0$. But $\alpha_n \to \phi(x) > 0$, so this yields the uniform convergence of (2).

Thus $1 \le \liminf_{n\to\infty} \langle D\left(x-y_n\right), z\rangle$. Since z strongly exposes $B(E^*)$

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at $d\phi(x)$, we have $D(x-y_n) \to d\phi(x)$. Now

$$\|x-y_n\| \geq \langle d\phi(x), x-y_n \rangle = \langle D(x-y_n), x-y_n \rangle + \langle d\phi(x)-D(x-y_n), x-y_n \rangle$$
$$\geq \|x-y_n\| - \|d\phi(x)-D(x-y_n)\| \cdot \|x-y_n\| ,$$

so $\left\langle d\phi(x), \|x-y_n\|^{-1}(x-y_n) \right\rangle \to 1$ as $n \to \infty$. By assumption *(i)*, this implies that $\|x-y_n\|^{-1}(x-y_n) \to z$; hence $y_n \to x - \phi(x)z$ as required.

(b) Suppose that (y_n) is a maximizing sequence for x. For each $t < 0$,

$$\psi(x+tz) - \psi(x) \geq \sup_n \|x+tz-y_n\| - \lim_{n\to\infty} \|x-y_n\|$$
$$\geq \limsup_{n\to\infty} \left(\|x+tz-y_n\|-\|x-y_n\|\right) .$$

From assumption *(i)* it is clear that

$$1 = \langle d\psi(x), z \rangle = \lim_{t\to 0} t^{-1}\left(\psi(x+tz)-\psi(x)\right) ,$$

so

$$1 \leq \liminf_{t\to 0-} \left[t^{-1} \limsup_{n\to\infty} \left(\|x+tz-y_n\|-\|x-y_n\|\right)\right]$$
$$= \liminf_{t\to 0-} \liminf_{n\to\infty} t^{-1}\left(\|x+tz-y_n\|-\|x-y_n\|\right) .$$

The rest of the proof is similar to part *(a)*.

COROLLARY 2.5. *Suppose that M is a closed subset of a Banach space E equipped with a norm which is Fréchet differentiable, is uniformly Gateaux differentiable and induces a Fréchet differentiable dual norm on E^*.*

(a) If $x \in E \sim M$ is a point of Gateaux differentiability of ϕ with $\|d\phi(x)\| = 1$ then each minimizing sequence for x converges and hence P is continuous at x.

(b) If M is bounded and x is a point of Gateaux differentiability of ψ with $\|d\psi(x)\| = 1$ then each maximizing sequence for x converges and hence F is continuous at x.

Proof. Our assumptions on the norm of E imply that every nonzero element of E (respectively E^*) strongly exposes $B(E^*)$ (respectively $B(E)$). Thus we can apply Theorem 2.4. The continuity of P (respectively F) follows immediately from the convergence of all minimizing (respectively maximizing) sequences.

If we strengthen the hypothesis on ϕ to Fréchet differentiability of ϕ at x, then we can obtain the convergence of minimizing sequences for x with weaker assumptions on the Banach space E.

THEOREM 2.6. *Suppose M is a closed subset of a Banach space E.*

(a) If $x \in E \sim M$ is a point of Fréchet differentiability of ϕ, then $\|\phi'(x)\| = 1$. If $\phi'(x)$ strongly exposes $B(E)$ at some point z, then every minimizing sequence for x converges to $x - \phi(x)z$, and the latter is the unique nearest point in M to x.

(b) If M is bounded and $x \in E$ is a point of Fréchet differentiability of ψ, then $\|\psi'(x)\| = 1$. If $\psi'(x)$ strongly exposes $B(E)$ at some point z, then every maximizing sequence for x converges to $x - \psi(x)z$, and the latter is the unique farthest point in M from x.

Proof. In order to prove *(a)* and *(b)* simultaneously, introduce a constant λ which is to be equal to 1 in part *(a)* and equal to -1 in part *(a)*. In part *(b)* let (y_n) be any minimizing sequence for x and let $\phi_1 = \phi$. In part *(b)*, let (y_n) be any maximizing sequence for x and let $\phi_{-1} = \psi$.

Choose a sequence (α_n) of positive numbers such that $\alpha_n \to 0$ and $\alpha_n^2 > \lambda\left(\|x-y_n\|-\phi_\lambda(x)\right)$ for every $n \in \mathbb{N}$. If $0 < t < 1$ then for each n,

$$\begin{aligned}\lambda\phi_\lambda\left(x+\lambda t\left(y_n-x\right)\right) &\le \lambda\|x+\lambda t\left(y_n-x\right)-y_n\| \quad \text{since } y_n \in M , \\ &= \lambda(1-\lambda t)\|x-y_n\| \\ &\le (1-\lambda t)\left\{\alpha_n^2+\lambda\phi_\lambda(x)\right\} ;\end{aligned}$$

hence

$$\lambda\phi_\lambda(x) - \lambda\phi_\lambda\left(x+\lambda t\left(y_n-x\right)\right) \ge t\phi_\lambda(x) - 2\alpha_n^2 . \tag{3}$$

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Now let $\varepsilon > 0$. By definition of $\phi'_\lambda(x)$, there is $\delta > 0$ such that whenever $\|y\| < \delta$ we have

$$|\phi_\lambda(x+y)-\phi_\lambda(x)-\langle \phi'_\lambda(x), y\rangle| \le \varepsilon\|y\| .$$

Let $t_n = \alpha_n\left(\|x-y_n\|\right)^{-1}$. For large n, $\alpha_n < \delta$, so taking $\lambda t_n\left(y_n-x\right)$ in place of y yields

$$\begin{aligned}\varepsilon t_n\|x-y_n\| - \lambda\langle \phi'_\lambda(x), \lambda t_n\left(y_n-x\right)\rangle &\ge \lambda\phi_\lambda(x) - \lambda\phi_\lambda\left(x+\lambda t_n\left(y_n-x\right)\right) \\ &\ge t_n\phi_\lambda(x) - 2\alpha_n^2\end{aligned}$$

by (3). Thus

$$\langle \phi'_\lambda(x), t_n\left(x-y_n\right)\rangle \ge -\varepsilon\alpha_n - 2\alpha_n^2 + t_n\phi_\lambda(x)$$

and dividing by α_n yields

$$\left\langle\phi'_\lambda(x), \left(\|x-y_n\|\right)^{-1}\left(x-y_n\right)\right\rangle \ge -\varepsilon - 2\alpha_n + \left(\|x-y_n\|\right)^{-1}\phi_\lambda(x) .$$

Since $\varepsilon > 0$ was arbitrary, $\alpha_n \to 0$ and $\|\phi'_\lambda(x)\| \le 1$ (by Lemma 2.1) we have

$$\begin{aligned}1 &\ge \liminf_{n\to\infty} \left\langle\phi'_\lambda(x), \left(\|x-y_n\|\right)^{-1}\left(x-y_n\right)\right\rangle \\ &\ge \liminf_{n\to\infty} \left(\|x-y_n\|\right)^{-1}\phi_\lambda(x) = 1 ;\end{aligned}$$

hence $\|\phi'_\lambda(x)\| = 1$ as required. Furthermore, if $\phi'_\lambda(x)$ strongly exposes $B(E)$ at z then $\left(\|x-y_n\|\right)^{-1}\left(x-y_n\right) \to z$ because

$$\left\langle\phi'_\lambda(x), \left(\|x-y_n\|\right)^{-1}\left(x-y_n\right)\right\rangle \to 1 = \|\phi'_\lambda(x)\| .$$

It follows that $y_n \to x - \phi_\lambda(x)z$.

COROLLARY 2.7. *Suppose M is a closed subset of a Banach space E such that the norm of E^* is Fréchet differentiable.*

(a) If ϕ is Fréchet differentiable at some $x \in E$, then every minimizing sequence for x converges, hence P is continuous at x.

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(b) If M *is bounded and* ψ *is Fréchet differentiable at some* $x \in E$ *, then every maximizing sequence for* x *converges, hence* F *is continuous at* x *.*

Proof. By the assumptions on E , every nonzero element of E^* strongly exposes $B(E)$. So Theorem 2.6 applies, except for the trivial case $x \in M$ in part *(a)*.

COROLLARY 2.8. *Suppose* M *is a closed bounded subset of a Banach space* E *such that the norm of* E^* *is Fréchet differentiable. Then the set of* x *in* E *which have every maximizing sequence for* x *converging to the unique farthest point in* M *for* x *is a residual subset of* E *.*

Proof. The function ψ is clearly convex and E is reflexive, so ψ is Fréchet differentiable at the points of some residual subset of E (see [*14*]). By Corollary 2.7 *(b)*, each of these points has the required property.

Corollary 2.8 generalizes a result of Asplund [*2*].

THEOREM 2.9. *Let* M *be a bounded subset of a Banach space* E *. Suppose that* $x \in E$ *and* $y \in F(x)$ *and that* $d\psi(x)$ *exists. Then the norm of* E *is Gateaux differentiable at* $x - y$ *, with derivative* $d\psi(x)$ *. If* $\psi'(x)$ *exists then the norm of* E *is Fréchet differentiable at* $x - y$ *.*

Proof. Suppose $\psi'(x)$ exists, and let $x^* \in D(x-y)$, so that for every $h \in E$,

$$\langle x^*, h\rangle \le \|x-y+h\| - \|x-y\| .$$

Since $\|x-y+h\| - \|x-y\| \le \psi(x+h) - \psi(x)$ we have that for every $\varepsilon > 0$ there is $\delta > 0$ such that $\|h\| < \delta$ implies

$$\langle x^*, h\rangle \le \langle \psi'(x), h\rangle + \varepsilon\|h\| ,$$

that is, whenever $\|z\| = 1$,

$$\langle x^*, z\rangle \le \langle \psi'(x), z\rangle + \varepsilon .$$

This being true for all $\varepsilon > 0$ we conclude that $x^* = \psi'(x)$. It follows that for each $\varepsilon > 0$ there is $\delta > 0$ such that $\|h\| < \delta$ implies

$$\langle x^*, h\rangle \le \|x-y+h\| - \|x-y\| \le \langle x^*, h\rangle + \varepsilon\|h\| ,$$

so $x^* = \psi'(x)$ is the Fréchet derivative of the norm at $x - y$. If only $d\psi(x)$ exists, a similar proof shows that $d\psi(x)$ is the Gateaux derivative of the norm at $x - y$.

Theorem 2.9 does not have an analogue for the function ϕ , as the following example shows.

EXAMPLE 2.10. Let $E = \mathbb{R}^2$ equipped with the norm $\|(x_1, x_2)\| = |x_1| + |x_2|$, and let M be the bounded set $\{(0, t) \in \mathbb{R}^2 : -1 \le t \le 1\}$. Then ϕ is (Fréchet) differentiable at each point $x = (x_1, x_2)$ such that $x_1 \ne 0$ and $-1 < x_2 < 1$: for such points, $\phi(x) = |x_1|$ and $P(x) = (0, x_2)$. However, the norm is not differentiable at $x - P(x) = (x_1, 0)$.

3. Sufficient conditions for differentiability of ϕ and ψ

It should not now be surprising that we need some continuity-like condition (such as "every maximizing sequence converges") in order to prove that ψ or ϕ is differentiable. Also, Theorem 2.9 shows that, at least for ψ , we need to assume the differentiability of the norm at $x - Fx$.

THEOREM 3.1. *Suppose* M *is a closed subset of a Banach space* E *and* $x \in E \sim M$.

(a) If every minimizing sequence in M *for* x *converges to* z *and the norm of* E *is Gateaux (respectively Fréchet) differentiable at* $x - z$ *then* ϕ *is Gateaux (respectively Fréchet) differentiable at* x .

(b) If $P(y)$ *is nonempty for a dense set of* y *in some neighborhood of* x *and if* P *is continuous at* x *, with the norm* E *Gateaux (respectively Fréchet) differentiable at* $x - Px$ *, then* ϕ *is Gateaux (Fréchet) differentiable at* x .

Abatzoglou [1] proved a result less general than Theorem 3.1 *(b)*: he assumed that P is continuous on an open set containing x .

THEOREM 3.2. *Suppose* M *is a closed bounded subset of a Banach space* E *and let* $x \in E$.

(a) If every maximizing sequence for x *converges to* z *and the norm of* E *is Gateaux (respectively Fréchet) differentiable at* $x - z$ *, then* ψ *is Gateaux (respectively Fréchet) differentiable at* x .

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(b) If $F(y)$ *is nonempty for a dense set of* y *in some neighborhood of* x *and* F *is continuous at* x *, with the norm of* E *being Gateaux (respectively Fréchet) differentiable at* $x - F(x)$ *, then* ψ *is Gateaux (respectively Fréchet) differentiable at* x *.*

To prove these theorems we will obtain a general result which contains both as special cases. Let $h : E \to \mathbb{R}$ be a Lipschitz function. If M is a subset of E define

$$\eta(x) = \inf\{h(x-m) : m \in M\} .$$

Recall the definition of the *Clarke subgradient* [7], ∂h of h : first let

$$h^0(x, y) = \limsup_{\substack{z \to x \\ t \to 0+}} \frac{h(z+ty)-h(z)}{t}$$

for $x, y \in E$, and then define

$$\partial h(x) = \{x^* \in E^* : \langle x^*, y \rangle \leq h^0(x, y) \text{ for all } y \in E\} .$$

Note that if $\partial h(x)$ is single valued then $dh(x)$ exists and $\partial h(x) = \{dh(x)\}$ (see [11]).

We need the following mean-value property for ∂h .

PROPOSITION 3.3 [11]. *If* x *and* y *are points of* E *then there is a point* z *of* $[x, y] = \{tx+(1-t)y : 0 \leq t \leq 1\}$ *and some* $z^* \in \partial h(z)$ *such that*

$$\langle z^*, y-x \rangle = h(y) - h(x) .$$

THEOREM 3.4. *Let* M, h *and* η *be as above. Suppose that* x *is a point of* E *where* η *is finite and that* z *is a point of* M *such that* $\eta(x) = h(x-z)$ *and* ∂h *is single-valued at* $x - z$ *. Further assume the following continuity-like condition: for every* y *in some neighborhood of zero in* E *we can assign an element* $m(y)$ *of* M *such that, as* $y \to 0$ *, both*

$$\|y\|^{-1}\{h\big(x+y-m(y)\big)-\eta(x+y)\} \to 0 \tag{4}$$

and

$$m(y) \to z . \tag{5}$$

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It then follows that η *is Gateaux differentiable at* x *and* $d\eta(x) = dh(x-z)$. *Moreover, if* ∂h *is continuous at* $x - z$ *then* η *is Fréchet differentiable at* x .

Proof. Let $y \in E$ and $t > 0$. Then define

$$(6) \qquad r(ty) = t^{-1}[\eta(x+ty)-\eta(x)-\langle dh(x-z),\ ty\rangle]$$
$$\leq t^{-1}[h(x+ty-z)-h(x-z)-\langle dh(x-z),\ ty\rangle]$$

since $z \in M$. Also if we define

$$\alpha(ty) = t^{-1}\left[\eta(x+ty)-h\left(x+ty-m(ty)\right)\right] ,$$

then $\alpha(ty) \to 0$ as $t \to 0$ by (4). Moreover,

$$(7) \quad r(ty) \geq t^{-1}\{h[x+ty-m(ty)]-h[x-m(ty)]\} + \alpha(ty) - \langle dh(x-z),\ y\rangle$$
$$= \langle x^*,\ y\rangle - \langle dh(x-z),\ y\rangle + \alpha(ty)$$

for some $x^* = x^*(ty) \in \partial h\left(w(ty)\right)$, where $w(ty) \in [x+ty-m(ty),\ x-m(ty)]$, by Proposition 3.3. As $t \to 0$ we have $m(ty) \to z$, by (5), so $w(ty)$ converges to $x - z$. Since ∂h is norm-weak* upper semicontinuous at $x - z$ (see [11]) we have $x^*(ty) \to dh(x-z)$ weak* as $t \to 0$. Thus $\langle x^*(ty)-dh(x-z),\ y\rangle \to 0$ as $t \to 0$, so (7) converges to zero as $t \to 0$. Also (6) converges to zero as $t \to 0$ since h is Gateaux differentiable at $x - z$. So we have $r(ty) \to 0$ as $t \to 0+$, for all $y \in E$, which implies that $d\eta(x)$ exists and is equal to $dh(x-z)$.

If ∂h is continuous at $x - z$, then all the assertions concerning convergence in this proof are valid uniformly for $y \in B(E)$, and η is Fréchet differentiable at x .

Proof of Theorem 3.1. It easily is seen that for h equal to the norm of E , the Clarke subgradient ∂h and the duality map D are identical, and that D is single-valued (respectively continuous) at a point x of E if and only if the norm is Gateaux (respectively Fréchet) differentiable at x . Hence, setting h equal to the norm, we only need to produce $m(y)$ satisfying the conditions (4) and (5) of Theorem 3.4.

(a) For each nonzero y in E take any $m(y) \in M$ such that

$$0 \leq \|x+y-m(y)\| - \phi(x+y) \leq \|y\|^2$$

and take $m(0) = z$; this choice of $m(y)$ clearly satisfies condition (4)

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of Theorem 3.4. Furthermore,

$$\begin{aligned}\phi(x) &\le \|x-m(y)\| \le \|x+y-m(y)\| + \|y\| \\ &\le \|y\|^2 + \phi(x+y) + \|y\| \quad \text{by choice of } m(y) , \\ &\le \|y\|^2 + \phi(x) + 2\|y\|\end{aligned}$$

By Lemma 2.1. This shows that if $y_n \to 0$ then $\|x-m(y_n)\| \to \phi(x)$, so $m(y_n) \to z$ since $(m(y_n))$ is a minimizing sequence for x . Thus condition (5) of Theorem 3.4 is also satisfied.

(b) Let U be a neighborhood of zero such that there is a dense subset A of $x + U$ on which nearest points exist. For each nonzero $y \in U$ take any $w(y) \in A$ such that

$$\|x+y-w(y)\| < \|y\|^2 ,$$

and take $w(0) = x$. Let $m(y)$ be any element of $P(w(y))$ for each $y \in U$. Then continuity of P at x implies that $m(y) \to P(x)$ as $y \to 0$. Also

$$\begin{aligned}0 &\le \|x+y-m(y)\| - \phi(x+y) \\ &\le \|x+y-w(y)\| + \|w(y)-m(y)\| - \phi(w(y)) + \|x+y-w(y)\| \\ &\qquad \text{by the triangle inequality and Lemma 2.1,} \\ &\le \|w(y)-m(y)\| - \phi(w(y)) + 2\|y\|^2 \quad \text{by choice of } w(y) , \\ &= 2\|y\|^2 \quad \text{since } m(y) \in P(w(y)) .\end{aligned}$$

Thus the conditions of Theorem 3.4 are satisfied.

Proof of Theorem 3.2. We note that

$$\psi(y) = -\inf\{-\|y-m\| : m \in M\}$$

and thus we can apply Theorem 3.4 to $h = -\|\cdot\|$ and $\eta = -\psi$. The details are similar to those of the Proof of Theorem 3.1.

COROLLARY 3.5. *Suppose that E is a Banach space such that the norms of E and E^* are both Fréchet differentiable.*

(a) If M is a closed subset of E and $x \in E \sim M$, then the following are equivalent:

(i) the metric projection is continuous at x ;

(ii) every minimizing sequence in M for x converges;

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(iii) the function ϕ *is Fréchet differentiable at* x .

(b) If M *is a closed bounded subset of* E *and* x *is a point of* E *, then the following are equivalent:*

(i) the metric antiprojection is continuous at x *;*

(ii) every maximizing sequence in M *for* x *converges;*

(iii) the function ψ *is Fréchet differentiable at* x .

Proof. *(a)* Lau [*10*] proved that $P(y)$ is nonempty for a dense set of y in E . Now, by Theorem 3.1 *(b)*, if P is continuous at x , then ϕ is Fréchet differentiable at x . Conversely, if ϕ is Fréchet differentiable at x , then Corollary 2.7 shows that every minimizing sequence for x converges, which in turn implies the continuity of P at x .

(b) By Corollary 2.8, there is a dense set of $y \in E$ such that $F(y)$ is nonempty. Now Theorem 3.2 *(b)* and Corollary 2.7 finish the proof, as in part *(a)*.

COROLLARY 3.6. *Suppose that* M *is a closed subset of a Banach space* E *such that the norm of* E *is both Fréchet differentiable and uniformly Gateaux differentiable and the norm of* E^* *is Fréchet differentiable.*

(a) The following are equivalent for a point x *of* $E \sim M$ *:*

(i) the function ϕ *is Fréchet differentiable at* x *;*

(ii) the function ϕ *is Gateaux differentiable at* x *and* $\|d\phi(x)\| = 1$ *;*

(iii) the metric projection onto M *is continuous at* x .

(b) If M *is bounded and* $x \in E$ *, the following are equivalent:*

(i) the function ψ *is Fréchet differentiable at* x *;*

(ii) the function ψ *is Gateaux differentiable at* x *and* $\|d\psi(x)\| = 1$ *;*

(iii) the metric antiprojection is continuous at x .

Proof. This is immediate from Theorem 2.6, Corollary 2.5 and Corollary 3.5.

Our interest in the differentiability of ϕ arose initially from an

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attempt to answer the following question. *Is every real valued locally Lipschitzian function on a separable reflexive Banach space necessarily Fréchet differentiable on a dense set?* This question appears to remain open. (The counterexample presented in [4] (and cited subsequently in [5], [15] and [22]) is in fact continuously Frechet differentiable, while the one presented in [13] is convex, hence differentiable on a dense G_δ set.) The next corollary shows that such a "generic" differentiability result is valid for ϕ in certain spaces.

COROLLARY 3.7. *Let M be a nonempty closed subset of a reflexive locally uniformly convex Banach space E. If E is smooth then ϕ is Gateaux differentiable except on a set of the first category, and if the norm of E is Fréchet differentiable then ϕ is Fréchet differentiable except on a set of the first category.*

Proof. Lau [10] has shown that there is a dense G_δ subset A of E such that, if $x \in A$, then every minimizing sequence in M for x converges. We can apply Theorem 3.1 *(a)* to get ϕ differentiable on $A \sim M$. However, ϕ is constant (zero) on the interior of M, so we have ϕ differentiable on $A \sim \text{boundary}(M)$, which is a dense G_δ set since the boundary of M is closed and nowhere dense.

We need not prove a corresponding result for ψ since ψ is convex and every reflexive space E is an *Asplund space* [14], that is, *every* continuous convex function on E is Fréchet differentiable on a dense G_δ subset of E.

4. Convexity of M when ϕ is differentiable

Suppose that M is a closed subset of a Banach space E. If ϕ is Fréchet differentiable (or Gateaux differentiable with norm-one derivative) at each point of $E \sim M$ then we give conditions on E which guarantee that M is convex.

A subset M of E is called a *Čebyšev set* if every point x of E has a unique nearest point in M, and the set M is *spproximatively compact* provided every minimizing sequence in M for each point x of E is relatively compact. We need the following results of Vlasov.

THEOREM 4.1 (Vlasov [21]). *(a) In a smooth locally uniformly convex Banach space* E *, every approximatively compact Čebyšev set is convex.*

(b) In a Banach space with uniformly Gateaux differentiable norm, every approximatively compact Čebyšev set is convex.

Our first result of this section is for ϕ Gateaux differentiable with norm-one derivative. However, if ϕ is Fréchet differentiable, then Theorem 2.6 shows that the derivative has norm equal to one, for each point of $E \sim M$.

THEOREM 4.2. *Suppose that* M *is a closed subset of a Banach space* E *equipped with a norm which is Fréchet differentiable, is uniformly Gateaux differentiable and induces a Fréchet differentiable norm on* E^* *. If* ϕ *is Gateaux differentiable at* x *and* $\|d\phi(x)\| = 1$ *, for all* $x \in E \sim M$ *, then* M *is convex.*

Proof. By Corollary 2.5, if x is a point of $E \sim M$ then x has a unique nearest point in M and every minimizing sequence for x converges, hence is relatively compact. If x is a point of M , then the same conclusions are obvious. Thus M is an approximatively compact Čebyšev set, and Theorem 4.1 *(b)* shows that M is convex.

THEOREM 4.3. *Let* M *be a closed subset of a smooth reflexive locally uniformly convex Banach space. If* ϕ *is Fréchet differentiable at each point of* $E \sim M$ *then* M *is convex.*

Proof. By Corollary 2.7, if $x \in E \sim M$ then x has a unique nearest point in M and every minimizing sequence for x converges, hence is relatively compact. So M is an approximatively compact Čebyšev set, and M is convex by Theorem 4.1 *(a)*.

The farthest distance function ψ for a closed nonempty bounded subset M of a reflexive space E can not be Gateaux differentiable with nonzero derivative at each point of E . (Since ψ is nonnegative-valued and convex, and E is reflexive, it is easily seen that ψ attains its minimum at some $z \in E$. But then $d\psi(z) = 0$.) Nor does taking ψ differentiable only for points not in M help.

EXAMPLE 4.4. There is, in any Hilbert space H , a bounded nonconvex subset M such that $\psi'(x)$ exists for all $x \in H \sim M$. In fact, we can take $M = S(H) \cup \{0\} = \{x \in H : \|x\| = 0 \text{ or } 1\}$. Then $\psi(x) = 1 + \|x\|$ for all $x \in H$, which is Fréchet differentiable at every nonzero $x \in H$, yet $0 \in M$.

5. More examples

EXAMPLE 5.1. Let E be the Hilbert space l_2 and M the closed subset $\left\{2e_1, 1\tfrac{1}{2}e_2, \ldots, \left(1+n^{-1}\right)e_n, \ldots\right\}$ where e_n is the nth coordinate unit vector. Then $0 \in E$ has no nearest point in M but ϕ is Gateaux differentiable at 0 with $d\phi(0) = 0$.

Proof. For $x \in E$ we have, as in Asplund [3],

$$\phi^2(x) = \inf\left\{\left\|x-\left(1+n^{-1}\right)e_n\right\|^2 : n = 1, 2, \ldots\right\}$$
$$= \|x\|^2 - f(x)$$

where $f(x) = \sup\left\{2\left(1+n^{-1}\right)\langle x, e_n\rangle-\left(1+n^{-1}\right)^2 : n = 1, 2, \ldots\right\}$, so f is continuous and convex. Also $f(x) = -1$ whenever

$$2\left(1+n^{-1}\right)\langle x, e_n\rangle < \left(1+n^{-1}\right)^2 - 1$$

for all n ; hence

(8) $$\phi^2(x) = \|x\|^2 + 1$$

provided

$$\langle x, e_n\rangle < (2n+1)\cdot\left(2n^2+2n\right)^{-1}$$

for all n . Since $\phi^2 = \|\cdot\|^2 - f$ is the difference of two convex functions, it is sufficient to check Gateaux differentiability of ϕ on a dense set of directions. Thus on the set

$$A = \left\{x \in E : \langle x, e_n\rangle < (2n+1)\cdot\left(2n^2+2n\right)^{-1}, n = 1, 2, \ldots\right\}$$

we have $d\phi(x) = \left(1+\|x\|^2\right)^{-\frac{1}{2}}x$ for all $x \in A$, by (8), since the derivative of $\|\cdot\|^2$ at x is equal to $2x$ for all $x \in l_2$.

It should also be noted that ϕ is not Fréchet differentiable at any

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point of A by Corollary 3.6, and that A is not a Gaussian null set in the sense of Phelps (see Lemma 3 of [16]); of course, since the closure of A has empty interior, it is of the first category in l_2 .

EXAMPLE 5.2. In $E = l_2$ let

$$M = \left\{e_1, 1\tfrac{1}{2}e_2, \ldots, \left(1+n^{-1}\right)e_n, \ldots\right\} .$$

Then $0 \in E$ has a unique nearest point in M (namely, e_1) but ϕ is *not* Gateaux differentiable at 0 , since Lemma 2.2 would then imply that $\|d\phi(0)\| = 1$ and Corollary 2.5 would show that every minimizing sequence for 0 converges to e_1 , which is not the case.

EXAMPLE 5.3. There exists a locally uniformly convex Asplund space E and a nonempty bounded closed convex subset M of E which admits no farthest points. This shows that Corollary 2.7 fails if the conditions on E are weakened to E being a locally uniformly convex Asplund space.

Proof. Cobzaş [8] defined an equivalent norm $|||\cdot|||$ on the Asplund space c_0 such that $E = \left(c_0, |||\cdot|||\right)$ is locally uniformly convex and if $M = \left\{x \in c_0 : \|x\|_\infty \le 1\right\}$ is the original unit ball of c_0 then no point of $E \sim M$ has a nearest point in M . Consequently, M admits no farthest points: if $x \in E$ and $y \in M$ is a farthest point from x , with $r = \|x-y\|$, say, then $r > 0$ and $M \subset x + rB(E)$. Let $z = 2y - x$. If $m \in M$, then $\|m-x\| \le r$ and therefore

$$\|z-x\| = \|2(y-x)-(m-x)\| \ge 2r - r = r = \|z-y\| ,$$

so y is a nearest point in M to z .

Nor does Corollary 2.7 work if E is only assumed to be reflexive and structly convex.

EXAMPLE 5.4. There exists a strictly convex reflexive Banach space E , a nonempty open subset U and a closed set M in E , such that ϕ is Fréchet differentiable throughout U but no point in U has a nearest point in M .

Proof. Edelstein [9] renormed $l_2 \oplus \mathbb{R}$ by taking

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$$|||(x, r)||| = \max(\|x\|, |r|) + \left(r^2 + \sum_n 2^{-2n} x_n^2\right)^{\frac{1}{2}}$$

for $(x, r) \in E$, and showed that **no** point in the open set

$$U = \{(u, r) : \|u\| < \tfrac{1}{2} \text{ and } |r| < \tfrac{1}{2}\}$$

has a nearest point in the set

$$M = \left\{\left(e_n, 2+n^{-1}\right) : n = 1, 2, \ldots\right\} .$$

However, for $(u, r) \in U$,

$$\phi(u, r) = 2 - r + \left((2-r)^2 + \sum_n 2^{-2n} u_n^2\right)^{\frac{1}{2}}$$

which is easily seen to be Fréchet differentiable on U.

References

[1] Theagenis Abatzoglou, "The metric projection on C^2 manifolds in Banach spaces", *J. Approx. Theory* 26 (1979), 204-211.

[2] Edgar Asplund, "Farthest points in reflexive locally uniformly rotund Banach spaces", *Israel J. Math.* 4 (1966), 213-216.

[3] Edgar Asplund, "Čebyšev sets in Hilbert spaces", *Trans. Amer. Math. Soc.* 144 (1969), 235-240.

[4] V.I. Averbukh and O.G. Smolyanov, "The theory of differentiation in linear topological spaces", *Russian Math. Surveys* 22 (1967), no. 6, 201-258.

[5] V.I. Averbukh and O.G. Smolyanov, "The various definitions of the derivative in linear topological spaces", *Russian Math. Surveys* 23 (1968), no. 4, 67-113.

[6] Jörg Blatter, "Weiteste Punkte und nächste Punkte", *Rev. Roumaine Math. Pures Appl.* 14 (1969), 615-621.

[7] Frank H. Clarke, "Generalized gradients and applications", *Trans. Amer. Math. Soc.* 205 (1975), 247-261.

[8] S. Cobzaş, "Antiproximinal sets in Banach spaces", *Math. Balkanica* **4** (1974), 79-82.

[9] Michael Edelstein, "Weakly proximinal sets", *J. Approx. Theory* **18** (1976), 1-8.

[10] Ka-Sing Lau, "Almost Chebyshev subsets in reflexive Banach spaces", *Indiana Univ. Math. J.* **27** (1978), 791-795.

[11] Gérard Lebourg, "Valeur moyenne pour gradient géneralisé", *C.R. Acad. Sci. Paris Ser. A* **281** (1975), 795-797.

[12] A.R. Lovaglia, "Locally uniformly convex Banach spaces", *Trans. Amer. Math. Soc.* **78** (1955), 225-238.

[13] F. Mignot, "Contrôle dans les inéquations variationelles elliptiques", *J. Funct. Anal.* **22** (1976), 130-185.

[14] I. Namioka and R.R. Phelps, "Banach spaces which are Asplund spaces", *Duke Math. J.* **42** (1975), 735-750.

[15] M.Z. Nashed, "Differentiability and related properties of nonlinear operators: some aspects of the role of differentials in non-linear functional analysis", *Nonlinear functional analysis and applications*, 103-309 (Proc. Adv. Seminar, University of Wisconsin, Madison, 1970. Mathematics Research Center Publication No. 26. Academic Press, New York, London, 1971).

[16] R.R. Phelps, "Gaussian null sets and differentiability of Lipzchitz map on Banach spaces", *Pacific J. Math.* **77** (1978), 523-531.

[17] V. Šmulian, "Sur la dérivabilité de la norme dans l'espace de Banach", *C.R. (Doklady) Acad. Sci. URSS (N.S.)* **27** (1940), 643-648.

[18] V.L. Šmulian, "Sur la structure de la sphère unitaire dans l'espace de Banach", *Rec. Math.* [*Mat. Sbornik*] *N.S.* **9** (51) (1941), 545-561.

[19] Francis Sullivan, "Geometrical properties determined by the higher duals of a Banach space", *Illinois J. Math.* **21** (1977), 315-331.

[20] С.Л. Троянски [S.L. Trojanski], "О равномерной выпуклости и гладкости в каждом направлении в несепарабельных пространствах Банахова с безусловным базисом" [Uniform convexity and smoothness in every direction in nonseparable Banach spaces with unconditional bases], *C.R. Acad. Bulgare Sci.* 30 (1977), no. 9, 1243-1246.

[21] L.P. Vlasov, "On Čebyšev sets", *Soviet Math. Dokl.* 8 (1967), 401-404.

[22] Eduardo H. Zarantonello, "Projections on convex sets in Hilbert space and spectral theory" [Part I: Projections on convex sets, 237-341; Part II: Spectral theory, 343-424], *Contributions to nonlinear functional analysis* (Proc. Sympos. University of Wisconsin, Madison, 1971. Mathematics Research Center Publication No. 27. Academic Press, New York, London, 1971).

[23] N.V. Zhivkov, "Metric projections and antiprojections in strictly convex normed spaces", *C.R. Acad. Bulgare Sci.* 31 (1978), no. 4, 369-372.

[24] Václav Zizler, "Banach spaces with the differentiable norms", *Comment. Math. Univ. Carolin.* 9 (1968), 415-440.

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