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Abstract

Desingularized fiber products of semi-stable elliptic surfaces with $H^3_{\text{etale}} = 0$ are classified. Such varieties may play a role in the study of supersingular threefolds, of the deformation theory of varieties with trivial canonical bundle, and of arithmetic degenerations of rigid Calabi–Yau threefolds.

Contents

1. Introduction

Let X be a smooth, projective, irreducible curve over an algebraically closed field k. Let $\pi : Y \to X$ and $\pi' : Y' \to X$ be relatively minimal elliptic surfaces with section. We assume that each is semi-stable and has at least one singular fiber. The fiber product, $\bar{W} = Y \times_X Y'$, is singular exactly at points $(y, y') \in \pi^{-1}(s)_{\text{sing}} \times (\pi')^{-1}(s)_{\text{sing}}$. Blowing up the reduced singular locus of $\bar{W}$ yields a non-singular projective threefold, W. This note is concerned with such

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threefolds W for which the third l-adic Betti number, $h^3(W, \mathbb{Q}_l)$, is zero ($l \neq \mathrm{char}(k)$). That such threefolds exist at all may come as a surprise. The fact that the Hodge number, $h^0(W, \Omega^3_{W/k})$, always turns out to be positive implies that they do not exist when $\mathrm{char}(k) = 0$. A first step towards understanding these objects is the following classification theorem.

Theorem 1.1. *Let W be constructed as above. Suppose that $h^3(W, \mathbb{Q}_l) = 0$. Then:*

(i) $\mathrm{char}(k) \in \{2, 3, 5, 7, 11, 17, 29, 31, 41, 73, 251, 919, 9001\}$;

(ii) *W may be defined over the prime subfield of k unless* $\mathrm{char}(k) = 2$, *in which case W may be defined over the field with four elements;*

(iii) *for each characteristic p in* (i) *and each pair $(n, n') \in (\mathbb{Z}_{\geq 0})^2$ there exists W as above with $h^0(W, \Omega^3_{W/k}) = p^n + p^{n'} - 1$; other values of $h^0(W, \Omega^3_{W/k})$ do not occur;*

(iv) *there are only finitely many isomorphism classes of varieties W as above with fixed Hodge number $h^0(W, \Omega^3_{W/k})$.*

The proof of the theorem begins with the formula for the third étale Betti number of a desingularized fiber product of semi-stable elliptic surfaces with section given in Proposition 3.1. The Betti number is zero precisely when X has genus zero, π and π' are not isogenous and have the same four places of bad reduction (cf. Corollary 3.2). The classification of semi-stable elliptic surfaces over $\mathbb{P}^1$ with four singular fibers is recalled in § 4. The task of arranging that the places of bad reduction coincide without forcing the elliptic fibrations to be isogenous is the focus of §§ 5 and 6. The desingularized fiber products are classified in § 7. Some Hodge numbers are computed in § 8 and Theorem 1.1 is proved in § 9. In § 10 resolutions of the singularities of $\bar{W}$ which do not introduce exceptional divisors are discussed. This leads to the construction of certain smooth projective threefolds, $\widehat{W}$, with trivial canonical sheaf and $H^3(\widehat{W}, \mathbb{Q}_l) = 0$. Deformations of these varieties are investigated in § 11. Other smooth projective threefolds with trivial canonical sheaf and $H^3(\widehat{W}, \mathbb{Q}_l) = 0$ were constructed by Hirokado [Hir99], Schröer [Sch04a], and Ekedahl [Eke04] using very different methods. Some of the differences and the questions that they raise are discussed in §§ 12 and 13.

2. Notation

We use the following notation in this paper:

- $\mathbb{N} = \mathbb{Z}_{\geq 1}$;
- k is an algebraically closed field;
- l is a prime number different from $\mathrm{char}(k)$;
- X is an irreducible curve, smooth and projective over k;
- g_X is the genus of X;
- $j : \eta = \mathrm{Spec}(k(X)) \to X$ the inclusion of the generic point of X;
- $\bar{\eta} = \mathrm{Spec}(\overline{k(X)})$, a geometric generic point which is algebraic over η;
- $G_\eta = \mathrm{Gal}(\bar{\eta}/\eta)$ the absolute Galois group of the field $k(X)$;
- semi-stable elliptic surfaces are assumed to be relatively minimal and to have a section;
- $\pi : Y \to X$, $\pi' : Y' \to X$ non-isotrivial semi-stable elliptic surfaces;
- $\bar{W} = Y \times_X Y'$;
- $W \xrightarrow{\sigma} \bar{W}$ is the blow-up of $\bar{W}$ along the reduced singular locus;

- $Q \subset W$ is the exceptional divisor for σ;
- $\bar{f} : \bar{W} \to X$ denotes the canonical projection. $f := \bar{f} \circ \sigma : W \to X$;
- S (respectively S') is the locus in X over which π (respectively π') fails to be smooth;
- m_s (respectively m'_s) number of irreducible components of $\pi^{-1}(s)$ (respectively $(\pi')^{-1}(s)$);
- two elliptic surfaces over a common base curve are *isogenous* if their generic fibers are;
- $\epsilon = \begin{cases} 1, & \text{if } Y \text{ and } Y' \text{ are isogenous} \\ 0, & \text{otherwise.} \end{cases}$

3. The third Betti number of desingularized fiber products

Keep the notation of the previous section.

Proposition 3.1. *The third Betti number of the desingularized fiber product is given by*

$$h^3(W, \mathbb{Q}_l) = 2\left(6g_X - 4 + |S \cap S'| + \epsilon + \sum_{s \in S - S \cap S'} m_s + \sum_{s \in S' - S \cap S'} m'_s\right).$$

Corollary 3.2. *We have $h^3(W, \mathbb{Q}_l) = 0$ exactly when $g_X = 0$, $\epsilon = 0$, $S = S'$ and $|S| = 4$.*

Proof. By the proposition, $h^3(W, \mathbb{Q}_l) = 0$ implies $g_X = 0$. According to [Bea81] the minimal number of singular fibers of a semi-stable elliptic surface over $\mathbb{P}^1$ is four. Thus, $4 \leq |S \cup S'| = |S \cap S'| + |S - S \cap S'| + |S' - S \cap S'|$ with equality only when $S = S'$ and $|S| = 4$. The assertion follows. $\square$

Proof of Proposition 3.1. The Betti number $h^3(W, \mathbb{Q}_l)$ is independent of the choice of prime $l \neq \mathrm{char}(k)$ (see [Mil80, Remark VI.12.5b]). The proof proceeds via the Leray spectral sequence for $f : W \to X$. Applying Poincaré duality [Mil80, Proposition V.2.2c] and ignoring Tate twists (since k is algebraically closed) gives

$$E_2^{2,1} \simeq H^2(X, R^1 f_* \mathbb{Q}_l) \simeq H^2(X, j_* j^* R^1 f_* \mathbb{Q}_l) \simeq H^0(X, j_* j^* R^1 f_* \mathbb{Q}_l^\vee) = 0,$$

since

$$H^1(f^{-1}(\bar{\eta}), \mathbb{Q}_l)^\vee \simeq H^1(\pi^{-1}(\bar{\eta}), \mathbb{Q}_l) \oplus H^1((\pi')^{-1}(\bar{\eta}), \mathbb{Q}_l)$$

has no non-zero G_η-invariants. Indeed, the existence of non-zero G_η-invariants would imply the existence of a dominant morphism from X to a tower of modular curves, $X_1(l^n)$, $n \to \infty$, of unbounded genus which is impossible.

Observe that $E_\infty^{1,2}$ is isomorphic to

$$E_2^{1,2} \simeq H^1(X, R^2 f_* \mathbb{Q}_l) \simeq H^1(X, j_* j^* R^2 f_* \mathbb{Q}_l),$$

where the second isomorphism is a consequence of the local invariant cycle theorem [Del80, 3.6.1]. Set $\mathfrak{R} := j_* j^* (R^1 \pi_* \mathbb{Q}_l \otimes R^1 \pi'_* \mathbb{Q}_l)$ and note that $j_* j^* (R^2 \pi_* \mathbb{Q}_l \otimes \pi'_* \mathbb{Q}_l) \simeq \mathbb{Q}_l(-1)$. Thus,

$$j_* j^* R^2 f_* \mathbb{Q}_l \simeq \mathbb{Q}_l(-1) \oplus \mathbb{Q}_l(-1) \oplus \mathfrak{R}. \tag{3.1}$$

Since $\mathfrak{R}$ is self-dual up to a Tate twist, Poincaré duality [Mil80, Proposition V.2.2c] yields,

$$h^1(X, \mathfrak{R}) = -e(X, \mathfrak{R}) + 2h^0(X, \mathfrak{R}).$$

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The isogeny theorem gives $h^0(X,\mathfrak{R})=\epsilon$ (see [FSW92, IV.1, Remark (ii)]). There is a formula for the euler characteristic,

$$e(X,\mathfrak{R})=e(X,\mathbb{Q}_l)\dim_{\mathbb{Q}_l}(\mathfrak{R}_{\bar\eta})-\sum_{s\in(S\cup S')}[\dim_{\mathbb{Q}_l}(\mathfrak{R}_{\bar\eta})-\dim_{\mathbb{Q}_l}(\mathfrak{R}_{\bar\eta}^{I_s})],\tag{3.2}$$

which will be verified in Lemma 3.3 below for certain primes l. Now $\dim_{\mathbb{Q}_l}(\mathfrak{R}_{\bar\eta})=4$. It follows easily from the known action of the inertia group at a fiber of multiplicative reduction [Sil94, V.Ex 5.13] that $\dim_{\mathbb{Q}_l}((\mathfrak{R}_{\bar\eta})^{I_s})=2$ for any $s\in S\cup S'$. Thus, (3.2) gives

$$h^1(X,\mathfrak{R})=-e(X,\mathfrak{R})+2h^0(X,\mathfrak{R})=-(4(2-2g_X)-2|S\cup S'|)+2\epsilon.$$

Adding in the two $h^1(X,\mathbb{Q}_l(-1))$ terms from (3.1) yields

$$h^1(X,j_*j^*R^2f_*\mathbb{Q}_l)=4g_X-(4(2-2g_X)-2|S\cup S'|)+2\epsilon.\tag{3.3}$$

Finally, consider the term $E_2^{0,3}=H^0(X,R^3f_*\mathbb{Q}_l)$. By the local invariant cycle theorem, the map on stalks,

$$(R^3f_*\mathbb{Q}_l)_s\to(j_*j^*R^3f_*\mathbb{Q}_l)_s\simeq(j_*j^*(R^1\pi_*\mathbb{Q}_l\otimes R^2\pi'_*\mathbb{Q}_l\oplus R^2\pi_*\mathbb{Q}_l\otimes R^1\pi'_*\mathbb{Q}_l))_s,\tag{3.4}$$

is surjective for any closed point $s\in X$. For $s\in S\cap S'$ $h^3(f^{-1}(s),\mathbb{Q}_l)=2$ (see [Sch02b, § 12]) and the target has dimension two, since $\dim_{\mathbb{Q}_l}(R^1\pi_*\mathbb{Q}_l)_{\bar\eta}^{I_s}=1$ at a place of multiplicative reduction and the same holds for π'. The proper base change theorem implies that (3.4) is an isomorphism. As $H^0(X,j_*j^*R^3f_*\mathbb{Q}_l)=0$, the global sections of $R^3f_*\mathbb{Q}_l$ are all supported on $(S\cup S')-(S\cap S')$. Set $U=X-(S\cap S')$ and observe that

$$R^3f_*\mathbb{Q}_l|_U\simeq R^3\bar f_*\mathbb{Q}_l|_U\simeq(R^1\pi_*\mathbb{Q}_l\otimes R^2\pi'_*\mathbb{Q}_l\oplus R^2\pi_*\mathbb{Q}_l\otimes R^1\pi'_*\mathbb{Q}_l)|_U$$

decomposes as

$$\left(R^1\pi_*\mathbb{Q}_l\otimes\left(\mathbb{Q}_l(-1)\oplus\sum_{s\in S'\cap U}i_{s*}\mathbb{Q}_l^{m'_s-1}\right)\right)\Bigg|_U\oplus\left(\left(\mathbb{Q}_l(-1)\oplus\sum_{s\in S\cap U}i_{s*}\mathbb{Q}_l^{m_s-1}\right)\otimes R^1\pi'_*\mathbb{Q}_l\right)\Bigg|_U.$$

Thus,

$$h^0(X,R^3f_*\mathbb{Q}_l)=\sum_{s\in S-(S\cap S')}2(m_s-1)+\sum_{s\in S'-(S\cap S')}2(m'_s-1).\tag{3.5}$$

Furthermore the differential $d_2^{0,3}:E_2^{0,3}\to E_2^{2,2}$ is zero. This may be seen by noting that f is defined over some finitely generated $\mathbb{Z}$-algebra, specializing to a suitable place with finite residue field and applying a weight argument [Kat05, 7.5.2].

Thus, $h^3(W,\mathbb{Q}_l)=h^1(X,j_*j^*R^2f_*\mathbb{Q}_l)+h^0(X,R^3f_*\mathbb{Q}_l)$ and the proposition follows from (3.3) and (3.5). □

It remains to justify (3.2).

Lemma 3.3. *For primes $l\neq\mathrm{char}(k)$ and $l\nmid 2\prod_{s\in S}m_s\prod_{s\in S'}m'_s$,*

$$e(X,\mathfrak{R})=e(X,\mathbb{Q}_l)\dim_{\mathbb{Q}_l}(\mathfrak{R}_{\bar\eta})-\sum_{s\in(S\cup S')}[\dim_{\mathbb{Q}_l}(\mathfrak{R}_{\bar\eta})-\dim_{\mathbb{Q}_l}(\mathfrak{R}_{\bar\eta}^{I_s})].$$

Proof. For an abelian group A, let $A[l]$ denote the kernel of multiplication by l. Define $\mathfrak{R}_n:=j_*j^*(R^1\pi_*\mathbb{Z}/l^n\otimes R^1\pi'_*\mathbb{Z}/l^n)$ and set $\varpi^i:=\dim_{\mathbb{Z}/l}((\varprojlim_n H^i(X,\mathfrak{R}_n))[l])$. Since π and π'

are semi-stable, the action of any inertia group on the stalk $\mathfrak{R}_{1,\bar{\eta}}$ factors through the tame quotient [Sil94, Theorem IV.10.2b] and [Mil80, Theorem V.2.12] gives the Euler characteristic formula,

$$e(X, \mathfrak{R}_1) = e(X, \mathbb{Z}/l) \dim_{\mathbb{Z}/l}(\mathfrak{R}_{1\ \bar{\eta}}) - \sum_{s \in (S \cup S')} [\dim_{\mathbb{Z}/l}(\mathfrak{R}_{1\ \bar{\eta}}) - \dim_{\mathbb{Z}/l}(\mathfrak{R}^{I_s}_{1\ \bar{\eta}})]. \tag{3.6}$$

The hypothesis on l implies that the right-hand sides of (3.6) and (3.2) are equal and that each $\mathfrak{R}_n$ is a flat sheaf of $\mathbb{Z}/l^n$-modules. From the exact sequence [Mil80, Proof of V.1.11],

$$\varprojlim_n H^i(X, \mathfrak{R}_n) \xrightarrow{l} \varprojlim_n H^i(X, \mathfrak{R}_n) \longrightarrow H^i(X, \mathfrak{R}_1)$$
$$\longrightarrow \varprojlim_n H^{i+1}(X, \mathfrak{R}_n) \xrightarrow{l} \varprojlim_n H^{i+1}(X, \mathfrak{R}_n)$$

one deduces a short exact sequence,

$$0 \longrightarrow (\mathbb{Z}/l)^{h^i(X,\mathfrak{R})+\varpi^i} \longrightarrow H^i(X, \mathfrak{R}_1) \longrightarrow (\mathbb{Z}/l)^{\varpi^{i+1}} \longrightarrow 0$$

for each i. The equality $e(X, \mathfrak{R}_1) = e(X, \mathfrak{R})$ and with it the lemma thus follow. $\square$

4. Semi-stable elliptic surfaces over $\mathbb{P}^1$ with four singular fibers

Semi-stable elliptic surfaces over $\mathbb{P}^1_k$ with four singular fibers were classified by Beauville [Bea82] when $\mathrm{char}(k) = 0$. They are all rational surfaces coming from pencils of cubic curves on $\mathbb{P}^2$. Table 1 contains Beauville's description of the pencil, the set $S \subset \mathbb{P}^1$ where the pencil has bad reduction and the Kodaira type of the singular fiber above each point of S. Each fibration is specified by an integer which we call the *level* and denote by L in the table. An identity section defined over $\mathbb{Z}[1/L]$ is specified in the third column by giving a base point for the pencil. This is a simple base point when $L \in \{3, 5, 6\}$. When $L \in \{4, 8\}$ the section comes from the unique infinitely near base point lying over the given base point and when $L = 9$ it comes from the second-order infinitely near base point. The next to last column gives the cross ratio orbit (c.r.o.) of S, that is the set of all $\lambda \in k$ such that there is an automorphism of $\mathbb{P}^1_k$ which takes the set S to $\{0, 1, \infty, \lambda\}$. If λ is one such element, then the others are $1/\lambda$, $1 - \lambda$, $1/(1 - \lambda)$, $\lambda/(\lambda - 1)$, $(\lambda - 1)/\lambda$. The final column gives the J-invariants of the elliptic fibrations (cf. [Sil86, III.1]). These may be computed starting from the Weierstrass equations in [MP86, Table 5.3].

In the fifth column μ_3 denotes the set of all third roots of one and $\Theta = \{\theta, \theta'\}$ indicates the roots of the polynomial $t^2 - 11t - 1$. In the sixth column ζ_6 is a primitive cube root of -1, $\mathfrak{C} = \{-8, -\frac{1}{8}, 9, \frac{1}{9}, \frac{9}{8}, \frac{8}{9}\}$ and $\mathfrak{K} = \{\kappa, 1/\kappa, 1 - \kappa, 1/(1 - \kappa), (\kappa - 1)/\kappa, \kappa/(\kappa - 1)\}$ where $\kappa = \theta/\theta'$ ($\kappa = (11 + 5\sqrt{5})/(11 - 5\sqrt{5})$ if $\mathrm{char}(k) \neq 2$).

PROPOSITION 4.1. *Table 1 gives a classification up to isomorphism of rational, semi-stable elliptic surfaces with four singular fibers in any characteristic with the one caveat that no surface of level L exists in characteristic p when $p|L$.*

Proof. [Lan91] Note that rational elliptic surfaces $\pi'' : Y'' \to \mathbb{P}^1$ and $\pi''' : Y''' \to \mathbb{P}^1$ are considered isomorphic if there exist isomorphisms, $\alpha : Y'' \to Y'''$ and $\beta : \mathbb{P}^1 \to \mathbb{P}^1$, satisfying $\beta \circ \pi'' = \pi''' \circ \alpha$. $\square$

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Table 1. Semi-stable rational elliptic surfaces with four singular fibers.

L	Equation	Section	Fibers	S	c.r.o.	J
3	$x^3+y^3+z^3+txyz$	$(1:-1:0)$	$\mathrm{I}_3\mathrm{I}_3\mathrm{I}_3\mathrm{I}_3$	$-3\mu_3,\infty$	ζ_6,ζ_6^{-1}	$\dfrac{(t^4-6^3t)^3}{(t^3+27)^3}$
4	$x(x^2+y^2+2zy)+tz(x^2-y^2)$	$(0:0:1)$	$\mathrm{I}_2\mathrm{I}_2\mathrm{I}_4\mathrm{I}_4$	$1,-1,0,\infty$	$-1,2,\dfrac{1}{2}$	$\dfrac{-2^8(t^4-t^2+1)^3}{t^4(t+1)^2(t-1)^2}$
5	$x(x-z)(y-z)+tzy(x-y)$	$(1:0:1)$	$\mathrm{I}_1\mathrm{I}_1\mathrm{I}_5\mathrm{I}_5$	$\Theta,0,\infty$	$\mathfrak{K}$	$\dfrac{(t^4-12t^3+14t^2+12t+1)^3}{t^5(t^2-11t-1)}$
6	$(x+y)(y+z)(z+x)+txyz$	$(1:-1:0)$	$\mathrm{I}_1\mathrm{I}_2\mathrm{I}_3\mathrm{I}_6$	$-8,1,0,\infty$	$\mathfrak{C}$	$\dfrac{-(t^4+8t^3-16t+16)^3}{t^3(t-1)^2(t+8)}$
8	$(x+y)(xy-z^2)+txyz$	$(0:0:1)$	$\mathrm{I}_1\mathrm{I}_1\mathrm{I}_2\mathrm{I}_8$	$-1,1,0,\infty$	$-1,2,\dfrac{1}{2}$	$\dfrac{2^4(16t^4-16t^2+1)^3}{t^2(t+1)(t-1)}$
9	$x^2y+y^2z+z^2x+txyz$	$(0:0:1)$	$\mathrm{I}_1\mathrm{I}_1\mathrm{I}_1\mathrm{I}_9$	$-3\mu_3,\infty$	ζ_6,ζ_6^{-1}	$\dfrac{t^3(t^3+24)^3}{t^3+27}$

Non-rational semi-stable elliptic surfaces over $\mathbb{P}^1_k$ with four singular fibers may be constructed in characteristic $p>0$ by Frobenius base change. We recall the construction: each entry in Table 1 with $p\nmid L$ determines an elliptic surface over the prime field, $\pi_0: Y_0 \to \mathbb{P}^1_{\mathbb{F}_p}$. Let $F_0: \mathbb{P}^1_{\mathbb{F}_p} \to \mathbb{P}^1_{\mathbb{F}_p}$ be the absolute Frobenius morphism, i.e. the identity on the topological space and the pth power map on the structure sheaf. For each $n\in\mathbb{Z}_{\geq 0}$ there is a commutative diagram,

$$\begin{tikzcd}
Y_{0,n} \arrow[r, "\varrho_{0,n}"] \arrow[d, "\pi_{n,0}"'] & \bar{Y}_{0,n} \arrow[r] \arrow[d, "\bar{\pi}_{n,0}"'] & Y_0 \arrow[d, "\pi_0"'] \\
\mathbb{P}^1_{\mathbb{F}_p} \arrow[r, equal] & \mathbb{P}^1_{\mathbb{F}_p} \arrow[r, "F_0^n"] & \mathbb{P}^1_{\mathbb{F}_p}
\end{tikzcd}$$

in which the right-hand square is Cartesian and $\varrho_{0,n}$ is a minimal resolution of singularities. Base change the diagram by $\mathrm{Spec}(k)\to\mathrm{Spec}(\mathbb{F}_p)$ and write $X=\mathbb{P}^1_k$.

$$\begin{tikzcd}
Y_n \arrow[r, "\varrho_n"] \arrow[d, "\pi_n"'] & \bar{Y}_n \arrow[r] \arrow[d, "\bar{\pi}_n"'] & Y \arrow[d, "\pi"'] \\
X \arrow[r, equal] & X \arrow[r, "F^n"] & X
\end{tikzcd} \tag{4.1}$$

Lemma 4.2. (i) *$\pi_n: Y_n \to X$ is an elliptic surface with four singular fibers;*

(ii) *the sets of critical values of π_n and π coincide;*

(iii) *to each singular fiber of π of type I_r there is a corresponding singular fiber of π_n of type $I_{p^n r}$;*

(iv) *the elliptic fibrations π_n and π are isogenous.*

Proof. (i) Let $S\subset X$ be the complement of the locus over which π is smooth. Then $|S|=4$ and π_n is smooth over $X-(F^n)^{-1}(S)$.

(ii) Since S is defined over the prime field, $S=(F^n)^{-1}(S)$.

(iii) The corresponding fiber of $\bar{\pi}_n$ is an r-gon of projective lines. A local calculation gives the result that the surface $\bar{Y}_n$ has an A_{p^n-1} singularity at each singular point of the r-gon. Blowing these up gives the result.

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(iv) The nth power of the absolute Frobenius map, $F^n_{Y_0}: Y_0 \to Y_0$, factors through the fiber product, $\bar{Y}_{0,n}$. The identity section maps to the identity section so the map on generic fibers is an isogeny. Now base changing gives a morphism, $Y \to \bar{Y}_n$, which is an isogeny on generic fibers. □

DEFINITION 4.3. Fix an algebraically closed field k. The elliptic fibrations which appear in Table 1 will be referred to as *BLIS fibrations of exponent zero*. In the event that $\mathrm{char}(k) > 0$, the elliptic fibration, π_n, constructed in (4.1) from the BLIS fibration of exponent zero, π, will be called a *BLIS fibration of exponent* n.

PROPOSITION 4.4. *Every semi-stable elliptic fibration over* $\mathbb{P}^1_k$ *with exactly four singular fibers is isomorphic to a unique BLIS fibration.*

Proof. Semi-stable elliptic surfaces over $\mathbb{P}^1_k$ with four singular fibers are *extremal* in the sense that the Picard number equals the second Betti number and the Mordell–Weil group is finite [Bea81, § 4.A]. Extremal elliptic surfaces were classified by Ito [Ito98, Ito02] and Schweizer [Sch00]. The assertion follows from this classification. □

Remark 4.5. BLIS is an abbreviation for Beauville–Lang–Ito–Schweizer.

Remark 4.6. The elliptic surfaces of levels 3 and 9 in Table 1 are isogenous. The same holds for the surfaces of levels 4 and 8.

5. Automorphisms of $\mathbb{P}^1$ which stabilize a set of four points

It is important to study automorphisms of $\mathbb{P}^1$ which stabilize a given set of four points for the following reason: suppose that $\pi : Y \to X = \mathbb{P}^1_k$ is a BLIS fibration, $\alpha \in \mathrm{Aut}(X)$ stabilizes the bad reduction locus S and the pullback π' of π by α is not isogenous to π. Then the desingularized fiber product of π and π' has $h^3 = 0$ by Corollary 3.2.

Given a finite subset of closed points, $T \subset \mathbb{P}^1_k$, $\mathrm{Aut}(T)$ will denote the set of permutations of the set while $\mathrm{Aut}(\mathbb{P}^1_k, T)$ denotes the subgroup of $\mathrm{Aut}_k(\mathbb{P}^1_k)$ which stabilizes T. If $|T| \geq 3$, then $\mathrm{Aut}(\mathbb{P}^1_k, T)$ is canonically identified with a subgroup of $\mathrm{Aut}(T)$. Assume henceforth that $|T| = 4$. Write $V_4 \subset A_4 \subset \mathfrak{S}_4$ for Klein's fourgroup, the alternating group and the symmetric group, respectively. Let D_4 denote the dihedral group with eight elements. When $\mathrm{char}(k) \neq 3$ write $\zeta_6 \in k^*$ for a primitive cube root of -1.

LEMMA 5.1. (i) *If* $\mathrm{char}(k) \neq 2$ *and the cross ratio orbit of* T *is* $\{-1, 2, 1/2\}$, *then* $\mathrm{Aut}(\mathbb{P}^1_k, T) \simeq D_4$, *except when* $\mathrm{char}(k) = 3$, *in which case* $\mathrm{Aut}(\mathbb{P}^1_k, T) \simeq \mathfrak{S}_4$.

(ii) *If* $\mathrm{char}(k) \neq 3$ *and the cross ratio orbit of* T *is* $\{\zeta_6, \zeta_6^{-1}\}$, *then* $\mathrm{Aut}(\mathbb{P}^1_k, T) \simeq A_4$.

(iii) *If the cross ratio orbit of* T *is different from cases (i) and (ii), then it consists of six distinct points and* $\mathrm{Aut}(\mathbb{P}^1_k, T) \simeq V_4$.

Proof. For $\lambda \in k^* - \{1\}$, the subgroup, $\mathfrak{V} \subset \mathrm{Aut}_k(\mathbb{P}^1_k)$, generated by $t \mapsto \lambda/t$ and $t \mapsto (t-\lambda)/(t-1)$ is isomorphic to V_4 and stabilizes $\{\infty, 0, 1, \lambda\}$. Any element of $\mathrm{Aut}_k(\mathbb{P}^1_k)$ which maps $\{\infty, 0, 1, \lambda\}$ to a set containing $\{\infty, 0, 1\}$ has the form $\mathfrak{s} \circ \mathfrak{v}$ where $\mathfrak{v} \in \mathfrak{V}$ and $\mathfrak{s} \in \mathrm{Aut}(\mathbb{P}^1_k, \{\infty, 0, 1\}) \simeq \mathfrak{S}_3$. If only the identity in $\mathrm{Aut}(\mathbb{P}^1_k, \{\infty, 0, 1\})$ fixes λ, then the cross ratio orbit consists of six distinct points and $\mathrm{Aut}(\mathbb{P}^1_k, \{\infty, 0, 1, \lambda\}) \simeq V_4$. The conjugacy class in $\mathrm{Aut}(\mathbb{P}^1_k, \{\infty, 0, 1\})$ of points of order two consists of the fractional linear transformations $t \mapsto 1/t$ (respectively $t/(t-1)$ and $1-t$) which fix the pairs of points $-1, 1$ (respectively $2, 0$ and

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Table 2. Automorphism groups.

τ	$\mathrm{Aut}(\tau,0)$	Exceptions
2	D_4	$(\mathfrak{S}_4;3)$
3	A_4	none
5	V_4	$(A_4;2,31),(D_4;11,251)$
6	V_4	$(D_4;5,7,17),(A_4;73)$

$1/2,\infty)$. Case (i) when $\mathrm{char}(k)\neq 3$ follows. The conjugacy class of elements of order three consists of the transformations, $t\mapsto 1/(1-t)$ (respectively $(t-1)/t$). These automorphisms fix only ζ_6 and ζ_6^{-1} and result (ii) follows. Finally when $\mathrm{char}(k)=3$, $GL(\mathbb{F}_3)/\pm Id\simeq\mathfrak{S}_4$ acts faithfully on $\{\infty,0,1,-1\}$. □

Definition 5.2. The *type* of a semi-stable elliptic surface over $X=\mathbb{P}^1_k$ with four singular fibers is the product of the distinct primes which divide the level of the corresponding BLIS fibration of exponent zero.

Remark 5.3. (i) The type is an element of the set $\{2,3,5,6\}$.

(ii) There are no elliptic surfaces of type τ in characteristic p if $p|\tau$.

(iii) Two BLIS fibrations of the same type over algebraically closed fields of the same characteristic $p\geq 0$ have the same set $S\subset\mathbb{P}^1(\mathbb{F}_{p^2})$ of bad reduction.

Definition 5.4. Write $\mathrm{Aut}(\tau,p)$ for $\mathrm{Aut}(X,S)$, where $\pi:Y\to X=\mathbb{P}^1_k$ is any BLIS fibration of type τ over an algebraically closed field k of characteristic p.

The canonical specialization map $\phi:\mathrm{Aut}(\tau,0)\to\mathrm{Aut}(\tau,p)$ is defined and injective for all primes $p\nmid\tau$ since S in Table 1 is finite and étale over $\mathrm{Spec}(\mathbb{Z}[1/\tau])$. In fact, ϕ is an isomorphism for all except finitely many characteristics p. The exceptional characteristics are indicated in Table 2. The notation $(G;p_1,\ldots,p_r)$ means that $\mathrm{Aut}(\tau,p)\simeq G$, when $p\in\{p_1,\ldots,p_r\}$.

For each exceptional value of (τ,p) in Table 2, Table 3 gives automorphisms, α, which represent the non-trivial left cosets of the image of $\mathrm{Aut}(\tau,0)\to\mathrm{Aut}(\tau,p)$ in $\mathrm{Aut}(\tau,p)$. The notation ζ_3 refers to a primitive cube root of one; S is as in Table 1.

Justification of Table 2. The justification is computational, but easy. We consider briefly the case $\tau=5$ to give the flavor. From Table 1, $S=\{\infty,0,\theta,\theta'\}$, where θ and θ' are the roots of $t^2-11t-1$. Now $\mathrm{Aut}(X,S)$ is strictly larger than V_4 precisely when $\lambda=\theta/\theta'$ lies in one of the two exceptional cross ratio orbits, $\{-1,2,1/2\}$ or $\{\zeta_6,\zeta_6^{-1}\}$. Suppose $\lambda=\zeta_6$. Then $\theta^3=-(\theta')^3$ which implies $0=11\theta^2+\theta+11(\theta')^2+\theta'$. As $\theta+\theta'=11$ and $\theta\theta'=-1$, the previous equation may be rewritten as $0=11(11^2+2)+11=2^2\cdot 11\cdot 31$. Characteristic 11 is ruled out, since then $\theta'=-\theta$ so $\lambda=-1$. Characteristics 2 and 31 really are exceptional as the explicit automorphisms α in Table 3 show. The case $\lambda\in\{-1,2,1/2\}$ is left to the reader.

Remark 5.5. A modular interpretation of the exceptional automorphisms is not apparent. At least in characteristics two, three and five these automorphisms do not stabilize the supersingular loci ($t=1$, $t=\pm\sqrt{-1}$ and $t\in\{3,4,\pm\sqrt{2}\}$, respectively).

Table 3. Exceptional automorphisms.

Characteristic	τ	S	α
2	5	∞, μ_3	$t \mapsto \zeta_3 t$; $t \mapsto \zeta_3^2 t$
3	2	$\infty, 0, 1, -1$	$t \mapsto 1/(1-t)$; $t \mapsto (t-1)/t$
5	6	$\infty, 0, 1, -8$	$t \mapsto t/(t-1)$
7	6	$\infty, 0, 1, -8$	$t \mapsto 1/t$
11	5	$\infty, 0, 1, -1$	$t \mapsto 1/t$
17	6	$\infty, 0, 1, -8$	$t \mapsto 1-t$
31	5	$\infty, 0, 5, 6$	$t \mapsto (5t-25)/t$; $t \mapsto 25/(5-t)$
73	6	$\infty, 0, 1, -8$	$t \mapsto 1/(1-t)$; $t \mapsto (t-1)/t$
251	5	$\infty, 0, 171, 91$	$t \mapsto 91-t$

6. Existence and non-existence of isogenies

Let $\pi : Y \to X = \mathbb{P}^1_k$ be a BLIS fibration of type τ. Set $p = \mathrm{char}(k)$. An element $\alpha \in \mathrm{Aut}(\tau, p)$ is said to be *generic* if it is in the image of the specialization map, $\phi : \mathrm{Aut}(\tau, 0) \to \mathrm{Aut}(\tau, p)$, otherwise it is said to be *exceptional*. Write $\pi' : Y' \to X$ for the base change of π with respect to the automorphism, $\alpha : X \to X$.

Proposition 6.1. *With notation as above we have the following results.*

(i) *If α is generic, then Y' is isogenous to Y.*

(ii) *If α is exceptional, then Y' is not isogenous to Y.*

Proof. (i) By Lemma 4.2(iv) it suffices to verify the assertion when π is one of the fibrations in Table 1. By Remark 4.6 it suffices to treat the cases of levels 3, 4, 5 and 6. Define $\dot{X} = X - S$. Adding a $\dot{}$ to the notation indicates base change by the inclusion, $\dot{X} \subset X$. For each entry in Table 1 $\dot{\pi} : \dot{Y} \to \dot{X}$ is the universal family of elliptic curves with a particular level structure. In the following we use standard notation from the theory of moduli of elliptic curves [Sil86, Appendix C, § 13].

Level 3. In this case $\dot{\pi} : \dot{Y} \to \dot{X}$ is the universal family of elliptic curves with a symplectic level three structure. $\mathrm{SL}(2, \mathbb{Z}/3)$ permutes the symplectic level 3 structures transitively. This action on the functor gives an action on the universal family, $\dot{\pi}$. The action on the modular curve, $\dot{X}$, is via $\mathrm{SL}(2, \mathbb{Z}/3)/\pm \mathit{Id} \simeq A_4$. Thus in level 3 the base-changed fibration, π', is actually isomorphic to π.

Level 4. The relevant level structure is a point of order four plus a point of order two which is not a multiple of the point of order four. The automorphism group of $\mathbb{Z}/2 \times \mathbb{Z}/4$ is isomorphic to a dihedral group, D_4. This group acts on the modular curve, $\dot{X}$, through the quotient by its center, $D_4/\pm \mathit{Id} \simeq \mathrm{Gal}(\dot{X}/X_1(4)) \times \mathrm{Gal}(\dot{X}/X(2))$. The generator of the first factor fixes the places of type I_2 reduction and interchanges the places of type I_4 reduction. The generator of the second factor fixes the places of type I_4 reduction, but interchanges the places of type I_2 reduction. Taking the quotient of Y by the subgroup generated by $2s$, where s is any section of order four, gives an isogeny from Y to the pullback of Y via an involution which takes places of type I_4 reduction to places of type I_2 reduction.

Level 5. The relevant level structure is a point of order five. The operation of $(\mathbb{Z}/5)^*$ permutes the possible level structures and acts on the modular curve $\dot{X}$ via $(\mathbb{Z}/5)^*/\pm Id \simeq \mathrm{Gal}(\dot{X}/X_0(5))$. This action interchanges the two places of type I_5 reduction as well as the two places of type I_1 reduction. Taking the quotient by the distinguished subgroup of order five gives rise to a model of the base change of Y via an Atkin–Lehner involution which interchanges places of type I_5 reduction with places of type I_1 reduction.

Level 6. The relevant level structure is a point of order six. There are four subgroups of the distinguished cyclic subgroup of sections of order six of π. Taking the quotient of Y by each of these yields a model for the base change of π with respect to each of the elements of $V_4 \simeq \mathrm{Aut}(6,0)$.

(ii) Suppose now that $\alpha \in \mathrm{Aut}(X,S)$ is an exceptional automorphism. To show that Y and Y' are not isogenous it suffices by Lemma 4.2(iv) and Remark 4.6 to treat the case that Y appears in Table 1 and has level 3, 4, 5 or 6. For type 3 there are no exceptional automorphisms and hence nothing to prove. In the remaining cases we claim that π and π' are not isomorphic. In all cases except the case $\tau = 5$ and $\mathrm{char}(k) = 11$ this is clear from the Kodaira types of the fibers at the places of bad reduction (cf. Tables 1 and 3). The remaining case is settled by a look at the J-invariant (again see Tables 1 and 3). Thus, if there is an isogeny, $\psi : Y \to Y'$, then there is an isogeny with non-trivial cyclic kernel. We consider the various possibilities.

Suppose that ψ exists and that $\mathrm{Ker}(\psi)$ has a non-trivial connected component. Taking the quotient by this component gives a BLIS fibration of type π_n with $n > 0$ which has fibers of Kodaria type $I_{p^n r}$ (cf. Lemma 4.2). Taking a further quotient by an étale group scheme of order prime to p yields an elliptic surface with type $I_{p^n r'}$ reduction for some r' prime to p. Since p is prime to the type, this surface cannot be isomorphic to Y'. This is a contradiction.

It remains to rule out the case that $\mathrm{Ker}(\psi)$ is a cyclic étale group scheme. Certain cyclic groups of this type occur naturally as part of the level structure in the elliptic surfaces of Table 1. By the proof of part (i) taking the quotient of Y by such a subgroup yields the pullback of Y by a generic automorphism in the level 5 and 6 cases. The pullback by a generic automorphism is not isomorphic to the pullback by an exotic isomorphism, since then Y itself would be isomorphic to the pullback by an exotic isomorphism. In the case of level 4 the quotient is isomorphic to the pullback of Y'' by a generic automorphism, where Y'' has either level 4 or 8, depending on the choice of subgroup. From the Kodaira types of the singular fibers neither is isomorphic to the pullback of Y by an exotic automorphism.

It remains to rule out the possibility that a surface in Table 1 acquires extra level structure in characteristic p in the form of a cyclic étale subgroup. Assume first that this group has order prime to p. There is a forget-the-extra-level-structure map between modular curves for the two moduli problems. It has positive degree and does not admit a section even when restricted to characteristic p which is prime to both level structures.

Finally we claim that Y_η has no étale subgroup of order p equal to $\mathrm{char}(k)$. If such were to exist, write E for the quotient elliptic curve. Now the dual isogeny, $E \to Y_\eta$, is purely inseparable and may be identified with the relative Frobenius morphism, $F_{E/\eta}$. This identifies Y_η with $E^{(p)}$ (see [Sil86, II.2.10]). It follows that the J-invariant of Y is a pth power. This contradicts the fact that the singular fibers of Y have Kodaira types I_r with $p \nmid r$. $\square$

TABLE 4. Fiber products of rational elliptic surfaces with $S = S'$.

Characteristic	Types	Kodaira types
2	3, 5	$\mathrm{I}_3-\mathrm{I}_5, \mathrm{I}_3-\mathrm{I}_5, \mathrm{I}_3-\mathrm{I}_1, \mathrm{I}_3-\mathrm{I}_1$
2	5, 5	$\mathrm{I}_5-\mathrm{I}_5, \mathrm{I}_5-\mathrm{I}_1, \mathrm{I}_1-\mathrm{I}_5, \mathrm{I}_1-\mathrm{I}_1$
3	2, 2	$\mathrm{I}_4-\mathrm{I}_4, \mathrm{I}_4-\mathrm{I}_2, \mathrm{I}_2-\mathrm{I}_4, \mathrm{I}_2-\mathrm{I}_2$
5	2, 6	$\mathrm{I}_4-\mathrm{I}_6, \mathrm{I}_2-\mathrm{I}_3, \mathrm{I}_4-\mathrm{I}_2, \mathrm{I}_2-\mathrm{I}_1$
5	6, 6	$\mathrm{I}_6-\mathrm{I}_2, \mathrm{I}_3-\mathrm{I}_3, \mathrm{I}_2-\mathrm{I}_6, \mathrm{I}_1-\mathrm{I}_1$
7	2, 6	$\mathrm{I}_4-\mathrm{I}_6, \mathrm{I}_4-\mathrm{I}_3, \mathrm{I}_2-\mathrm{I}_2, \mathrm{I}_2-\mathrm{I}_1$
7	6, 6	$\mathrm{I}_6-\mathrm{I}_3, \mathrm{I}_3-\mathrm{I}_6, \mathrm{I}_2-\mathrm{I}_2, \mathrm{I}_1-\mathrm{I}_1$
11	2, 5	$\mathrm{I}_4-\mathrm{I}_5, \mathrm{I}_2-\mathrm{I}_5, \mathrm{I}_4-\mathrm{I}_1, \mathrm{I}_2-\mathrm{I}_1$
11	5, 5	$\mathrm{I}_5-\mathrm{I}_5, \mathrm{I}_5-\mathrm{I}_5, \mathrm{I}_1-\mathrm{I}_1, \mathrm{I}_1-\mathrm{I}_1$
17	2, 6	$\mathrm{I}_2-\mathrm{I}_6, \mathrm{I}_4-\mathrm{I}_3, \mathrm{I}_4-\mathrm{I}_2, \mathrm{I}_2-\mathrm{I}_1$
17	6, 6	$\mathrm{I}_6-\mathrm{I}_6, \mathrm{I}_3-\mathrm{I}_2, \mathrm{I}_2-\mathrm{I}_3, \mathrm{I}_1-\mathrm{I}_1$
29	5, 6	$\mathrm{I}_5-\mathrm{I}_6, \mathrm{I}_5-\mathrm{I}_2, \mathrm{I}_1-\mathrm{I}_3, \mathrm{I}_1-\mathrm{I}_1$
31	3, 5	$\mathrm{I}_3-\mathrm{I}_5, \mathrm{I}_3-\mathrm{I}_5, \mathrm{I}_3-\mathrm{I}_1, \mathrm{I}_3-\mathrm{I}_1$
31	5, 5	$\mathrm{I}_5-\mathrm{I}_5, \mathrm{I}_5-\mathrm{I}_1, \mathrm{I}_1-\mathrm{I}_5, \mathrm{I}_1-\mathrm{I}_1$
41	5, 6	$\mathrm{I}_5-\mathrm{I}_6, \mathrm{I}_5-\mathrm{I}_2, \mathrm{I}_1-\mathrm{I}_3, \mathrm{I}_1-\mathrm{I}_1$
73	3, 6	$\mathrm{I}_3-\mathrm{I}_6, \mathrm{I}_3-\mathrm{I}_3, \mathrm{I}_3-\mathrm{I}_2, \mathrm{I}_3-\mathrm{I}_1$
73	6, 6	$\mathrm{I}_6-\mathrm{I}_2, \mathrm{I}_3-\mathrm{I}_6, \mathrm{I}_2-\mathrm{I}_3, \mathrm{I}_1-\mathrm{I}_1$
251	2, 5	$\mathrm{I}_2-\mathrm{I}_5, \mathrm{I}_4-\mathrm{I}_5, \mathrm{I}_4-\mathrm{I}_1, \mathrm{I}_2-\mathrm{I}_1$
251	5, 5	$\mathrm{I}_5-\mathrm{I}_5, \mathrm{I}_5-\mathrm{I}_1, \mathrm{I}_1-\mathrm{I}_5, \mathrm{I}_1-\mathrm{I}_1$
919	5, 6	$\mathrm{I}_5-\mathrm{I}_6, \mathrm{I}_5-\mathrm{I}_3, \mathrm{I}_1-\mathrm{I}_2, \mathrm{I}_1-\mathrm{I}_1$
9001	5, 6	$\mathrm{I}_5-\mathrm{I}_2, \mathrm{I}_5-\mathrm{I}_3, \mathrm{I}_1-\mathrm{I}_6, \mathrm{I}_1-\mathrm{I}_1$

7. Classification of desingularized fiber products with $h^3 = 0$

Let W be a desingularized fiber product of semi-stable elliptic surfaces with vanishing third Betti number. By Corollary 3.2 and § 4, W is isomorphic to the desingularization of a fiber product, $Y \times_X Y'$, constructed as follows: $\pi : Y \to X = \mathbb{P}^1_k$ is a BLIS fibration and $\pi' : Y' \to X$ is the base change of a BLIS fibration, $\pi'' : Y'' \to X$, by an element $\alpha \in \mathrm{Aut}(X)$ with $\alpha(S) = S''$. (We write $\pi' = \alpha^*\pi''$ and $Y' = \alpha^* Y''$.) Furthermore Y and Y' must not be isogenous. The results of §§ 5 and 6 permit us to list the possibilities. This is done in Table 4.

Explanation of Table 4. The rows in the table are in bijective correspondence with *equivalence classes* of desingularized fiber products of semi-stable elliptic surfaces with vanishing third Betti number defined over the algebraic closure of the prime field. Two desingularized fiber products, W_1 and W_2, are said to be *equivalent* if one of the following two relations exists among the pairs of semi-stable elliptic surfaces, (π_1, π_1') and (π_2, π_2'), from which they are constructed; there exists $\alpha \in \mathrm{Aut}(X)$ such that:

(i) π_1 is isogenous to $\alpha^*\pi_2$ and π_1' is isogenous to $\alpha^*\pi_2'$; or

(ii) π_1 is isogenous to $\alpha^*\pi_2'$ and π_1' is isogenous to $\alpha^*\pi_2$.

For each equivalence class of desingularized fiber products with vanishing third Betti number, the first two columns of Table 4 indicate the characteristic of the base field and the types of the BLIS fibrations involved. The third column indicates the nature of the four singular fibers in the fiber product. For simplicity this information is only given when the exponents of the BLIS fibrations are both zero and the level is minimal for the given type. A complete list of singular fibers may be obtained by replacing the 4-tuple of Kodaira types in the table with the 4-tuple of Kodaira types of isogenous elliptic surfaces. For example, applying this to the final line in the table, $p = 9001$, gives the complete list of singular fibers, $(n, n') \in (\mathbb{Z}_{\geq 0})^2$:

$$\mathrm{I}_{p^n 5} - \mathrm{I}_{p^{n'} 2}, \mathrm{I}_{p^n 5} - \mathrm{I}_{p^{n'} 3}, \mathrm{I}_{p^n} - \mathrm{I}_{p^{n'} 6}, \mathrm{I}_{p^n} - \mathrm{I}_{p^{n'}},$$

and

$$\mathrm{I}_{p^n 5} - \mathrm{I}_{p^{n'} 6}, \mathrm{I}_{p^n 5} - \mathrm{I}_{p^{n'}}, \mathrm{I}_{p^n} - \mathrm{I}_{p^{n'} 2}, \mathrm{I}_{p^n} - \mathrm{I}_{p^{n'} 3}.$$

Justification of Table 4. Consider first the case when the BLIS fibrations π and π'' have the same type τ. Then $S = S''$ and $\alpha \in \mathrm{Aut}(X, S) = \mathrm{Aut}(\tau, p)$, where $p = \mathrm{char}(k)$. By Proposition 6.1 α cannot be generic, but any exceptional α gives rise a desingularized fiber product W with $H^3(W, \mathbb{Q}_l) = 0$. If $\alpha' \in \mathrm{Aut}(\tau, p)$ is also exceptional and π''' is the base change of π'' with respect to α', then π and π''' are isogenous if and only if Y' is isogenous to its base change with respect to $\alpha^{-1} \circ \alpha'$, which is equivalent to α and α' lying in the same left coset of $\mathrm{Aut}(\tau, 0)$ in $\mathrm{Aut}(\tau, p)$. A list of all possible primes p and left cosets of $\mathrm{Aut}(\tau, 0)$ in $\mathrm{Aut}(\tau, p)$ is given in Table 3. If $\mathrm{Aut}(\tau, 0)$ has index two in $\mathrm{Aut}(\tau, p)$, there is only one non-trivial coset, so there is only one equivalence class of desingularized fiber product with $h^3 = 0$ for the given choice of characteristic and type. When $\mathrm{Aut}(\tau, 0)$ has index three the elements of $\mathrm{Aut}(\tau, p)$ which appear in Table 3 have the form α and α^2, where α has order 3. Base changing the fiber product $Y \times_X \alpha^* Y$ by α^{-1} gives the fiber product $(\alpha^2)^* Y \times_X Y$ which is equivalent to $Y \times_X (\alpha^2)^* Y$. Again there is only one equivalence class for the given characteristic and type.

Suppose now that the types τ and τ'' of the BLIS fibrations π and π'' are different. There are only finitely many characteristics p in which there is an element $\alpha \in \mathrm{Aut}(X)$ with $\alpha(S) = S''$. In fact, this occurs exactly when the cross ratio orbits of S and S'' coincide in characteristic p and $p \nmid \tau\tau''$. When $(\tau, \tau'') = (2, 3)$, these conditions are never met. When $\tau \in \{2, 3\}$ and $\tau'' \in \{5, 6\}$, then we are dealing with the exceptional primes for types 5 and 6 listed in Table 2. To find the characteristics p in the case $\tau = 5, \tau' = 6$ we refer to Table 1 and ask when $\mathfrak{K} = \mathfrak{C}$, i.e. when $\kappa = (11 + 5\sqrt{5})/(11 - 5\sqrt{5}) \in \{-8, -\frac{1}{8}, 9, \frac{1}{9}, \frac{9}{8}, \frac{8}{9}\}$. It is easy to analyze the various possibilities. For example, if $\kappa = 8/9$, then $9(11 + 5\sqrt{5}) = 8(11 - 5\sqrt{5})$ or, equivalently, $11 = -85\sqrt{5}$. Squaring both sides gives $0 = 36004 = 2^2 \cdot 9001$. The case $p = 2$ is ruled out, since two divides the level 6, so $p = 9001$ is the only solution. The case $\kappa = 9/8$ also gives $p = 9001$, while $\kappa = -8$ or $-1/8$ yields $p = 919$ and $\kappa = 9$ yields $p = 29$ or 41 as does $\kappa = 1/9$.

When $\tau < \tau''$, Y and Y' are not isogenous. This follows from the fact that if Y_η has a subgroup of prime order l, then so does any isogenous elliptic curve. To see that each line in Table 4 with $\tau < \tau''$ corresponds to at most one equivalence class observe that the choice of $\alpha \in \mathrm{Aut}(X)$ identifying S with S'' is unique up to precomposing with elements of $\mathrm{Aut}(S, \tau)$. Since the characteristics which appear in the lines with $\tau < \tau''$ are never exceptional for τ in the sense of Table 2, the effect of altering the choice of α is to replace Y by an isogenous elliptic surface.

8. The canonical sheaf and some Hodge numbers

We continue to use the notation of the previous section. In particular, W is a desingularized fiber product with $H^3(W, \mathbb{Q}_l) = 0$ constructed from BLIS fibrations π and π'' in characteristic p of exponents $n, n' \in \mathbb{Z}_{\geq 0}$. Furthermore $\pi' = \alpha^* \pi''$ where $\alpha \in \mathrm{Aut}(X)$ with $\alpha(S) = S''$.

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Proposition 8.1. *We have* $h^0(W, \Omega^3_{W/k}) = p^n + p^{n'} - 1$.

Proof. The morphisms $\pi : Y \to X$, $\pi' : Y' \to X$ and $\bar{f} : \bar{W} \to X$ are projective, local complete intersection morphisms. It follows from [Kle80, Example 7(ii) and Corollary 19] that relative dualizing sheaves $\omega_{Y/X}$, $\omega_{Y'/X}$ and $\omega_{\bar{W}/X}$ exist, that each is an invertible sheaf, and that the restriction to any open subscheme which is smooth over X is isomorphic to the determinant of the relative Kähler differentials. Write $q : \bar{W} \to Y$ and $q' : \bar{W} \to Y'$ for the projections. From the behavior of relative Kähler differentials in fiber products and the fact that the non-smooth locus is in codimension two we conclude that $\omega_{\bar{W}/X} \simeq q^*\omega_{Y/X} \otimes (q')^*\omega_{Y'/X}$. A similar argument shows that there is a relative dualizing sheaf, $\omega_{\bar{W}/k}$, which is isomorphic to $\bar{f}^*\Omega_{X/k} \otimes \omega_{\bar{W}/X}$. The adjunction formula gives $\Omega^3_{W/k} \simeq \sigma^*\omega_{\bar{W}/k} \otimes \mathcal{O}_W(Q)$, where $Q \subset W$ is the exceptional divisor of the blow-up, $\sigma : W \to \bar{W}$. The canonical map, $\pi^*\pi_*\omega_{Y/X} \to \omega_{Y/X}$, is an isomorphism, as can be verified by a local calculation [Liu02, 9.4.3.5]. Thus,

$$\Omega^3_{W/k} \simeq f^*(\Omega_{X/k} \otimes \pi_*\omega_{Y/X} \otimes \pi'_*\omega_{Y'/X}) \otimes \mathcal{O}_W(Q). \tag{8.1}$$

Relative duality gives an isomorphism, $\pi_*\omega^\vee_{Y/X} \simeq R^1\pi_*\mathcal{O}_Y$. Now

$$12\deg(\pi_*\omega_{Y/X}) = -12\deg(R^1\pi_*\mathcal{O}_Y) = 12(\chi(\mathcal{O}_X) - \chi(R^1\pi_*\mathcal{O}_Y)) = 12\chi(\mathcal{O}_Y) = e(Y),$$

where $\chi(\mathcal{F})$ is the Euler characteristic of the coherent sheaf $\mathcal{F}$, $e(Y)$ is the l-adic Euler characteristic for Y, the second to last equality comes from the Leray spectral sequence for π, the last equality is Noether's theorem and the fact that $K_Y^2 = 0$, which follows from $\Omega^2_{Y/k} \simeq \pi^*(\Omega_{X/k} \otimes \pi_*\omega_{Y/X})$. Since π is semi-stable and in particular tamely ramified, $e(Y)$ is the sum of the Euler characteristics of the singular fibers [Sch02a, 3.2]. Thus, $e(Y) = 12p^n$. Since $X \simeq \mathbb{P}^1_k$, $\pi_*\omega_{Y/X} \simeq \mathcal{O}_{\mathbb{P}^1}(p^n)$. Apply the projection formula and the fact that $f_*\mathcal{O}_W(Q) \simeq \mathcal{O}_{\mathbb{P}^1}$ to (8.1) to conclude

$$H^0(W, \Omega^3_{W/k}) \simeq H^0(X, f_*\Omega^3_{W/k}) \simeq H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1/k}(p^n + p^{n'})) \simeq k^{p^n + p^{n'} - 1}. \qquad \square$$

Proposition 8.2. *We have* $h^1(W, \mathcal{O}_W) = 0$, $h^2(W, \mathcal{O}_W) = p^n + p^{n'} - 2$ *and* $h^3(W, \mathcal{O}_W) = p^n + p^{n'} - 1$.

Proof. The third assertion follows from the previous proposition and Serre duality.

The first assertion follows from the Leray spectral sequence: since $g_X = 0$, $H^1(X, f_*\mathcal{O}_W) \simeq H^1(X, \mathcal{O}_X) = 0$ and

$$R^1f_*\mathcal{O}_W \simeq R^1\bar{f}_*\mathcal{O}_{\bar{W}} \simeq R^1\pi_*\mathcal{O}_Y \oplus R^1\pi'_*\mathcal{O}_{Y'},$$

where the degrees of the two invertible sheaves on the right is negative by the proof of the previous proposition.

For the second assertion observe that $R^2f_*\mathcal{O}_W$ is an invertible sheaf of negative degree and hence has no non-zero global sections. Indeed

$$R^2f_*\mathcal{O}_W \simeq R^2\bar{f}_*\mathcal{O}_W \simeq R^1\pi_*R^1q_*(q')^*\mathcal{O}_{Y'} \simeq R^1\pi_*\pi^*R^1\pi'_*\mathcal{O}_{Y'} \simeq R^1\pi_*\mathcal{O}_Y \otimes R^1\pi'_*\mathcal{O}_{Y'},$$

where the second isomorphism comes from the Leray spectral sequence for $\bar{f} = \pi \circ q$, the third isomorphism is flat base change [Har77, III.9.3] and the fourth is the projection formula [Har77, III.Ex 8.3]. Thus

$$H^2(W, \mathcal{O}_W) \simeq H^1(X, R^1\pi_*\mathcal{O}_Y \oplus R^1\pi'_*\mathcal{O}_{Y'}) \simeq k^{p^n + p^{n'} - 2},$$

since $\deg(R^1\pi_*\mathcal{O}_Y) = -e(Y)/12 = -p^n$ and $\deg(R^1\pi'_*\mathcal{O}_Y) = -p^{n'}$. $\square$

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9. Proof of Theorem 1.1

Part (i) of the theorem was established in the justification of Table 4.

For part (ii), recall from § 7 that W is constructed from data (π, π'', α), where π and π'' are BLIS fibrations and $\alpha \in \mathrm{Aut}(X)$ satisfies $\alpha(S) = S''$. Note that every BLIS fibration is defined over the prime field. If π and π'' have the same type, then α appears in Table 3. It is defined over the prime field except when $\mathrm{char}(k) = 2$, in which case it is defined over the field with four elements. Suppose that the types satisfy $\tau < \tau''$. In the case of the first line of Table 4 we may take $\alpha = Id$. In the remaining cases α is defined over the prime field since the cross ratio of S is defined over the prime field. Finally the blow-up of the fiber product along $\bar{W}_{\mathrm{sing}}$, $\sigma : W \to \bar{W}$, is defined over the field of definition of $\bar{W}$.

Part (iii) follows from Proposition 8.1.

Part (iv) follows from Proposition 8.1 and the finiteness of the number of lines in Table 4.

10. Projective threefolds with trivial canonical sheaf and $h^3 = 0$

It is remarkable that there exist non-singular projective threefolds, V, with trivial canonical sheaf and $H^3(V, \mathbb{Q}_l) = 0$. Such varieties do not exist in characteristic zero by Hodge theory. The first example was constructed by Hirokado [Hir99] in characteristic three. More recently, Schröer [Sch04a] and then Ekedahl [Eke04] have constructed a few further examples in characteristics two and three; some of these have moduli.

Proposition 10.1. (i) *The fiber product of level* $(4, 4)$ *in characteristic three (see line 3 of Table 4) admits several non-isomorphic small resolutions which are projective threefolds with trivial canonical sheaf and* $h^3 = 0$.

(ii) *The fiber products of other types of rational elliptic surfaces in Table 4 do not admit small projective resolutions. Small resolutions do exist as algebraic spaces.*

(iii) *There exist projective threefolds with trivial canonical sheaf and* $h^3 = 0$ *which admit a pencil of Kummer surfaces in which the general fiber is not supersingular.*

Proof. (i) In characteristic three consider the fiber product of the level 4 surface in Table 1 with its base change by an exceptional automorphism (line 1 of Table 2 and line 3 of Table 4). Every irreducible component of every fiber of the map, $\bar{f} : \bar{W} \to X$, is non-singular. Ordering the components of the singular fibers and then successively blowing up one component at a time in the order chosen gives rise to a morphism, $\gamma : \widehat{W} \to \bar{W}$, which collapses finitely many rational curves in $\widehat{W}$. The variety $\widehat{W}$ is projective, non-singular and has trivial canonical sheaf. There is a birational morphism, $\varpi : W \to \widehat{W}$, which on every component, $\mathbb{P}^1 \times \mathbb{P}^1$, of the exceptional divisor $Q \subset W$ is projection onto one factor. It follows from the Leray spectral sequence for ϖ that $h^3(\widehat{W}, \mathbb{Q}_l) = 0$.

To verify that the isomorphism class of $\widehat{W}$ depends on the order in which the blow-ups are performed, observe first that each singular fiber of $\bar{f}$ is the product of a Kodaira type I_2 fiber with a Kodaira type I_4 fiber or is the self-product of an I_2 fiber or an I_4 fiber. The components of I_2 and I_4 which meet the identity section will be called identity components. These components together with the component of I_4 which does not meet the identity component will be called *even*. For $s \in S$ a component of $\bar{f}^{-1}(s)$ is called even if it is the product of even components of $\pi^{-1}(s)$ and $(\pi')^{-1}(s)$. Every singular point of $\bar{W}$ is contained in an even component of a singular fiber. Thus, blowing up the even components already gives a non-singular variety, $\widehat{W}_{\mathrm{even}}$. Blowing up the strict transforms of the remaining components has no effect, as they are Cartier divisors. Since the even components are disjoint, the isomorphism class of $\widehat{W}_{\mathrm{even}}$ is independent of the order in which the even components are blown up. The inversion maps on the generic fibers of π and π' extend to biregular involutions of Y and Y'. The product gives a biregular involution of $\bar{W}$ which lifts to a biregular involution, ι, of $\widehat{W}_{\mathrm{even}}$. To construct a resolution $\widehat{W} \to \bar{W}$ such that inversion is not biregular fix a singular fiber of $\bar{W}$ of type $\mathrm{I}_2 - \mathrm{I}_4$. Blow-up the component which is the product of the identity component of the I_2 fiber with a component of the I_4 fiber which does not meet any section of order two. Then blow up the component which meets the identity section of $\bar{f}$. No matter in what order the remaining fiber components are blown up, inversion will not give a regular involution of the resulting variety, $\widehat{W}$.

(ii) The obstruction is the existence of a fiber of Kodaira type I_1 in either Y or Y'. The proof of [Sch88, 3.1(iii)] goes through in this case. The assumption in [Sch88, 3.1(iii)] that $S \neq S'$ does not hold here, but all one needs is that Y and Y' are not isogenous. There is no obstruction to small resolution in the category of algebraic spaces [Art73, Chapitre VI]. Such a small resolution, call it V, has trivial canonical sheaf. Let $E \subset V$ denote the exceptional curve in the small resolution and let $E_{\bar{w}}$ denote the component of E which lies over a node, $\bar{w}$, contained in a I_1–I_n fiber of $\bar{f} : \bar{W} \to X$. By [Art73, Chapitre VI], $E_{\bar{w}}$ has intersection number zero with each divisor on V. Thus, no point of $E_{\bar{w}}$ admits an open affine neighborhood and V is not a scheme. (For a very similar argument see [Art73, pp. 177–185].)

(iii) Write $\tilde{W}_{\mathrm{even}}$ for the blow-up of $\widehat{W}_{\mathrm{even}}$ along the fixed point locus of ι. Then ι lifts to a biregular involution $\tilde{\iota} \in \mathrm{Aut}(\tilde{W}_{\mathrm{even}})$. Denote the quotient by $\mathfrak{W}$. Since the fixed locus of $\tilde{\iota}$ is a non-singular divisor and the characteristic is different from two, $\mathfrak{W}$ is non-singular. Any non-zero global section of $\Omega^3_{\tilde{W}_{\mathrm{even}}/k}$ descends to give a nowhere vanishing section of $\Omega^3_{\mathfrak{W}/k}$. Blowing up $\widehat{W}_{\mathrm{even}}$ along 16 disjoint $\mathbb{P}^1$'s to produce $\tilde{W}_{\mathrm{even}}$ does not change the third Betti number. The map $H^3(\mathfrak{W}, \mathbb{Q}_l) \to H^3(\tilde{W}_{\mathrm{even}}, \mathbb{Q}_l)$ is injective as one can see by applying the projection formula [Mil80, Vi.11.6a,d] with one factor being $1 \in H^0(\tilde{W}_{\mathrm{even}}, \mathbb{Q}_l)$. Thus, $H^3(\mathfrak{W}, \mathbb{Q}_l) = 0$. □

Open question 10.2. (i) Do there exist non-singular projective threefolds with trivial canonical sheaf and $h^3 = 0$ in characteristics other than two and three?

11. Deformations and lifting to characteristic zero

The smooth projective threefolds with trivial canonical sheaf constructed by Hirokado, Schröer and Ekedahl all have obstructed deformations. This manifests itself when one tries to lift these varieties to characteristic zero [Hir99, Sch04a, Eke04]. This result stands in notable contrast to the situation in characteristic zero where the deformation theory of smooth, projective threefolds with trivial canonical sheaf is unobstructed.

Proposition 11.1. *Let W be a desingularized fiber product of semi-stable rational elliptic surfaces with $h^3(W, \mathbb{Q}_l) = 0$. Then W does not admit a formal lifting to characteristic zero.*

Proof. The argument in [Sch04a, § 2] works here with minor modifications. By Proposition 8.2, $H^1(W, \mathcal{O}_W) = 0$. The assumption that the elliptic surfaces are rational (i.e. exponent zero) implies $H^2(W, \mathcal{O}_W) = 0$. As in [Sch04a, § 2] it follows that if a formal lifting were to exist, then such a lifting would exist over a Noetherian, integral base, B, and it would be isomorphic to a formal completion of a smooth projective scheme, $\mathbf{f} : \mathcal{W} \to B$. Furthermore, restriction would give an isomorphism, $\mathrm{Pic}(\mathcal{W}) \to \mathrm{Pic}(W)$. Since $\Omega^3_{W/k} \simeq \mathcal{O}_W(Q)$ and $H^i(Q, \mathcal{N}_{Q/W}) = 0$ for all i, the relative Hilbert scheme, $\mathrm{Hilb}_{\mathcal{W}/B}$, is locally étale over B at Q. Thus, Q lifts to a unique effective divisor $\mathcal{Q} \subset \mathcal{W}$. The isomorphism of Picard groups gives that $\Omega^3_{\mathcal{W}/B} \simeq \mathcal{O}_{\mathcal{W}}(\mathcal{Q})$. Passing to the fraction field K of B gives a smooth, projective threefold, $\mathcal{W}_K$ over a field of characteristic zero with $h^0(\mathcal{W}_K, \Omega^3_{\mathcal{W}_K/K}) \neq 0$. By base-change theorems, $R^3\mathbf{f}_*\mathbb{Z}/l^n$, is a constant sheaf on B for all n. Thus, $h^3(\mathcal{W}_{\bar{K}}, \mathbb{Q}_l) = 0$. Passing to an associated complex manifold and applying Hodge theory gives a contradiction. □

Schröer's argument [Sch04a, § 2] shows that the projective threefolds described in parts (i) and (iii) of Proposition 10.1 do not lift to characteristic zero. The referee has remarked that the algebraic spaces constructed in Proposition 10.1(ii) should also not lift to characteristic zero. We can offer the following sketch, referring to [Knu71] as a reference for algebraic spaces: Let T be the spectrum of a non-equicharacteristic discrete valuation ring with residue field k. Suppose that there exists a smooth, proper algebraic space, $\mathcal{V} \to T$, whose closed fiber, V, is isomorphic to a small resolution of $\bar{W}$ as in Proposition 10.1(ii). Write $E \subset V$ for the collection of exceptional curves with normal sheaves isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ which contract to $\bar{W}_{\mathrm{sing}}$. The relative Hilbert functor of $\mathcal{V}$ over T is representable by an algebraic space over T, $\mathrm{Hilb}_{\mathcal{V}/T}$ (see [Art69a, § 6]). Since $H^i(E, \mathcal{N}_{E/V}) = 0$ for all i, E has a unique lift to each infinitesimal neighborhood of V in $\mathcal{V}$ (see [Art69a, § 6] and also [Ser06, § 3.2.4]). Thus the point $[E] \in \mathrm{Hilb}_{V/k}$ lifts to a formal section of $\mathrm{Hilb}_{\mathcal{V}/T}$. By Artin's approximation theorem [Art73, Théorème 2.3] there is an étale neighborhood T' of the closed point of T such that $[E]$ lifts to a section of $\mathrm{Hilb}_{\mathcal{V}'/T'}$, where $\mathcal{V}' = \mathcal{V} \times_T T'$. Thus, E lifts to a relative curve, $\mathcal{E}' \subset \mathcal{V}'$, which is smooth over T as the closed fiber is smooth over k. The familiar notion from scheme theory of blowing up a sheaf of ideals extends to algebraic spaces, since blowing up is local in the étale topology (see the introduction in [Art70]). Blowing up the sheaf of ideals of $\mathcal{E}'$ yields a smooth, proper algebraic space, $\mathcal{W}$, whose closed fiber is isomorphic to the blow-up of V along E. However, this is none other than the blow-up, W, of the fiber product, $\bar{W}$, along $\bar{W}_{\mathrm{sing}}$ (cf. [Art73, pp. 177–185]). Since the formal completion of $\mathcal{W}$ along W (see [Knu71, p. 216]) is a formal scheme (see [GD71, 10.6.3] and [Art69b, Theorem 3.2]), this contradicts Proposition 11.1.

It seems likely that the existence of $\mathcal{V}$ could be ruled out without blowing up by contradicting the Hodge decomposition for the cohomology of Moisezon manifolds in the spirit of the proof of Proposition 11.1. Some useful references for this approach are [Art70, § 7], [Moi67, Theorem 3, p. 98] and [Lew91, Proposition 13.2]. One would need to generalize standard comparison and base-change theorems (cf. [Mil80, III.3.12, VI.2.1 and VI.4.2], [Art69a, Lemma 6.9] and [Art73, Chapitre VII]) from schemes to algebraic spaces over an appropriate base. The author is not aware that the necessary generalizations have appeared in print.

The next result shows that the threefolds, $\widehat{W}$, constructed in the previous section are rigid. Define the tangent sheaf, $\mathcal{T}_{\widehat{W}/k} := \mathcal{H}om_{\mathcal{O}_{\widehat{W}}}(\Omega_{\widehat{W}/k}, \mathcal{O}_{\widehat{W}})$.

Theorem 11.2. *Let $\widehat{W}$ be a small projective resolution of a fiber product of semi-stable elliptic surfaces with section and at least one singular fiber. Suppose that $\Omega^3_{\widehat{W}/k} \simeq \mathcal{O}_{\widehat{W}}$. If $h^3(\widehat{W}, \mathbb{Q}_l) = 0$, then $H^1(\widehat{W}, \mathcal{T}_{\widehat{W}/k}) = 0$.*

Proof. By the triviality of the canonical sheaf there is an isomorphism, $\Omega^2_{\widehat{W}/k} \simeq \mathcal{T}_{\widehat{W}/k}$. By Serre duality it suffices to show $H^2(\widehat{W}, \Omega^1_{\widehat{W}/k}) = 0$. As verifying this seems no easier than computing all of the Hodge numbers of $\widehat{W}$ we prove the following.

PROPOSITION 11.3. *The only non-zero Hodge numbers of $\widehat{W}$ are $h^{00}(\widehat{W}) = h^{33}(\widehat{W}) = 1$, $h^{03}(\widehat{W}) = h^{30}(\widehat{W}) = 1$ and $h^{11}(\widehat{W}) = h^{22}(\widehat{W}) = 36$.*

Proof. By Proposition 8.2 and Serre duality it suffices to compute $H^i(\widehat{W}, \Omega_{\widehat{W}/k})$ for each i. As in [Sch02b, § 11] the cohomology of $\Omega_{\widehat{W}/k}$ may be computed using the Leray spectral sequence for $\hat{f} : \widehat{W} \to X$. In particular, $H^0(\widehat{W}, \Omega_{\widehat{W}/k}) = 0$, since the natural map, $\Omega_X \to \hat{f}_*\Omega_{\widehat{W}/k}$ is an isomorphism [Sch02b, 11.8(iv)]. Write m_s (respectively m'_s) for the number of irreducible components in $\pi^{-1}(s)$ (respectively $(\pi')^{-1}(s)$). We use the following technical lemma.

LEMMA 11.4. *We have $R^1\hat{f}_*\Omega_{\widehat{W}/X} \simeq \mathcal{O}_X^3 \oplus \mathcal{O}_X(-S) \oplus_{s\in S} i_{s*}\mathcal{O}_s^{m_s m'_s - 1}$.*

The proof of the lemma is given at the end of the section. To use it to prove the proposition, consider the short exact sequence

$$0 \longrightarrow \hat{f}^*\Omega_{X/k} \longrightarrow \Omega_{\widehat{W}/k} \longrightarrow \Omega_{\widehat{W}/X} \longrightarrow 0$$

and the following corresponding long exact sequence of derived functors,

$$R^1\hat{f}_*\Omega_{\widehat{W}/k} \xrightarrow{c} R^1\hat{f}_*\Omega_{\widehat{W}/X} \xrightarrow{e} R^2\hat{f}_*\hat{f}^*\Omega_{X/k} \longrightarrow R^2\hat{f}_*\Omega_{\widehat{W}/k} \longrightarrow R^2\hat{f}_*\Omega_{\widehat{W}/X}\,. \tag{11.1}$$

Write $\mathcal{L}$ for $R^1\pi_*\mathcal{O}_Y$ and $\mathcal{L}'$ for $R^1\pi'_*\mathcal{O}_{Y'}$. By [Sch02b, 11.8(iii)] and the fact that Y and Y' are rational elliptic surfaces,

$$R^2\hat{f}_*\hat{f}^*\Omega_{X/k} \simeq \Omega_{X/k} \otimes R^2\hat{f}_*\mathcal{O}_{\widehat{W}} \simeq \Omega_{X/k} \otimes \mathcal{L} \otimes \mathcal{L}' \simeq \mathcal{O}_{\mathbb{P}^1}(-4).$$

Furthermore, c is injective since

$$0 \longrightarrow \Omega_{X/k} \xrightarrow{\alpha} \hat{f}_*\Omega_{\widehat{W}/k} \longrightarrow \hat{f}_*\Omega_{\widehat{W}/X} \xrightarrow{e''} \Omega_X \otimes R^1\hat{f}_*\mathcal{O}_{\widehat{W}} \longrightarrow \operatorname{Ker}(c) \longrightarrow 0$$

is exact, α is an isomorphism and

$$\hat{f}_*\Omega_{\widehat{W}/X} \simeq \mathcal{L}^{-1}(-S) \oplus (\mathcal{L}')^{-1}(-S) \simeq \mathcal{O}_{\mathbb{P}^1}(-3)^{\oplus 2} \simeq \Omega_X \otimes (\mathcal{L} \oplus \mathcal{L}') \simeq \Omega_X \otimes R^1\hat{f}_*\mathcal{O}_{\widehat{W}}$$

by [Sch02b, 11.8(ii) and 11.14(i)].

As the ranks of the first two $\mathcal{O}_X$-modules in (11.1) are three and four, the map e is non-zero. By Lemma 11.4 it must be the projection of $R^1\hat{f}_*\Omega_{\widehat{W}/X}$ onto the direct summand $\mathcal{O}_X(-S) \simeq \mathcal{O}_{\mathbb{P}^1}(-4)$. Thus,

$$R^1\hat{f}_*\Omega_{\widehat{W}/k} \simeq \mathcal{O}_X^{\oplus 3} \oplus_{s\in S} i_{s*}\mathcal{O}_s^{m_s m'_s - 1},$$

where $i_s : s \to X$ is the inclusion. Consequently,

$$h^1(\widehat{W}, \Omega_{\widehat{W}/k}) = h^1(X, \hat{f}_*\Omega_{\widehat{W}/k}) + h^0(X, R^1\hat{f}_*\Omega_{\widehat{W}/k}) = 1 + 3 + \sum_{s\in S}(m_s m'_s - 1).$$

By Proposition 10.1 and line 3 of Table 4, $\sum_{s\in S} m_s m'_s = 16 + 8 + 8 + 4 = 36$. Thus, $h^1(\widehat{W}, \Omega_{\widehat{W}/k}) = 36$. Now (11.1) and [Sch02b, 11.15] yield

$$R^2\hat{f}_*\Omega_{\widehat{W}/k} \simeq R^2\hat{f}_*\Omega_{\widehat{W}/X} \simeq \mathcal{L} \oplus \mathcal{L}' \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}.$$

Thus, $h^3(\widehat{W}, \Omega_{\widehat{W}/k}) = h^1(X, R^2\hat{f}_*\Omega_{\widehat{W}/k}) = 0$ and

$$h^2(\widehat{W}, \Omega_{\widehat{W}/k}) = h^0(X, R^2\hat{f}_*\Omega_{\widehat{W}/k}) + h^1(X, R^1\hat{f}_*\Omega_{\widehat{W}/k}) = 0.$$

It remains only to prove the lemma.

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Proof of Lemma 11.4. Observe that the sheaf $\Omega_{\widehat{W}/X}$ is torsion free and hence flat over X. This follows from the fact that for a closed point $w \in \widehat{W}$ with image $x \in X$ there are local parameters $x_1, \dots, x_3 \in \mathcal{O}_{\widehat{W},w}$ such that the pullback of a uniformizing parameter $t \in \mathcal{O}_{X,x}$ satisfies $t = x_1 \dots x_i$ for some $i \leq 3$ which implies that $dt \in \Omega_{\widehat{W}/k \cdot w} \simeq \oplus_{i=1}^{3} \mathcal{O}_{\widehat{W},w}\, dx_i$ is an indivisible element [Har77, II.8.3A].

Write $E \subset \widehat{W}$ for the exceptional locus of the small projective resolution of singularities, $\gamma : \widehat{W} \to \bar{W}$. In the exact sequence [Sch02b, 11.11–11.12]

$$\begin{aligned} 0 \longrightarrow R^1 \hat{f}_* \gamma^* \Omega_{\bar{W}/X} \xrightarrow{\alpha} R^1 \hat{f}_* \Omega_{\widehat{W}/X} \longrightarrow R^1 \hat{f}_* \Omega_{E/k} \\ \longrightarrow R^2 \hat{f}_* \gamma^* \Omega_{\bar{W}/X} \longrightarrow R^2 \hat{f}_* \Omega_{\widehat{W}/X} \longrightarrow 0\,. \end{aligned}$$

we have

$$\begin{aligned} R^1 \hat{f}_* \Omega_{E/k} &\simeq \oplus_{s\in S} i_{s*} \mathcal{O}_s^{\oplus m_s m_s'} =: \mu, \\ R^1 \hat{f}_* \gamma^* \Omega_{\bar{W}/X} &\simeq \mathcal{O}_X^2 \oplus \mathcal{O}_X^2(-S) \oplus \nu \quad \text{(see [Sch02b, 11.11-13])} \\ R^2 \hat{f}_* \gamma^* \Omega_{\bar{W}/X} &\simeq \mathcal{L} \oplus \mathcal{L}' \oplus \nu \quad \text{(see [Sch02b, 11.13v])}, \end{aligned}$$

where $\nu = \bigoplus_{s\in S} i_{s*} \mathcal{O}_s^{\oplus m_s + m_s' - 2}$. The direct summand $\mathcal{O}_X^2 \subset R^1 \hat{f}_* \gamma^* \Omega_{\bar{W}/X}$ is generated by the Hodge cohomology classes of the inverse images of the identity sections of π and π' via the projections $q : \widehat{W} \to Y$ and $q' : \widehat{W} \to Y'$. It is also a direct summand of $R^1 \hat{f}_* \Omega_{\widehat{W}/X}$. As $R^2 \hat{f}_* \Omega_{\widehat{W}/X} \simeq \mathcal{L} \oplus \mathcal{L}'$ (see [Sch02b, 11.15]), the above exact sequence gives rise to an exact sequence,

$$0 \longrightarrow \mathcal{O}_X^2(-S) \oplus \nu \longrightarrow R^1 \hat{f}_* \Omega_{\widehat{W}/X} / \mathcal{O}_X^2 \longrightarrow \mu \longrightarrow \nu \longrightarrow 0\,.$$

Set $F = \hat{f}^{-1}(s)$. By [Sch02b, 11.15], $R^2 \hat{f}_* \Omega_{\widehat{W}/X}$ is locally free and the natural map, $R^2 \hat{f}_* \Omega_{\widehat{W}/X} \otimes i_{s*} \mathcal{O}_s \to H^2(F, \Omega_{F/k})$, is an isomorphism. It follows that the natural map, $R^1 \hat{f}_* \Omega_{\widehat{W}/X} \otimes i_{s*} \mathcal{O}_s \to H^1(F, \Omega_{F/k})$ is also an isomorphism [Har77, 12.11 and 8.2a]. For $s \in S$, $\dim_k(H^1(F, \Omega_{F/k})) = m_s m_s' + 3$ (see [Sch08]). Since the rank of $R^1 \hat{f}_* \Omega_{\widehat{W}/X}$ is four, the torsion subsheaf, $\tau \subset R^1 \hat{f}_* \Omega_{\widehat{W}/X}$ satisfies, $\tau \otimes i_{s*} \mathcal{O}_s \simeq i_{s*} \mathcal{O}_s^{\oplus m_s m_s' - 1}$. The Hodge cohomology classes of the irreducible components of F generate a subsheaf $\theta_1 \subset R^1 \hat{f}_* \Omega_{\widehat{W}/X}$ isomorphic to $\bigoplus_{s\in S} i_{s*} \mathcal{O}_s^{\oplus m_s m_s' - 1}$. The -1 appears in the exponent because the class of F is zero. The composition, $\theta_1 \to \tau \to \tau \otimes i_{s*} \mathcal{O}_s$, is an isomorphism. It follows that $\theta_1 \simeq \tau$. Thus, the previous exact sequence yields the following short exact sequence.

$$0 \longrightarrow \mathcal{O}_X^2(-S) \longrightarrow R^1 \hat{f}_* \Omega_{\widehat{W}/X} / (\mathcal{O}_X^2 \oplus \tau) \longrightarrow \bigoplus_{s\in S} i_{s*} \mathcal{O}_s \longrightarrow 0\,.$$

The map e in (11.1) gives rise to the composition

$$\mathcal{O}_X^2(-S) \longrightarrow R^1 \hat{f}_* \Omega_{\widehat{W}/X} / (\mathcal{O}_X^2 \oplus \tau) \longrightarrow \Omega_X \otimes \mathcal{L} \otimes \mathcal{L}'\,,$$

which is non-zero. In the case at hand

$$\mathcal{O}_X(-S) \simeq \mathcal{O}_{\mathbb{P}^1}(-4) \simeq \Omega_X \otimes \mathcal{L} \otimes \mathcal{L}',$$

so this map is in fact a split surjection. Now the lemma follows from the following short exact sequence,

$$0 \longrightarrow \mathcal{O}_X(-S) \longrightarrow R^1\hat{f}_*\Omega_{\widehat{W}/X}/(\mathcal{O}_X(-S)\oplus\mathcal{O}_X^2\oplus\tau) \longrightarrow \bigoplus_{s\in S} i_{s*}\mathcal{O}_s \longrightarrow 0\,. \qquad \square\square\square$$

Remark 11.5. It follows from Proposition 12.1 that the Picard group of $\widehat{W}$ is free of rank $35 = h^{1,1}(\widehat{W}) - 1$.

12. Supersingular threefolds

There is a well-established notion of supersingular abelian variety in positive characteristic. If the dimension is at least two and the base field is algebraically closed, an abelian variety, A, is supersingular if and only if the rank of the Néron–Severi group, $\rho(A)$, equals $h^2(A,\mathbb{Q}_l)$. Shioda [Shi77] has suggested that a smooth projective surface V over an algebraically closed field be called supersingular if $\rho(V) = h^2(V,\mathbb{Q}_l)$. Every unirational surface has this property. Shioda asks if conversely every supersingular surface with $\pi_1(V) = \{1\}$ is unirational. It seems interesting to consider these issues in higher dimensions as well.

Proposition 12.1. *Let W be a desingularized fiber product of semi-stable elliptic surfaces with vanishing third Betti number. Then:*

(i) $\pi_1(W) = \{1\}$;

(ii) $\rho(W) = h^2(W,\mathbb{Q}_l)$.

Proof. (i) First note that $\pi_1(Y) = \pi_1(Y') = \{1\}$, for the elliptic surfaces from which W is constructed. This is clear if Y is rational (i.e. if the exponent is zero). In the general case one needs only note that Y contains an open dense subset which is a purely inseparable finite cover of the variety, U, which is obtained from a BLIS fibration of exponent zero by removing the finite set of points where the fibration is not smooth (see [Mil80, I.5.2h] and [Gro71, IX.4.10]). Write $i : \dot{X} := X - S = X - S' \to X$ for the inclusion and denote base change by i by adding a $\dot{}$ to the notation. For a closed point $x \in \dot{X}$, there is an exact sequence,

$$\pi_1(f^{-1}(x)) \longrightarrow \pi_1(\dot{W}) \longrightarrow \pi_1(\dot{X}) \longrightarrow \{1\}$$

and analogous exact sequences for $\dot{Y}$ and $\dot{Y}'$ (see [Gro71, X.1.4]). It follows that $\pi_1(\dot{W})$ is generated by the images of the groups $\pi_1(\dot{Y})$ and $\pi_1(\dot{Y}')$ under the maps given by the identity sections. The surjective map, $\pi_1(\dot{W}) \to \pi_1(W)$, must be zero, since the composition, $\pi_1(\dot{Y}) \to \pi_1(\dot{W}) \to \pi_1(W)$, factors through $\pi_1(Y)$ and the same holds for Y'.

(ii) Every desingularized fiber product of semi-stable elliptic surfaces has the property that its l-adic euler characteristic is the sum of the l-adic euler characteristics of the singular fibers [Sch02a, 8.13]. Write n and n' for the exponents of the semi-stable elliptic surfaces Y and Y' used to construct W. It is easy to compute the l-adic euler characteristics, $e(\bar{W}) = \sum_{s\in S} p^n m_s p^{n'} m'_s$ and $e(W) = 4\sum_{s\in S} p^n m_s p^{n'} m'_s$. Thus,

$$h^2(W,\mathbb{Q}_l) = \frac{1}{2}(e(W) - 2 + 2h^1(W) + h^3(W)) = -1 + 2\sum_{s\in S} m_s p^n m'_s p^{n'}.$$

The rank of the Néron–Severi group, which is equal to the Picard group, may be computed by the localization sequence as the sum of the rank of the Picard group of the generic fiber plus the contribution from the components of the closed fibers [Sch88, 3.2]. The contribution from the generic fiber is two since Y and Y' are not isogenous and each has Mordell–Weil rank zero [Bea81, § 4.A]. The only relation of rational equivalence among the components of the closed fibers is that all fibers are rationally equivalent. This fact may be proved by intersecting fiber components with the fiber components of a sufficiently general very ample hypersurface in W and applying a standard result on the intersection matrix of the fiber components of a fibered surface [Liu02, 9.1.23]. It follows that the contribution from the closed fibers is $2\sum_{s\in S} m_s p^n m'_s p^{n'} - 3$. Thus, $\rho(W) = h^2(W, \mathbb{Q}_l)$. □

The proposition shows that there is no naive obstruction in l-adic cohomology to the unirationality of W. However, it is not even known whether W is uniruled. In contrast, the threefold of Hirokado is unirational as it was constructed by desingularizing a quotient of $\mathbb{P}^3_{\mathbb{F}_3}$ by the action of an appropriate vector field [Hir99]. The threefolds of Schröer and Ekedahl admit pencils of $K3$ surfaces with Picard number 22, so-called supersingular $K3$ surfaces, which in characteristic three are Kummer surfaces. These are uniruled since supersingular $K3$ surfaces over an algebraically closed field of characteristic two are unirational [RS78] as are supersingular Kummer surfaces over an algebraically closed field of characteristic three [Shi77, Proposition 8].

Unirationality or uniruledness of W would have consequences for the structure of the Chow group. Since $h^3(W, \mathbb{Q}_l) = 0$, the l-adic Abel–Jacobi map is zero. Is the Chow group of nullhomologous one cycles also zero?

13. Arithmetic degeneration of rigid Calabi–Yau threefolds

Let V be a smooth, projective threefold with trivial canonical sheaf and no first-order deformations defined over a field of characteristic zero. We may and do assume that V is defined over a number field, K. By deformation theory $0 = H^1(V, T_{V/K}) \simeq H^2(V, \Omega_{V/K})$. By Hodge theory and comparison theorems, $\dim(H^3(V_{\bar{K}}, \mathbb{Q}_l)) = 2$. It is interesting to ask how the degenerations of V might relate to the classical reduction theory of elliptic curves. A crude form of this reduction theory may be stated as follows: let $\mathfrak{o}$ be the strict Henselization of the integers of a number field K at a place $\mathfrak{p}$ and let $\mathbf{K}$ be the fraction field of $\mathfrak{o}$. Let $\bar{\mathbf{K}}$ be an algebraic closure of $\mathbf{K}$ and let $I \subset \mathrm{Gal}(\bar{\mathbf{K}}/\mathbf{K})$ denote the inertia group. Let l be a prime distinct from the residue characteristic.

Theorem 13.1. *An elliptic curve $E/\mathbf{K}$ has a relatively minimal regular projective model $\mathcal{E}$ over $\mathfrak{o}$, which is unique up to isomorphism.*

(i) *If I acts trivially on $H^\bullet(E_{\bar{K}}, \mathbb{Q}_l)$, then $\mathcal{E}$ is smooth over $\mathfrak{o}$.*

(ii) *If I acts non-trivially, but unipotently, then the closed fiber of $\mathcal{E}$ is an n-gon of projective lines.*

(iii) *After replacing $\mathfrak{o}$ by a finite extension if necessary, either (i) or (ii) holds.*

A number of rigid threefolds with trivial canonical sheaf may be constructed as fiber products of rational semi-stable elliptic surfaces [Sch04b] and [Sch88, § 7]. The few examples studied so far admit a smooth, projective regular model at places where the inertia group acts trivially even when the elliptic fibrations degenerate. At places where the inertial action on $H^\bullet(V_{\bar{\mathbf{K}}}, \mathbb{Q}_l)$ is non-trivial, but unipotent, one is frequently able to construct a projective regular model in which a desingularized fiber product of semi-stable rational elliptic surfaces with vanishing third Betti number appears as a component of the closed fiber. A first impression is that such varieties may be playing a role in three dimensions which is analogous to the role played by n-gons of projective lines in dimension one. The author hopes to return to this question in the future.

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Remark 13.2. It is interesting to note that reducing rigid Calabi–Yau varieties (or related varieties) modulo certain primes of bad reduction is a technique for producing smooth projective threefolds over a finite field which do not lift to characteristic zero.

14. Fiber products of more general elliptic surfaces

Fiber products of semi-stable elliptic surfaces are attractive because it is very easy to resolve the singularities. On the other hand it has become apparent that fiber products of non-semi-stable elliptic surfaces give further insight into the issues raised in the previous four sections. For example, such fiber products may be used to construct a one-dimensional family of smooth, projective threefolds with trivial canonical sheaf and $h^3 = 0$. In the context of § 12 the construction yields examples of threefolds which can be shown to be inseparably uniruled although not separably uniruled. This construction also arises naturally in the study of degenerations of rigid Calabi–Yau threefolds. The author hopes to say more about these more general fiber products in forthcoming work. Fiber products involving quasi-elliptic surfaces have been studied by Hirokado [Hir01].

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Note added in Proof.

Since this article was submitted for publication the author has become aware of the following related work:

S. Cynk and D. van Straten, *Small resolutions and non-liftable Calabi–Yau threefolds*, Preprint (2008), arXiv:0804.0668.

This preprint contains an answer to Open Question 10.2 and a different approach to finding obstructions to lifting.

The following two articles contain further constructions of Calabi–Yau threefolds with vanishing third Betti number in characteristics two and three.

M. Hirokado, H. Ito and N. Saito, *Calabi–Yau threefolds arising from fiber products of rational quasi-elliptic surfaces. I.* Ark. Mat. **45** (2007), 279–296.

M. Hirokado, H. Ito and N. Saito, *Calabi–Yau threefolds arising from fiber products of rational quasi-elliptic surfaces. II.* Manuscripta Math. **125** (2008), 325–343.

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