INTEGRAL OF AN E-FUNCTION EXPRESSED AS A SUM OF TWO E-FUNCTIONS

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§ 1. *Introductory.* The formula to be proved is

$$\frac{1}{\Gamma(\rho_{q+1}-\alpha_{p+1})}\int_0^1 t^{-\rho_{q+1}}(1-t)^{\rho_{q+1}-\alpha_{p+1}-1}E(p\,;\ \alpha_r:q\,;\ \rho_s:xt)\,dt$$
$$=\frac{\sin(\alpha_{p+1}\pi)}{\sin(\rho_{q+1}\pi)}E(p+1\,;\ \alpha_r:q+1\,;\ \rho_s:x)$$
$$+\frac{\sin(\alpha_{p+1}-\rho_{q+1})\pi}{\sin(\rho_{q+1}\pi)}x^{\rho_{q+1}-1}E(p+1\,;\ \alpha_r-\rho_{q+1}+1:2-\rho_{q+1},\rho_1-\rho_{q+1}+1,\ldots,\rho_q-\rho_{q+1}+1:x),\ (1)$$

where $\rho_{q+1}-\alpha_{p+1}>0$, $\alpha_r-\rho_{q+1}+1>0$, $r=1, 2, \ldots, p$, and $p\geqq q+1$.

The following formulae are required in the proof:

$$E(p\,;\ \alpha_r:q\,;\ \rho_s:x)=\sum_{r=1}^{p}P(\alpha_r\,;\ p-1\,;\ \alpha_s:q\,;\ \rho_t:x),\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where $p\geqq q+1$ and

$$P(\alpha_r\,;\ p-1\,;\ \alpha_s:q\,;\ \rho_t:x)=\frac{\prod_{s=1}^{p}{}'\,\Gamma(\alpha_s-\alpha_r)}{\prod_{t=1}^{q}\Gamma(\rho_t-\alpha_r)}\Gamma(\alpha_r)\,x^{\alpha_r}$$
$$\times F\begin{Bmatrix}\alpha_r,\alpha_r-\rho_1+1,\ldots,\alpha_r-\rho_q+1\,;\ (-1)^{p-q}x\\ \alpha_r-\alpha_1+1,\ldots*\ldots,\alpha_r-\alpha_p+1\end{Bmatrix},\ \ldots\ldots\ldots\ldots\ldots(3)$$

$r=1, 2, 3, \ldots, p$; when $p=q+1$, $|x|<1$:

$$\sin(\gamma\pi)\sin(\rho-\alpha)\pi+\sin(\gamma-\rho)\pi\sin(\alpha\pi)=\sin(\rho\pi)\sin(\gamma-\alpha)\pi.\ \ldots\ldots\ldots\ldots\ldots(4)$$

§ 2. *Proof of the Formula.* On applying (2) on the L.H.S. of (1), it becomes $\sum_{r=1}^{p}I_r$, where

$$I_r=\frac{\Gamma(\alpha_r-\rho_{q+1}+1)}{\Gamma(\alpha_r-\alpha_{p+1}+1)}\frac{\prod_{s=1}^{p}{}'\,\Gamma(\alpha_s-\alpha_r)}{\prod_{t=1}^{q}\Gamma(\rho_t-\alpha_r)}\Gamma(\alpha_r)\,x^{\alpha_r}$$
$$\times F\begin{Bmatrix}\alpha_r,\alpha_r-\rho_1+1,\ldots,\alpha_r-\rho_{q+1}+1\,;\ (-1)^{p-q}x\\ \alpha_r-\alpha_1+1,\ldots*\ldots,\alpha_r-\alpha_{p+1}+1\end{Bmatrix}$$

$r=1, 2, 3, \ldots, p$.

Now the R.H.S. of (1) is equal to

$$\sum_{r=1}^{p}\left[\frac{\sin(\alpha_{p+1}\pi)}{\sin(\rho_{q+1}\pi)}\frac{\sin(\rho_{q+1}-\alpha_r)\pi}{\sin(\alpha_{p+1}-\alpha_r)\pi}+\frac{\sin(\alpha_{p+1}-\rho_{q+1})\pi}{\sin(\rho_{q+1}\pi)}\frac{\sin(\alpha_r\pi)}{\sin(\alpha_{p+1}-\alpha_r)\pi}\right]I_r$$
$$+\left[\frac{\sin(\alpha_{p+1}\pi)}{\sin(\rho_{q+1}\pi)}+\frac{\sin(\alpha_{p+1}-\rho_{q+1})\pi}{\sin(\rho_{q+1}\pi)}\frac{\sin(\alpha_{p+1}\pi)}{\sin(\rho_{q+1}-\alpha_{p+1})\pi}\right]P(\alpha_{p+1}\,;\ p\,;\ \alpha_s:q+1\,;\ \rho_t:x).$$

Here the last term is zero, and the sum of the remaining terms, by (4), is $\sum_{r=1}^{p}I_r$. Thus the formula has been proved.

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