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PRODUCTS OF REFLECTIONS IN THE GROUP $SO^*(2n)$

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Introduction. Let $SO^*(2n)$ be the group of quaternionic $n \times n$ matrices A satisfying $A^*JA = J$, where J is a fixed skew-hermitian invertible matrix. An element $R \in SO^*(2n)$ is called a reflection if $R^2 = I_n$ and $R - I_n$ has rank one. We assume that $n \geqq 2$, in which case $S^*(2n)$ is generated by reflections. The length of $A \in SO^*(2n)$ is the smallest integer $k(\geqq 0)$ such that A can be written as $A = R_1R_2 \dots R_k$ where $R_1, \dots, R_k$ are reflections. In this paper, for each $A \in SO^*(2n)$, we compute its length $l(A)$. Set $r(A) = \text{rank } (A - I_n)$. Already in Section 3 we are able to show that the difference $\delta = l(A) - r(A)$ can take only three values 0, 1, or 2. The remainder of the paper deals with the problem of separating these three possibilities. The main results are stated in Section 4 and proved in Section 6. The intermediate Section 5 consists of a sequence of lemmas which are needed for the proof. Clearly $l(A)$ depends only on the conjugacy class of A and the main results in Section 4 are stated in terms of conjugacy classes. For the description of conjugacy classes in $SO^*(2n)$ we refer the reader to [**1**]. The present paper relies heavily on our previous paper [**5**] where the analogous problem was solved for the groups $U(p, q)$. It is worth remarking that only the various lemmas from that paper were used but not the main theorem.

So far we have solved the above problem for the following groups: $U(n)$ in [**3**], $Sp(n)$ in [**4**], $U(p, q)$ in [**5**], and $O(p, q)$ in [**6**] (see also [**2**]). The groups $U(p, q)$ contain two conjugacy classes of reflections and the problem is solved for a single conjugacy class of reflections as well as for the set of all reflections. In the case of $O(p, q)$ in both [**6**] and [**2**] only one conjugacy class of reflections is used. If one makes use of all reflections then a more general result is known [**7**].

Finally let us mention that the same problem for the groups $Sp(p, q)$ seems to be much harder and is still open.

By 1, i, j, k we denote the standard basis of the real quaternions **H**.

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1. Reflections and the basic equation. We denote by $SO^*(2n)$ the group of automorphisms of the n-dimensional right quaternionic vector space $V = \mathbf{H}^n$ which preserve a fixed nondegenerate skew-hermitian form $f: V \times V \to \mathbf{H}$. We remark that the choice of f is not important because any such two forms are equivalent. If a basis of V is chosen then f is represented by a matrix J which is skew-hermitian and invertible. An $n \times n$ quaternionic matrix A belongs to $SO^*(2n)$ if and only if $A^*JA = J$.

An element $r \in SO^*(2n)$ is called a *reflection* if r fixes pointwise a nondegenerate hyperplane W and $r(a) = -a$ for $a \in W^\perp$. If $a \in W^\perp \setminus \{0\}$ is fixed then we have

$$r(x) = x - 2af(a, a)^{-1}f(a, x)$$

for all x. Note that $f(a, a) \neq 0$ because W is nondegenerate. Since r is determined by a, we shall write $r = r_a$. Since $f(a, a)$ is a nonzero pure quaternion, we can normalize a so that $f(a, a) = i$.

Reflections form a single conjugacy class of $SO^*(2n)$ and they generate $SO^*(2n)$ if $n \geqq 2$. We shall denote by $l(u)$ the *length* of $u \in SO^*(2n)$ with respect to this generating set. Explicitly, $l(u)$ is the smallest integer k such that u can be expressed as a product $r_1 r_2 \ldots r_k$ of reflections. (For $u = 1$ we have $l(u) = 0$.) As the title suggests, the purpose of this paper is to compute $l(u)$ for all elements $u \in SO^*(2n)$.

If $u(x) = x\lambda$ for some nonzero vector x and $\lambda \in \mathbf{H}$ then we say that λ is an eigenvalue of u. More precisely, λ should be replaced by the conjugacy class $\{\mu\lambda\mu^{-1}: \mu \in \mathbf{H}^*\}$ but it will be convenient to restrict λ to be a complex number. If $\lambda \in \mathbf{C}$ then the intersection of this conjugacy class with $\mathbf{C}$ is $\{\lambda, \bar{\lambda}\}$ and for that reason we shall view λ and $\bar{\lambda}$ as the same eigenvalue of u. The *multiplicity* of the eigenvalue λ is equal to

$$\dim \operatorname{Ker} (u-\lambda)^n \quad \text{if } \lambda \in \mathbf{R}$$

and to

$$\dim \operatorname{Ker} (u^2 - (\lambda + \bar{\lambda})u + |\lambda|^2)^n \quad \text{if } \bar{\lambda} \neq \lambda.$$

The sum of multiplicities of all eigenvalues of u is equal to n. If 1 is the unique eigenvalue of u then u is *unipotent*, i.e., $(u - 1)^n = 0$.

For $u \in SO^*(n)$ we shall write

$$E(u) = \operatorname{Ker} (u - 1) \quad \text{and}$$

$$r(u) = n - \dim E(u) = \dim \operatorname{Im} (1 - u).$$

We conclude this section with two elementary but important lemmas.

LEMMA (1.1). *Let $u \in SO^*(2n)$, $a \in V$, $f(a, a) = i$, and $v = r_a u$. Then*

(i) $\operatorname{Re} \operatorname{tr} v = \operatorname{Re} \operatorname{tr} u + 2 \operatorname{Re} (i.f(a, u(a)))$,
(ii) $E(u)^\perp = \operatorname{Im} (1 - u)$,

(iii) $a \notin E(u)^{\perp} \Rightarrow E(v) = E(u) \cap a^{\perp}$ *and* $r(v) = r(u) + 1$,
(iv) $a \in E(u)^{\perp} \Rightarrow E(v) \supset E(u)$ *and* $r(v) = r(u)$ *or* $r(u) - 1$.

Proof. (i) follows from $v(x) = u(x) + 2aif(a, u(x))$. The proofs of the other three parts are easy modifications of the proof of [**5**, Lemma (4.2)].

LEMMA (1.2). *Let* $u \in SO^*(2n)$, $x \in V$, *and* $a = (1 - u)x$. *Then the following are equivalent*:
(i) $f(x, (1 - u)x) = i/2$,
(ii) $f(a, a) = i$ *and* $r(r_a u) = r(u) - 1$.

Moreover if (i) *holds then* $E(r_a u) = E(u) \oplus x\mathbf{H}$.

Proof. This is an easy modification of the proof of [**5**, Theorem (4.3)].

We shall refer to equation (i) of Lemma (1.2) as the *basic equation* of u. As an immediate consequence of Lemma (1.1), we have

(1.3) $\quad l(u) \geqq r(u), \forall\, u \in SO^*(2n)$.

2. SO^*-types and conjugacy classes. We consider triples (V, f, u) where V is a finite-dimensional right $\mathbf{H}$-vector space, $f: V \times V \to \mathbf{H}$ a nondegenerate skew-hermitian form and $u: V \to V$ an $\mathbf{H}$-linear automorphism of V such that

$$f(u(x), u(y)) = f(x, y) \quad \text{for all } x, y \in V.$$

Two such triples (V, f, u) and (V', f', u') are said to be equivalent, and we write

$$(V, f, u) \approx (V', f', u'),$$

if there exists an $\mathbf{H}$-linear isomorphism $v: V \to V'$ such that $v \circ u = u' \circ v$ and

$$f'(v(x), v(y)) = f(x, y) \quad \text{for all } x, y \in V.$$

Clearly $\approx$ is an equivalence relation and the corresponding equivalence classes are called *types*. More precisely, these types will be called *SO^*-types*. Besides these types we shall also need types introduced in our previous paper [**5**] where we studied the length problem for complex unitary groups $U(p, q)$. We shall refer to those types as *U-types*.

As in [**5**] we can transfer various properties of triples (V, f, u) to their types Δ. Thus if $(V, f, u) \in \Delta$ then $\dim \Delta = \dim V$, $l(\Delta) = l(u)$, $r(\Delta) = r(u)$, an eigenvalue of Δ is simply an eigenvalue of u, Δ is *unipotent* if u is unipotent, Δ is *trivial* if $u = 1$, and $\Delta = 0$ if $\dim V = 0$. Given two types Δ' and Δ'' one defines their *sum* $\Delta = \Delta' + \Delta''$ in the obvious way. In that case we say that Δ *contains* Δ' and write $\Delta \supset \Delta'$ or $\Delta' \subset \Delta$. A type Δ is *indecomposable* if $\Delta \neq 0$ and $\Delta = \Delta' + \Delta''$ implies that $\Delta' = 0$ or $\Delta'' = 0$.

Every type is uniquely expressible as a sum of indecomposable types, see [1]. Thus if $\Delta \supset \Delta'$ there is a unique Δ'' such that $\Delta = \Delta' + \Delta''$ and we write $\Delta'' = \Delta - \Delta'$.

We shall now describe all indecomposable SO^*-types. They are denoted as follows:

(2.1) $\Delta_m(\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1})$, $\lambda \neq 0$, $|\lambda| \neq 1$;

(2.2) $\Delta_m^\epsilon(\lambda, \bar{\lambda})$, $|\lambda| = 1$, $\operatorname{Im} \lambda \neq 0$, $\epsilon = \pm$;

(2.3) $\Delta_m^\epsilon(1, 1)$, $\Delta_m^\epsilon(-1, -1)$, $\epsilon = \pm$, m odd;

(2.4) $\Delta_m(1, 1)$, $\Delta_m(-1, -1)$, m even;

where $\lambda \in \mathbf{C}$ and $m(\geqq 0)$ is an integer. When we write $\Delta_m^\epsilon(\lambda, \bar{\lambda})$ it should be understood that $\epsilon = \pm$, $\lambda \in \mathbf{C}$, $|\lambda| = 1$, and if $\lambda = \pm 1$ that m is odd. By definition we have

$$\Delta_m^\epsilon(\lambda, \bar{\lambda}) = \Delta_m^\epsilon(\bar{\lambda}, \lambda).$$

Similar remarks apply to the types (2.1).

Given a SO^*-type Δ we choose $(V, f, u) \in \Delta$ and a basis of V. Let A (resp. J) be the matrix of u (resp. f), with respect to the chosen basis. Then $J^* = -J$, $A^*JA = J$, and we say that the matrix pair (A, J) *represents* the triple (V, f, u) (and the type Δ).

The type (2.1) is represented by

$$(2.5) \quad A = \begin{pmatrix} B & O \\ O & B^{*-1} \end{pmatrix} \quad \text{and} \quad J = i\begin{pmatrix} O & I \\ I & O \end{pmatrix},$$

where B is a Jordan block of size $m + 1$ with eigenvalue λ.

For the types $\Delta_m^\epsilon(\lambda, \bar{\lambda})$ we have

$$\Delta_m^\epsilon(\lambda, \bar{\lambda}) = \Delta_m^\epsilon(\bar{\lambda}, \lambda)$$

and so we may (and we do) assume that $\operatorname{Im} \lambda \geqq 0$. Since $|\lambda| = 1$, we can choose $\zeta \in i\mathbf{R}$ such that $\lambda = \exp \zeta$. Then the types (2.2)-(2.4) are represented by the pair (A, J) where $A = \exp X$ and X and J are matrices of size $m + 1$ given by

$$(2.6) \quad X = \begin{pmatrix} \zeta & & & & 0 \\ i & \zeta & & & \\ & i & \ddots & & \\ & & \ddots & \zeta & \\ & & & i & \zeta \end{pmatrix}, J = i^\epsilon \begin{pmatrix} & & & & 1 \\ & 0 & & 1 & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & \cdot^{\cdot^{\cdot}} & & 0 & \\ 1 & & & & \end{pmatrix}.$$

In the case (2.4) one can choose $\epsilon = \pm$ arbitrarily.

Assume that $(V, f, u) \in \Delta_m^\epsilon(\lambda, \bar{\lambda})$ and that we want to compute ϵ from the triple (V, f, u). This can be done by using the following lemma. Set

$$v = \begin{cases} \dfrac{u-1}{u+1}(\lambda + \bar{\lambda} - u - u^{-1})^m, & \lambda \neq \pm 1, \\ (-1)^{(m+1)2}\lambda(\lambda - u)^m, & \lambda = \pm 1. \end{cases}$$

LEMMA (2.7) *The form g defined by* $g(x, y) = \epsilon f(x, v(y))$ *is hermitian positive semi-definite and nonzero.*

Proof. We may assume that $\operatorname{Im} \lambda \geqq 0$. Then (V, f, u) is represented by the pair (A, J) given above. A simple computation shows that the matrix of $\lambda + \bar{\lambda} - u - u^{-1}$ (resp. $\lambda - u$) has the form

$$\begin{pmatrix} 0 & & & & 0 \\ \mu & 0 & & & \\ & \mu & 0 & & \\ & & & \ddots & \\ & * & & & 0 \\ & & & \mu & 0 \end{pmatrix}$$

where $\mu = i(\bar{\lambda} - \lambda)$ (resp. $\mu = -i\lambda$). It follows that the matrix B of v has all entries zero except the entry ν in the bottom left hand corner which is equal to $\mu^m(\lambda - 1)(\lambda + 1)^{-1}$ if $\lambda \neq \pm 1$, and to i if $\lambda = \pm 1$. The matrix ϵJB of g is diagonal with all diagonal entries zero except the first which is equal to $-i\nu$. Since $\operatorname{Im} \lambda \geqq 0$, we have $-i\nu > 0$ which completes the proof.

Now let $(V, f, u) \in \Delta$ where Δ is a U-type. Thus V is a finite-dimensional complex vector space, $f: V \times V \rightarrow \mathbf{C}$ is a nondegenerate hermitian form and $u: V \rightarrow V$ is a $\mathbf{C}$-linear automorphism such that

$$f(u(x), u(y)) = f(x, y) \quad \text{for all } x, y \in V.$$

Let V' be the right $\mathbf{H}$-vector space $V' = V \otimes_{\mathbf{C}} \mathbf{H}$. There is a skew-hermitian form $f': V' \times V' \rightarrow \mathbf{H}$ which is characterized by

$$f'(x \otimes \lambda, y \otimes \mu) = -\bar{\lambda}\, if(x, y)\mu$$

where $x, y \in V$ and $\lambda, \mu \in \mathbf{H}$. It is easy to check that it is nondegenerate. Let $u': V' \rightarrow V'$ be the $\mathbf{H}$-linear automorphism such that

$$u'(x \otimes \lambda) = u(x) \otimes \lambda.$$

Then u' preserves the form f' and the triple (V', f', u') determines an SO^*-type Δ'. The type Δ' depends only on Δ and we say that Δ' is the *quaternionization* of Δ. If Δ is indecomposable it turns out that Δ' is also indecomposable. Moreover every indecomposable SO^*-type is the quaternionization of an indecomposable U-type, but the latter is not unique in general. We now recall the notation for indecomposable U-types:

$$\Delta_m(\lambda, \bar{\lambda}^{-1}), \quad \lambda \neq 0, |\lambda| \neq 1;$$

and

$$\Delta_m^{\epsilon}(\lambda), \quad |\lambda| = 1, \epsilon = \pm;$$

where $m(\geqq 0)$ is an integer and $\lambda \in \mathbf{C}$. Whenever we write $\Delta_m(\lambda, \bar{\lambda}^{-1})$ it will be understood that $\lambda \in \mathbf{C}$, $\lambda \neq 0$, and $|\lambda| \neq 1$. Similarly, when we write $\Delta_m^{\epsilon}(\lambda)$ it should be understood that $\epsilon = \pm$, $\lambda \in \mathbf{C}$, and $|\lambda| = 1$. Furthermore, we have

$$\Delta_m(\lambda, \bar{\lambda}^{-1}) = \Delta_m(\bar{\lambda}^{-1}, \lambda).$$

For the explicit description of these types we refer to our paper [**5**].

LEMMA (2.8). *If Δ is an indecomposable U-type then its quaternionization Δ' is the SO^*-type given in the last column of Table* 1.

TABLE 1

Δ	Restrictions	Δ'
$\Delta_m(\lambda, \bar{\lambda}^{-1})$	$\lambda \neq 0, \lvert\lambda\rvert \neq 1$	$\Delta_m(\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1})$
$\Delta_m^{\epsilon}(\lambda)$	$\lvert\lambda\rvert = 1, \operatorname{Im} \lambda > 0$	$\Delta_m^{\epsilon}(\lambda, \bar{\lambda})$
$\Delta_m^{\epsilon}(\lambda)$	$\lvert\lambda\rvert = 1, \operatorname{Im} \lambda < 0$	$\Delta_m^{(-1)^{m-1}\epsilon}(\lambda, \bar{\lambda})$
$\Delta_m^{\epsilon}(\lambda)$	$\lambda = \pm 1$, m odd	$\Delta_m^{\epsilon}(\lambda, \lambda)$
$\Delta_m^{\epsilon}(\lambda)$	$\lambda = \pm 1$, m even	$\Delta_m(\lambda, \lambda)$

Proof. The assertion is obvious for the first and last row of the table. Let Δ be the U-type represented by the matrix pair (A, iJ) where $A = \exp X$ and X and J are as in (2.6). We set $\lambda = \exp \zeta$. An easy computation shows that the matrix

$$\epsilon(-i\bar{\lambda})^m iJ(A - \lambda I)^m$$

has all entries zero except the one in the upper left hand corner which is equal to 1. Hence by [**5**, Section 1] we have $\Delta = \Delta_m^{\epsilon}(\lambda)$. The quaternionization Δ' of Δ is represented by the matrix pair (A, J). Hence if $\operatorname{Im} \lambda \geqq 0$ then, by definition, we have $\Delta' = \Delta_m^{\epsilon}(\lambda, \bar{\lambda})$. If $\operatorname{Im} \lambda < 0$ then the matrix ϵJB from the proof of Lemma (2.7) has the nonzero entry

$$-i\nu = -i\frac{\lambda - 1}{\lambda + 1}[i(\bar{\lambda} - \lambda)]^m.$$

Thus $(-1)^{m-1}(-i\nu) > 0$ and consequently

$$\Delta' = \Delta_m^{(-1)^{m-1}\epsilon}(\lambda, \bar{\lambda}).$$

In closing this section we mention the connection between types and conjugacy classes. Let $u, u' \in SO^*(2n)$ and let f be the form on $V(=\mathbf{H}^n)$ which defines $SO^*(2n)$. Then u and u' are conjugate in $SO^*(2n)$ if and only if $(V, f, u) \approx (V, f, u')$. Thus there is a bijection between the conjugacy classes of $SO^*(2n)$ and the SO^*-types Δ satisfying $\dim \Delta = n$.

3. Pseudo-loxodromic types. Let Δ be an SO^*-type, $(V, f, u) \in \Delta$, $a \in V$ a non-isotropic vector and $v = r_a u$. Then $(V, f, v) \in \Delta'$ for some SO^*-type Δ'. In this situation we shall write $u \to v$ and $\Delta \to \Delta'$. By Lemma (1.1) we have

$$|r(\Delta') - r(\Delta)| \leqq 1$$

and so we can refine the concept $\Delta \to \Delta'$ by writing:

$$\Delta \xrightarrow{+} \Delta' \text{ if } r(\Delta') = r(\Delta) - 1,$$

$$\Delta \xrightarrow{0} \Delta' \text{ if } r(\Delta') = r(\Delta), \text{ and}$$

$$\Delta \xrightarrow{-} \Delta' \text{ if } r(\Delta') = r(\Delta) + 1.$$

Similar notations will be used for $u \to v$.

We say that Δ is *loxodromic* if it has an eigenvalue λ with $|\lambda| \neq 1$. We say that λ is *pseudo-loxodromic* if there exists a sequence

$$\Delta = \Delta^{(0)} \xrightarrow{+} \Delta^{(1)} \xrightarrow{+} \dots \xrightarrow{+} \Delta^{(k)} = \Delta'$$

with $k \geqq 0$ and Δ' loxodromic. We say that Δ is *effective* if $\Delta \not\supset \Delta_0(1, 1)$, or equivalently if $E(u)$ is totally isotropic. Any type Δ contains a unique effective type Δ_e such that

$$\Delta = \Delta_e + m\Delta_0(1, 1) \quad \text{for some } m (\geqq 0).$$

We say that Δ_e is the *effective part* of Δ.

LEMMA (3.1). *For an SO^*-type Δ we have*
(i) $l(\Delta) = r(\Delta)$ *if Δ is pseudo-loxodromic,*
(ii) $l(\Delta) \leqq r(\Delta) + 2$ *if* $\dim \Delta \geqq 2$.

Proof. (i) We have $r(\Delta) \geqq 2$. If $r(\Delta) = 2$ then

$$\Delta_e = \Delta_0(\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1}) \quad \text{for some } \lambda$$

and Δ_e is represented by the pair (A, J) where

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda}^{-1} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

We have $A = R_1 R_2$ where

$$R_1 = \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} \quad \text{and} \quad R_2 = \begin{pmatrix} 0 & j\bar{\lambda}^{-1} \\ -j\lambda & 0 \end{pmatrix}$$

are reflections in $SO^*(4)$, and so $l(\Delta) = r(\Delta) = 2$. Now let $r(\Delta) > 2$; we proceed by induction on $r(\Delta)$. If Δ is loxodromic then it follows from **[5**, Lemmas (8.1) and (8.5)], via quaternionization, that $\Delta \xrightarrow{+} \Delta'$ with Δ' loxodromic. Otherwise we have $\Delta \xrightarrow{+} \Delta'$ with Δ' pseudo-loxodromic, by definition of these types. Hence in both cases, the induction hypothesis gives $l(\Delta') = r(\Delta')$, and so

$$l(\Delta) \leqq l(\Delta') + 1 = r(\Delta).$$

In view of (1.3) this proves (i).

(ii) Since $\dim \Delta \geqq 2$, it follows from [**5**, Lemma (4.6) (i)] that $\Delta \rightarrow \Delta'$ with Δ' loxodromic. Hence

$$l(\Delta) \leqq l(\Delta') + 1 = r(\Delta') + 1 \leqq r(\Delta) + 2.$$

In view of (1.3) and Lemma (3.1) (ii) we have

$$l(\Delta) - r(\Delta) = 0, 1, \text{ or } 2 \quad \text{if } \dim \Delta \geqq 2.$$

The rest of this paper is devoted to the computation of this difference. The first part of Lemma (3.1) gives the answer when Δ is pseudo-loxodromic. This raises the question of recognizing which types Δ are pseudo-loxodromic. It is clear that if $\Delta \supset \Delta'$ and Δ' is pseudo-loxodromic then Δ itself is pseudo-loxodromic. In the next lemma we supply an extensive list of pseudo-loxodromic types.

LEMMA (3.2). *The following types are pseudo-loxodromic*:

a) $\Delta_m(\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1})$;

b) $\Delta_m^\epsilon(\lambda, \bar{\lambda})$, $\lambda \neq \pm 1$, $m \geqq 2$;

c) $\Delta_{2m-1}^\epsilon(1, 1)$, $m \geqq 2$; $\Delta_{2m}(1, 1)$, $m \geqq 2$; $2\Delta_2(1, 1)$;

d) $\Delta_{2m-1}^\epsilon(-1, -1)$, $m \geqq 2$; $\Delta_{2m}(-1, -1)$, $m \geqq 1$;

e) $\Delta_2(1, 1) + \Delta_0(-1, -1)$; $\Delta_2(1, 1) + \Delta_0^\epsilon(\lambda, \bar{\lambda})$;

$$\Delta_2(1, 1) + \Delta_1^\epsilon(\lambda, \bar{\lambda}), \lambda \neq 1;$$

f) $\Delta_1^{\epsilon_1}(\lambda, \bar{\lambda}) + \Delta_1^{\epsilon_2}(\mu, \bar{\mu})$ *with restrictions* $\lambda = \mu = -1 \Rightarrow \epsilon_2 = -\epsilon_1$ *and* $(\lambda = 1$ *or* $\mu = 1) \Rightarrow (\lambda \neq \mu$ *and* $\epsilon_2 = \epsilon_1)$;

g) $\Delta_1^{\epsilon_1}(\lambda, \bar{\lambda}) + \Delta_0^{\epsilon_2}(\mu, \bar{\mu})$, $\lambda \neq 1$, *with restriction*

$$\mathrm{Re}\,(\lambda - \mu) \leqq 0 \Rightarrow \epsilon_2 = \epsilon_1;$$

h) $\Delta_1^\epsilon(\lambda, \bar{\lambda}) + \Delta_0(-1, -1)$, $\lambda \neq \pm 1$;

i) $\Delta_1^+(1, 1) + \Delta_1^-(1, 1) + \Delta_0^{\epsilon_1}(\lambda, \bar{\lambda}) + \Delta_0^{\epsilon_2}(\mu, \bar{\mu})$;

j) $\Delta_1^+(1, 1) + \Delta_1^-(1, 1) + \Delta_0(-1, -1) + \Delta_0^\epsilon(\lambda, \bar{\lambda})$;

k) $\Delta_1^+(1, 1) + \Delta_1^-(1, 1) + 2\Delta_0(-1, -1)$;

l) $\Delta_1^\epsilon(1, 1) + \Delta_0^{-\epsilon}(\lambda, \bar{\lambda}) + \Delta_0^{\epsilon_1}(\mu, \bar{\mu})$ *with restriction*

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$$\operatorname{Re}(\lambda - \mu) \leqq 0 \Rightarrow \epsilon_1 = -\epsilon;$$

m) $\Delta_1^{\epsilon}(1, 1) + \Delta_0^{-\epsilon}(\lambda, \bar{\lambda}) + \Delta_0(-1, -1)$;

n) $\Delta_0^{\epsilon}(\lambda, \bar{\lambda}) + \Delta_0^{-\epsilon}(\mu, \bar{\mu}) + \Delta_0^{\epsilon_1}(\nu, \bar{\nu})$, $\operatorname{Re} \lambda \geqq \operatorname{Re} \mu \geqq \operatorname{Re} \nu$, $\operatorname{Re} \lambda > \operatorname{Re} \nu$, *and with further restriction*

$$\operatorname{Re} \mu = \operatorname{Re} \nu \Rightarrow \epsilon_1 = -\epsilon;$$

o) $\Delta_0^{+}(\lambda, \bar{\lambda}) + \Delta_0^{-}(\mu, \bar{\mu}) + \Delta_0(-1, -1)$;

p) $2\Delta_0^{+}(\lambda, \bar{\lambda}) + 2\Delta_0^{-}(\lambda, \bar{\lambda})$.

Proof. The types a) are in fact loxodromic. Using various lemmas from [**5**] and Table 1 we can show that each of the types b)-p) is pseudo-loxodromic. We indicate in each case which lemmas of [**5**] should be used: b) Lemmas (9.1) and (9.4); c) Lemmas (9.7), (9.8), and (9.10); d) Lemmas (9.1) and (9.4); e) Lemmas (9.9) and (9.11); f) Lemmas (9.12) and (9.15); g) and h) Lemma (9.14); i) Lemmas (9.13) and (9.17); j) Lemma (9.13); k) Lemma (11.17); l) and m) Lemma (9.16); n) and o) Lemma (9.18); and p) Lemma (9.19).

We give a complete proof only in case l). We may assume (and we do) that $\operatorname{Im} \lambda > 0$ and $\operatorname{Im} \mu > 0$. Define the U-type Δ as follows: if $\operatorname{Re}(\lambda - \mu) \leqq 0$ then

$$\Delta = \Delta_1^{\epsilon}(1) + \Delta_0^{\epsilon}(\bar{\lambda}) + \Delta_0^{-\epsilon}(\mu),$$

and if $\operatorname{Re}(\lambda - \mu) > 0$ then

$$\Delta = \begin{cases} \Delta_1^{+}(1) + \Delta_0^{+}(\lambda) & + \begin{cases} \Delta_0^{-}(\bar{\mu}) \text{ if } \epsilon_1 = \epsilon = +, \\ \Delta_0^{-}(\mu) \text{ if } \epsilon_1 = -\epsilon = -, \end{cases} \\[2ex] \Delta_1^{-}(1) + \Delta_0^{+}(\lambda) & + \begin{cases} \Delta_0^{-}(\bar{\mu}) \text{ if } \epsilon_1 = -\epsilon = +, \\ \Delta_0^{-}(\mu) \text{ if } \epsilon_1 = \epsilon = -. \end{cases} \end{cases}$$

By [**5**, Lemma (9.16)] Δ is pseudo-loxodromic. Since $\epsilon_1 = -\epsilon$ if $\operatorname{Re}(\lambda - \mu) \leqq 0$, it follows from Table 1 that the type l) is the quaternionization of Δ in all cases. Since Δ is pseudo-loxodromic, this implies that l) is also pseudo-loxodromic.

If a type Δ is not pseudo-loxodromic then it cannot contain any of the types a)-p) from Lemma (3.2). The next lemma describes such types.

LEMMA (3.3). *A type Δ does not contain any of the types* a)-p) *if and only if its effective part Δ_e has one of the forms*

$$(3.4) \quad \Delta_e = m\Delta_1^{\epsilon}(1, 1) + \sum_{k=1}^{n} \Delta_0^{\epsilon}(\lambda_k, \bar{\lambda}_k)$$

$$+\begin{cases} \Delta_1^{-\epsilon}(\mu, \bar{\mu}),\ \mu \neq \pm 1,\ \mathrm{Re}\,(\lambda_k - \mu) \geqq 0\ \forall\ k, \\ \Delta_0^{-\epsilon}(\mu, \bar{\mu}),\ \mathrm{Re}\,(\lambda_k - \mu) \geqq 0\ \forall\ k, \\ p\Delta_1^{-\epsilon}(-1, -1) + q\Delta_0(-1, -1), \end{cases}$$

or

$$(3.5)\quad \Delta_e = m\Delta_1^{+}(1, 1) + n\Delta_1^{-}(1, 1) + \begin{cases} 0, \\ \Delta_2(1, 1), \\ \Delta_0^{\epsilon}(\lambda, \bar{\lambda}), \\ \Delta_0(-1, -1). \end{cases}$$

Proof. If Δ_e is given by (3.4) or (3.5) then, by inspection, we see that Δ does not contain any of the types a)-p).

Conversely let Δ be a type not containing any of the types a)-p). We shall show that Δ_e has the form (3.4) or (3.5) by examining a number of cases. When proceeding from one case to the next we shall assume that all the previous cases are ruled out.

Case 1. $\Delta \supset \Delta_2(1, 1)$. By considering the types e) we infer that

$$\Delta \not\supset \Delta_0(-1, -1);\ \Delta_0^{\epsilon}(\lambda, \bar{\lambda});\ \Delta_1^{\epsilon}(\lambda, \bar{\lambda}),\ \lambda \neq 1.$$

Since Δ does not contain any of the types a)-d), we have

$$\Delta_e = m\Delta_1^{+}(1, 1) + n\Delta_1^{-}(1, 1) + \Delta_2(1, 1).$$

Case 2. $\Delta \supset \Delta_1^{-\epsilon}(-1, -1)$. We have

$$\Delta \not\supset \Delta_1^{\pm}(\lambda, \bar{\lambda}),\ \lambda \neq \pm 1;\ \Delta_1^{\epsilon}(-1, -1);\ \Delta_1^{-\epsilon}(1, 1) \quad \text{by f)},$$

and $\Delta \not\supset \Delta_0^{-\epsilon}(\lambda, \bar{\lambda})$ by g). Consequently

$$\Delta_e = m\Delta_1^{\epsilon}(1, 1) + \sum_{k=1}^{n} \Delta_0^{\epsilon}(\lambda_k, \bar{\lambda}_k) + p\Delta_1^{-\epsilon}(-1, -1) + q\Delta_0(-1, -1).$$

Case 3. $\Delta \supset \Delta_1^{-\epsilon}(\mu, \bar{\mu}),\ \mu \neq \pm 1$. By f) we have

$$\Delta - \Delta_1^{-\epsilon}(\mu, \bar{\mu}) \not\supset \Delta_1^{-\epsilon}(1, 1);\ \Delta_1^{\pm}(\lambda, \bar{\lambda}),\ \lambda \neq \pm 1.$$

By g) we have

$$\Delta \not\supset \Delta_0^{\epsilon}(\lambda, \bar{\lambda}),\ \mathrm{Re}\,(\lambda - \mu) < 0;\ \Delta_0^{-\epsilon}(\lambda, \bar{\lambda}).$$

Consequently we have

$$\Delta_e = m\Delta_1^{\epsilon}(1, 1) + \sum_{k=1}^{n} \Delta_0^{\epsilon}(\lambda_k, \bar{\lambda}_k) + \Delta_1^{-\epsilon}(\mu, \bar{\mu})$$

where $\mathrm{Re}\,(\lambda_k - \mu) \geqq 0$ for all k's.

In the remaining cases we have

$$\Delta_e = m\Delta_1^{\epsilon}(1, 1) + m'\Delta_1^{-\epsilon}(1, 1) + \sum_{k=1}^{n} \Delta_0^{\epsilon}(\lambda_k, \bar{\lambda}_k)$$

$$+ \sum_{k=1}^{n'} \Delta_0^{-\epsilon}(\mu_k, \bar{\mu}_k) + q\Delta_0(-1, -1).$$

Case 4. $\Delta \supset \Delta_1^{+}(1, 1) + \Delta_1^{-}(1, 1)$. By k) we have $q = 0$ or 1. If $q = 1$ then j) implies that $n = n' = 0$ and so

$$\Delta_e = m\Delta_1^{\epsilon}(1, 1) + m'\Delta_1^{-\epsilon}(1, 1) + \Delta_0(-1, -1).$$

Now let $q = 0$. By i) we have $n + n' \leqq 1$ and so

$$\Delta_e = m\Delta_1^{\epsilon}(1, 1) + m'\Delta_1^{-\epsilon}(1, 1) + \Delta'$$

where Δ' is 0 or $\Delta_0^{\pm}(\nu, \bar{\nu})$ for some ν.

Case 5. $\Delta \supset \Delta_0(-1, -1)$. By a) we have $n = 0$ or $n' = 0$. Since ϵ is arbitrary, we may assume that $n' = 0$. If also $n = 0$ then, taking into account that $m = 0$ or $m' = 0$, we see that Δ_e has the required form. Otherwise $n > 0$ and m) implies that $m' = 0$. Hence

$$\Delta_e = m\Delta_1^{\epsilon}(1, 1) + \sum_{k=1}^{n} \Delta_0^{\epsilon}(\lambda_k, \bar{\lambda}_k) + q\Delta_0(-1, -1).$$

From now on we may assume that

$$\Delta_e = m\Delta_1^{\epsilon}(1, 1) + \sum_{k=1}^{n} \Delta_0^{\epsilon}(\lambda_k, \bar{\lambda}_k) + \sum_{k=1}^{n'} \Delta_0^{-\epsilon}(\mu_k, \bar{\mu}_k).$$

Case 6. $\Delta \supset \Delta_1^{\epsilon}(1, 1)$. By l) we have $n' \leqq 1$ and if $n' = 1$ then

$$\operatorname{Re}(\mu_1 - \lambda_k) \leqq 0 \quad \text{for } 1 \leqq k \leqq n.$$

Hence Δ_e has one of the forms (3.4).

Thus we may assume now that $m = 0$.

Case 7. $\Delta \supset \Delta_0^{\epsilon}(\lambda_1, \bar{\lambda}_1) + \Delta_0^{\epsilon}(\lambda_2, \bar{\lambda}_2)$. If $n' = 0$ the assertion is obvious, so let $n' > 0$.

We may assume that $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_{n'}$ have positive imaginary parts and that $\operatorname{Re}\lambda_1 \geqq \operatorname{Re}\lambda_2 \geqq \ldots \geqq \operatorname{Re}\lambda_n$ and $\operatorname{Re}\mu_1 \geqq \operatorname{Re}\mu_2 \geqq \ldots \geqq \operatorname{Re}\mu_{n'}$. Since

$$\Delta \supset \Delta_0^{\epsilon}(\lambda_1, \lambda_1) + \Delta_0^{\epsilon}(\lambda_n, \bar{\lambda}_n) + \Delta_0^{-\epsilon}(\mu_1, \bar{\mu}_1),$$

it follows by considering the types n) that $\operatorname{Re}\mu_1 \leqq \operatorname{Re}\lambda_n$. Thus if $n' = 1$, Δ_e has the form (3.4). If $n' > 1$ then since

$$\Delta \supset \Delta_0^{-\epsilon}(\mu_1, \bar{\mu}_1) + \Delta_0^{-\epsilon}(\mu_{n'}, \bar{\mu}_{n'}) + \Delta_0^{\epsilon}(\lambda_1, \bar{\lambda}_1),$$

the same argument shows that Re $\lambda_1 \leqq$ Re $\mu_{n'}$. Hence we have

$$\lambda_1 = \lambda_2 = \ldots = \lambda_n = \mu_1 = \ldots = \mu_{n'}.$$

This is a contradiction since Δ does not contain a type of the form p).

This completes the proof.

4. Statement of the main results. Our main result, Theorem (4.2) below, gives the value of $l(\Delta) - r(\Delta)$. As a corollary we obtain a simple description of pseudo-loxodromic and non-pseudo-loxodromic types. The proof of the theorem and its corollary are postponed until Section 6 and are based on a sequence of lemmas of Section 5.

LEMMA (4.1). *If Δ is an SO^*-type then one and only one of the following holds*:

(i) Δ_e *contains at least one of the types* a)-p) *listed in Lemma* (3.2);

(ii) $\Delta_e = m\Delta_0(-1, -1)$;

(iii) $\Delta_e = \Delta_0^+(\lambda, \bar{\lambda}) + \Delta_0^-(\lambda, \bar{\lambda})$;

(iv) $\Delta_e = m\Delta_1^+(1, 1) + m\Delta_1^-(1, 1) + \begin{cases} \Delta_0(-1, -1), & m > 0, \\ \Delta_2(1, 1), & m \geqq 0; \end{cases}$

(v) $\Delta_e = m\Delta_1^+(1, 1) + m\Delta_1^-(1, 1) + \Delta_1^\epsilon(1, 1) + \begin{cases} \Delta_0(-1, -1), \\ \Delta_2(1, 1); \end{cases}$

(vi) $\Delta_e = m\Delta_1^\epsilon(1, 1) + \sum_{k=1}^{n} \Delta_0^\epsilon(\lambda_k, \bar{\lambda}_k) + p\Delta_1^{-\epsilon}(-1, -1)$

$+ q\Delta_0(-1, -1)$ *with restrictions*

(a) *if* $p = q = 0$ *then* $n > 1$, *and*

(b) *if* $p = n = 0$ *then* $m > 0$ *and* $q > 1$;

(vii) $\Delta_e = m\Delta_1^\epsilon(1, 1) + \sum_{k=1}^{n} \Delta_0^\epsilon(\lambda_k, \bar{\lambda}_k) + \Delta_1^{-\epsilon}(\mu, \bar{\mu}), \mu \neq \pm 1$,

Re $(\lambda_k - \mu) \geqq 0$ *for all* k*'s*;

(viii) $\Delta_e = m\Delta_1^\epsilon(1, 1) + \sum_{k=1}^{n} \Delta_0^\epsilon(\lambda_k, \bar{\lambda}_k) + \Delta_0^{-\epsilon}(\mu, \bar{\mu}), n \geqq 1$,

Re $(\lambda_k - \mu) \geqq 0$ *for all* k*'s and with additional restriction*

(c) *if* $n = 1$ *and* Re $\lambda_1 =$ Re μ *then* $m > 0$;

(ix) $\Delta_e = m\Delta_1^+(1, 1) + n\Delta_1^-(1, 1), m + n > 0$;

(x) $\Delta_e = m\Delta_1^+(1, 1) + n\Delta_1^-(1, 1) + \begin{cases} \Delta_0(-1, -1), \\ \Delta_2(1, 1), \end{cases} \quad |m-n| > 1$;

(xi) $\Delta_e = m\Delta_1^+(1, 1) + n\Delta_1^-(1, 1) + \Delta_0^\epsilon(\lambda, \bar{\lambda})$.

Proof. It is evident that in cases (ii)-(xi) Δ_e has one of the forms (3.4) or (3.5). In view of Lemma (3.3) it suffices to verify that the cases (ii)-(xi) are disjoint and cover all the possible types Δ_e given by formulas (3.4) and (3.5). The verification is easy and we omit the details.

THEOREM (4.2). *Let Δ be an SO^*-type,* $\dim \Delta \geqq 2$, *and* (i)-(xi) *the cases listed in Lemma* (4.1). *Then $l(\Delta) = r(\Delta) + \delta$ where $\delta = 0$ in cases* (i)-(iv), $\delta = 1$ *in cases* (v)-(viii), *and $\delta = 2$ in cases* (ix)-(xi).

Let us say that a pseudo-loxodromic type Δ is *minimal* if $\Delta \supset \Delta'$ and $\Delta \neq \Delta'$ imply that Δ' is not pseudo-loxodromic.

COROLLARY (4.3). *We have*

(i) *a type is pseudo-loxodromic if and only if it contains one of the types* a)-p) *of Lemma* (3.2);

(ii) *the minimal types are precisely the types* a)-p) *of Lemma* (3.2);

(iii) *a type Δ is not pseudo-loxodromic if and only if Δ_e has one of the forms* (3.4) *or* (3.5).

5. Lemmas about $\Delta \rightarrow \Delta'$. Unless stated otherwise all types will be SO^*-types and $\epsilon = \pm$.

LEMMA (5.1). *Let $(V, f, u) \in \Delta$ and define the form $g: V \times V \rightarrow \mathbf{H}$ by*

$$g(x, y) = \epsilon f(x, (u^{-1} - u)y).$$

Then

(i) *g is hermitian and its radical is* $\operatorname{Ker}(u^2 - 1)$,

(ii) *$g \geqq 0$ if and only if Δ contains only the indecomposable types:*

$$\Delta_1^{\epsilon}(1, 1), \Delta_0(1, 1), \Delta_1^{-\epsilon}(-1, -1), \Delta_0(-1, -1), \Delta_0^{\epsilon}(\lambda, \bar{\lambda}).$$

Proof. (i) We have

$$\begin{aligned}\epsilon g(y, x) &= f(y, (u^{-1} - u)x) = f((u - u^{-1})y, x) \\ &= \overline{-f(x, (u - u^{-1})y)} = \overline{\epsilon g(x, y)},\end{aligned}$$

and so g is hermitian. Since f is non-degenerate, the radical of g is the kernel of $u^{-1} - u$, i.e., $\operatorname{Ker}(u^2 - 1)$.

(ii) We may assume that Δ is indecomposable. If (A, J) represents Δ then the matrix of g is

$$K = \epsilon J(A^{-1} - A).$$

If $\Delta = \Delta_m(\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1})$ we have

$$A = \begin{pmatrix} B & 0 \\ 0 & B^{*-1} \end{pmatrix}, \quad J = i\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

$$K = i\epsilon\begin{pmatrix} 0 & B - B^{-1} \\ B^* - B^{*-1} & 0 \end{pmatrix}$$

where B is a Jordan block with eigenvalue λ and of size $m + 1$. Since $B \neq B^{-1}$, K is not positive semidefinite.

Now let Δ be one of the types $\Delta_m^{\epsilon_1}(\lambda, \bar{\lambda})$, $\operatorname{Im} \lambda \geqq 0$. Recall that if $\lambda = \pm 1$ then m is odd. Then (see Section 2) A and J can be taken in the form

$$A = \begin{pmatrix} \lambda & & & 0 \\ i\lambda & \lambda & & \\ & \ddots & \ddots & \\ & & & \lambda \\ * & & i\lambda & \lambda \end{pmatrix}, \quad J = -i\epsilon_1 \begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & & \cdot & \\ & \cdot & & \\ 1 & & & 0 \end{pmatrix};$$

the matrices are of size $m + 1$. An easy computation shows that

$$K = -i\epsilon\epsilon_1 \begin{pmatrix} * & & \beta & \alpha \\ & & \alpha & \\ & \cdot^{\cdot^{\cdot}} & & \\ \beta\ \alpha & & & \\ \alpha & & & 0 \end{pmatrix}$$

where $\alpha = \lambda - \bar{\lambda}$ and $\beta = i(\lambda + \bar{\lambda})$. If $\operatorname{Im} \lambda > 0$ then $K \geqq 0$ if and only if $m = 0$ and $\epsilon_1 = \epsilon$. If $\lambda = \pm 1$ then $K \geqq 0$ if and only if $m = 1$ and $\epsilon\epsilon_1\lambda > 0$.

Finally let $\Delta = \Delta_m(\lambda, \lambda)$, $\lambda = \pm 1$, m even. Then we can take A and J as in the previous case, where now ϵ_1 may be chosen arbitrarily, say $\epsilon_1 = +$. Clearly $K \geqq 0$ if and only if $m = 0$.

This completes the proof.

LEMMA (5.2). *Let Δ be a type with*

$$\Delta_e = m\Delta_1^{\epsilon}(1, 1) + \sum_{k=1}^{n} \Delta_0^{\epsilon}(\lambda_k, \bar{\lambda}_k) + p\Delta_1^{-\epsilon}(-1, -1) + q\Delta_0(-1, -1).$$

The following are valid:

(i) *if* $\Delta \xrightarrow{+} \Delta'$ *then* $q > 0$ *and* $\Delta' = \Delta - \Delta_0(-1, -1) + \Delta_0(1, 1)$,

(ii) *if* $m + n + p > 0$ *then* $l(\Delta) > r(\Delta)$.

Proof. (i) Let $(V, f, u) \in \Delta$ and let g be the form defined in Lemma (5.1). By that lemma we have $g \geqq 0$. We can choose $x \in V$ satisfying the basic equation $f(x, (1 - u)x) = i/2$ and such that, with $a = (1 - u)x$ and $v = r_a u$, we have $(V, f, v) \in \Delta'$. Since

$$0 = f(x, (1 - u)x) + \overline{f(x, (1 - u)x)} = f(x, (u^{-1} - u)x),$$

we have $g(x, x) = 0$. Since $g \geqq 0$, we conclude that x belongs to the radical of g, i.e., $(u^2 - 1)x = 0$. Consequently, we have $u(a) = -a$ and (i) is proved.

(ii) We use induction on q. Choose Δ' such that $\Delta \to \Delta'$ and $l(\Delta) = l(\Delta') + 1$. If $r(\Delta') \geqq r(\Delta)$ (this is so by part (i) if $q = 0$) then

$$l(\Delta) \geqq r(\Delta') + 1 > r(\Delta).$$

Otherwise we have $\Delta \xrightarrow{+} \Delta'$ and, by part (i),

$$\Delta' = \Delta - \Delta_0(-1, -1) + \Delta_0(1, 1).$$

By induction hypothesis $l(\Delta') > r(\Delta')$ and again

$$l(\Delta) = l(\Delta') + 1 > r(\Delta') + 1 = r(\Delta).$$

LEMMA (5.3). *Let*

$$\Delta = k_1\Delta_2(1, 1) + k_2\Delta_1^+(1, 1) + k_3\Delta_1^-(1, 1)$$

where $k_1 \leqq 1 \leqq k_1 + k_2 + k_3$, *and* $(V, f, u) \in \Delta$. *If* W *is a hyperplane in* V *then there exists a* u-*invariant nondegenerate subspace* X *such that* $X^\perp \subset W$ *and* $u|X$ *is of type* $\Delta_2(1, 1)$ $\Delta_1^\pm(1, 1)$, *or* $\Delta_1^+(1, 1) + \Delta_1^-(1, 1)$.

Proof. We shall refer to u-invariant and u-indecomposable subspaces of V as Jordan subspaces. We claim that every 3-dimensional Jordan subspace, say Y, is nondegenerate. Otherwise the 1-dimensional subspace $Y_1 = Y \cap E(u)$ is contained in the radical of Y. Since

$$V = Y + \operatorname{Ker}(1 - u)^2 \quad \text{and} \quad Y_1 = (1 - u)^2 Y,$$

it follows that $Y_1 \perp V$, which is a contradiction.

Now let Y be a Jordan subspace containing the 1-dimensional subspace $W^\perp$ and having maximal dimension. If Y is nondegenerate we can take $X = Y$. Otherwise $\dim Y = 2$ and we choose $b \in Y$ such that $d = (1 - u)b \neq 0$. By maximality of Y we have $b \notin \operatorname{Im}(1 - u)$. Since $\operatorname{Im}(1 - u) = E(u)^\perp$, there exists $c \in E(u)$ such that $f(b, c) = 1$. Since $E(u) \subset \operatorname{Im}(1 - u)$, we can choose $a \in V$ such that $c = (1 - u)a$. From

$$(1 - u^{-1})a = -u^{-1}(1 - u)a = -u^{-1}(c) = -c,$$

we obtain

$$\begin{aligned} f(d, a) &= f((1 - u)b, a) = f(b, (1 - u^{-1})a) \\ &= -f(b, c) = -1. \end{aligned}$$

Since Y is degenerate, we have $f(b, d) = 0$. Finally

$$E(u) \subset \operatorname{Im}(1 - u) = E(u)^\perp$$

implies that the subspace $\langle c, d\rangle$ is totally isotropic. Hence the Gram matrix J of a, b, c, d has the form

$$J = \begin{pmatrix} * & & & 1 \\ & & 1 & \\ & -1 & 0 & \\ -1 & & & \end{pmatrix}.$$

It follows that the subspace $X = \langle a, b, c, d\rangle$ is 4-dimensional, u-invariant, and nondegenerate. The restriction of u to X has the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

Since $J(A^{-1} - A)$ is indefinite, Lemma (5.1) implies that $u|X$ is of type $\Delta_1^+(1, 1) + \Delta_1^-(1, 1)$.

LEMMA (5.4). *Let*

$$\Delta = m\Delta_1^+(1, 1) + n\Delta_1^-(1, 1) + \Delta_2(1, 1).$$

(i) *If* $\Delta \xrightarrow{0} \Delta'$ *then, for some* $\epsilon = \pm$,

$$\Delta' = m\Delta_1^+(1, 1) + n\Delta_1^-(1, 1) + \Delta_1^\epsilon(1, 1) + \Delta_0(-1, -1).$$

(ii) *If* $\Delta \xrightarrow{\pm} \Delta'$ *then*

$$\Delta' = m\Delta_1^+(1, 1) + n\Delta_1^-(1, 1) + 2\Delta_0(1, 1) + \Delta_0(-1, -1).$$

Proof. Choose $(V, f, u) \in \Delta$ and $a \in V$ such that $f(a, a) = i$ and, with $v = r_a u$, $(V, f, v) \in \Delta'$. By Lemma (5.3) applied to the hyperplane $W = a^\perp$, we may assume that Δ is one of the types:

$$\Delta_2(1, 1), \Delta_1^+(1, 1), \Delta_1^-(1, 1), \Delta_1^+(1, 1) + \Delta_1^-(1, 1).$$

Since $r(\Delta) \geqq r(\Delta')$, Lemma (1.1) implies that $a \in E(u)^\perp$ and consequently we must have $\Delta = \Delta_2(1, 1)$. Since $E(v) \supset E(u)$ and $E(u)$ is totally isotropic, the eigenvalue 1 of v has multiplicity $\geqq 2$. Since $u(a) - a \in E(u)$, we have

$$f(a, u(a)) = f(a, a) = i,$$

and Lemma (1.1) gives Re tr $v = 1$. Thus -1 is also an eigenvalue of v and both (i) and (ii) follow.

LEMMA (5.5). *Let* $\Delta \to \Delta'$ *and* $r(\Delta') \leqq r(\Delta)$.

(i) *If* $\Delta = m\Delta_1^+(1, 1) + n\Delta_1^-(1, 1) + \Delta_0(-1, -1)$ *then* Δ' *is one of the types*

$$(m - 1)\Delta_1^+(1, 1) + n\Delta_1^-(1, 1) + \Delta_2(1, 1),$$
$$m\Delta_1^+(1, 1) + (n - 1)\Delta_1^-(1, 1) + \Delta_2(1, 1),$$
$$m\Delta_1^+(1, 1) + n\Delta_1^-(1, 1) + \Delta_0(1, 1),$$
$$(m - 1)\Delta_1^+(1, 1) + (n - 1)\Delta_1^-(1, 1) + 2\Delta_0(1, 1) + \Delta_2(1, 1).$$

(ii) *If* $\Delta = m\Delta_1^{\epsilon}(1, 1) + n\Delta_1^{-\epsilon}(1, 1) + \Delta_0^{\epsilon}(\lambda, \bar{\lambda})$ *then* Δ' *is one of the types*

$$m\Delta_1^{\epsilon}(1, 1) + n\Delta_1^{-\epsilon}(1, 1) + \Delta_0^{-\epsilon}(-\lambda, -\bar{\lambda}),$$

$$(m + 1)\Delta_1^{\epsilon}(1, 1) + (n - 1)\Delta_1^{-\epsilon}(1, 1) + \Delta_0^{-\epsilon}(-\lambda, -\bar{\lambda}),$$

$$m\Delta_1^{\epsilon}(1, 1) + (n - 1)\Delta_1^{-\epsilon}(1, 1) + 2\Delta_0(1, 1) + \Delta_0^{-\epsilon}(-\lambda, -\bar{\lambda}).$$

Proof. By replacing λ with $\bar{\lambda}$ (if necessary) we may assume that $\epsilon \operatorname{Im} \lambda < 0$ in case (ii). For uniformity set $\lambda = -1$ in case (i). Let $(V, f, u) \in \Delta$, $a \in V$, $f(a, a) = i$, $v = r_a u$, and $(V, f, v) \in \Delta'$. Let $b \in V$ be chosen so that $b \neq 0$ and $u(b) = b\lambda$. Since

$$f(u(b), b) = f(b, u^{-1}(b)),$$

it follows that in case (ii) $f(b, b) = it$ where $t \in \mathbf{R}$, $t \neq 0$. In fact the condition $\epsilon \operatorname{Im} \lambda < 0$ implies that $t > 0$. Hence in case (ii) we may assume that $f(b, b) = i$; this is also true in case (i).

Since $r(\Delta') \leqq r(\Delta)$, Lemma (1.1) implies that $a = b\xi + c$ where $c \in E(u)$. Since

$$i = f(a, a) = \bar{\xi}f(b, b)\xi = \bar{\xi}i\xi,$$

we have $|\xi| = 1$ and $\xi \in \mathbf{C}$. Clearly we may assume that $\xi = 1$ and so $a = b + c$.

From $E(v) \supset E(u)$, $E(u) \subset E(u)^{\perp}$, and $\dim E(u) = m + n$ it follows that 1 is an eigenvalue of v of multiplicity $\geqq 2(m + n)$. Since $\dim V = 2(m + n) + 1$, there is only one additional eigenvalue of v, and so it must lie on the unit circle. Since

$$f(a, u(a)) = f(b + c, b\lambda + c) = f(b, b)\lambda = i\lambda,$$

Lemma (1.1) gives

$$\operatorname{Re} \operatorname{tr} v = 2(m + n) - \operatorname{Re} \lambda.$$

Therefore the remaining eigenvalue of v is $-\lambda$. Thus in case (i) v is unipotent. In case (ii) a simple computation shows that $v(x) = -x\lambda$ for

$$x = b + c\frac{2\lambda}{1+\lambda}.$$

Since $f(x, x) = f(b, b) = i$ and $\epsilon \operatorname{Im}(-\lambda) > 0$, it follows that

$$\Delta' \supset \Delta_0^{-\epsilon}(-\lambda, -\bar{\lambda}).$$

If $c = 0$ the assertions of the lemma are obviously true. Otherwise by applying Lemma (5.3) to the space $b^{\perp} = \operatorname{Ker}(1 - u)^2$ and its hyperplane

$W = b^{\perp} \cap a^{\perp}$, we see that it suffices to consider the cases when $m \leqq 1$ and $n \leqq 1$. We shall now treat cases (i) and (ii) separately.

(i) Let first $\Delta = \Delta_1^{\epsilon}(1, 1) + \Delta_0(-1, -1)$ where $\epsilon = \pm$. If $r(\Delta') = r(\Delta)$, then, since v is unipotent, we must have $\Delta' = \Delta_2(1, 1)$. If $r(\Delta') = r(\Delta) - 1$ then it is easy to check that the basic equation implies that $c = 0$, which we have already considered.

Now let $m = n = 1$. If $r(\Delta') = r(\Delta)$ then $E(v) = E(u) \subset E(u)^{\perp}$, and $\dim E(u) = 2$ imply that

$$\Delta' = \Delta_2(1, 1) + \Delta_1^{\mp}(1,1).$$

Now let $r(\Delta') = r(\Delta) - 1$ and so $\dim E(v) = 3$. If X is the radical of $E(v)$ then $X \subset E(u)$ and $\dim X = 1$ or 2. If $\dim X = 1$ then $\Delta' \supset 2\Delta_0(1, 1)$, and so

$$\Delta' = 2\Delta_0(1, 1) + \Delta_2(1, 1).$$

If $\dim X = 2$ then $X = E(u)$ and since $E(v) \subset E(u)^{\perp}$ we obtain $E(v) = E(u) + b\mathbf{H}$. It follows from this and Lemma (1.1) that $a \in b\mathbf{H}$, i.e., $c = 0$.

(ii) Choose $d \in V$ such that $u(d) = c + d$. The scalar $\alpha = f(d, (u - 1)d) = f(d, c)$ is real because

$$\overline{f(d, c)} = -f(c, d) = f(\,(1 - u)d, d) = f(d, (1 - u^{-1})d)$$
$$= f(d, u^{-1}(u - 1)d) = f(d, u^{-1}(c)\,) = f(d, c).$$

A simple computation shows that the vector

$$y = d - b2i\alpha(1 + \lambda)^{-1}$$

satisfies

$$(v - 1)y = c\mu, \ \mu = 1 - 2i\alpha\frac{1 - \lambda}{1 + \lambda}.$$

If $\alpha = 0$ then necessarily

$$\Delta = \Delta_1^{+}(1, 1) + \Delta_1^{-}(1, 1) + \Delta_0^{\epsilon}(\lambda, \bar{\lambda}).$$

Since $f(y, (v - 1)y) = f(y, c\mu) = \alpha\mu$, it follows that when $\alpha = 0$ we have

$$\Delta' = \Delta_1^{+}(1, 1) + \Delta_1^{-}(1, 1) + \Delta_0^{-\epsilon}(-\lambda, -\bar{\lambda}).$$

If $\alpha \neq 0$ then the subspace $W = \langle b, c, d\rangle$ is nondegenerate and since $a \in W$ we may assume that $V = W$. If $\Delta = \Delta_1^{\epsilon}(1, 1) + \Delta_0^{\epsilon}(\lambda, \bar{\lambda})$ then $\epsilon\lambda > 0$. Since $\epsilon \operatorname{Im} \lambda < 0$, we have

$$i\alpha(1 - \lambda)(1 + \lambda)^{-1} < 0$$

and so $\mu > 0$. Hence

$$\epsilon f(y, (v - 1)y) = \epsilon\alpha\mu > 0$$

and so

$$\Delta' = \Delta_1^{\epsilon}(1, 1) + \Delta_0^{-\epsilon}(-\lambda, -\bar{\lambda}).$$

If $\Delta = \Delta_1^{-\epsilon}(1, 1) + \Delta_0^{\epsilon}(\lambda, \bar{\lambda})$ then clearly Δ' is one of the types

$$\Delta_0^{-\epsilon}(-\lambda, -\bar{\lambda}) + \Delta_1^{\pm}(1, 1) \quad \text{or} \quad \Delta_0^{-\epsilon}(-\lambda, \bar{\lambda}) + 2\Delta_0(1, 1).$$

LEMMA (5.6). *Let* $\mu \in \mathbf{C}$, *with* $|\mu| = 1$ *and* $\operatorname{Im} \mu > 0$, *be fixed and let*

$$\Delta = m\Delta_1^{\epsilon}(1, 1) + \sum_{k=1}^{n} \Delta_0^{\epsilon}(\lambda_k, \bar{\lambda}_k) + p\Delta_1^{-\epsilon}(\mu, \bar{\mu}) + q\Delta_0^{-\epsilon}(\mu, \bar{\mu})$$

with $\operatorname{Re}(\lambda_k - \mu) \geqq 0$ *for all k's. If* $\Delta \xrightarrow{+} \Delta'$ *and* -1 *is an eigenvalue of* Δ' *then* $n > 0$, $q > 0$, $\operatorname{Re}(\lambda_k - \mu) = 0$ *for some k, say for* $k = n$, *and*

$$\Delta' = \Delta - \Delta_0^{\epsilon}(\lambda_n, \bar{\lambda}_n) - \Delta_0^{-\epsilon}(\mu, \bar{\mu}) + \Delta_0(-1, -1) + \Delta_0(1, 1).$$

Proof. Let $(V, f, u) \in \Delta$, $x \in V$,

(5.7) $\quad f(x, (1 - u)x) = i/2,$

$a = (1 - u)x$, $v = r_a u$, and $(V, f, v) \in \Delta'$. Let $y \in V$, $y \neq 0$, satisfy $v(y) = -y$. Then

$$u(y) = -r_a(y) = -y - 2aif(a, y),$$

i.e., $y = (1 + u)^{-1}a\xi$ with

$$\xi = -2if(a, y) = -2if(a, (1 + u)^{-1}a)\xi.$$

Since $\xi \neq 0$ we obtain

$$\begin{aligned} i/2 &= f(a, (1 + u)^{-1}a) = f((1 - u)x, (1 + u)^{-1}(1 - u)x) \\ &= f((1 - u)x, x) - 2f((1 - u)x, u(1 + u)^{-1}x) \\ &= i/2 + 2f(x, (1 + u)^{-1}(1 - u)x), \end{aligned}$$

i.e.,

(5.8) $\quad f\left(x, \dfrac{1 - u}{1 + u}\, x\right) = 0.$

By taking real parts in (5.7) we get

(5.9) $\quad f(x, (u^{-1} - u)x) = 0.$

By multiplying (5.8) by $2 + \mu + \bar{\mu}$ and subtracting from (5.9) we get

(5.10) $\quad f\left(x, \dfrac{1 - u}{1 + u}(u + u^{-1} - \mu - \bar{\mu})x\right) = 0.$

We claim that the form $g: V \times V \to \mathbf{H}$ defined by

$$g(x, y) = -\epsilon f\left(x, \frac{1 - u}{1 + u}(u + u^{-1} - \mu - \overline{\mu})y\right)$$

is hermitian positive semidefinite and its radical is

(5.11) $\quad X = \operatorname{Ker}(u - 1) + \operatorname{Ker}(u^2 - (\mu + \overline{\mu})u + 1).$

It suffices to prove this claim for each of the indecomposable types contained in Δ, i.e., we may assume that Δ is indecomposable. If the matrix pair (A, J) represents Δ then the matrix of g is

$$K = -\epsilon J \frac{I - A}{I + A}(A + A^{-1} - (\mu + \overline{\mu})I).$$

The results of computations, assuming also that $\operatorname{Im} \lambda > 0$, are given in Table 2.

TABLE 2

Δ	A	J	K
$\Delta_1^{\epsilon}(1, 1)$	$\begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}$	$-i\epsilon\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\operatorname{Re}(1 - \mu) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
$\Delta_0^{\epsilon} \lambda, \overline{\lambda})$	(λ)	$(-i\epsilon)$	$4 \operatorname{Im} \lambda \cdot \lvert 1 + \lambda\rvert^{-2} \operatorname{Re}(\lambda - \mu)$
$\Delta_0^{-\epsilon}(\mu, \overline{\mu})$	(μ)	$(i\epsilon)$	(0)
$\Delta_1^{-\epsilon}(\mu, \overline{\mu})$	$\mu\begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}$	$i\epsilon\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$2 \operatorname{Re}(1 - \mu) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Hence our claim about g is true in each of these cases.

By (5.10) we have $g(x, x) = 0$ and so $x \in X$ and

$$a = (1 - u)x \in \operatorname{Ker}(u^2 - (\mu + \overline{\mu})u + 1) = Y.$$

Write $x = x_1 + x_2$ with $x_1 \in Y$ and $x_2 \perp Y$. By replacing x with x_1, we may assume that $x \in Y$. This implies that the subspace $W = \langle x, u(x) \rangle$ is u-invariant. Note that $a = (1 - u)x$ and $y = (1 + u)^{-1}a$ are also in W. Since $v(x) = x$ and $v(y) = -y$, we have

$$r_a(x) = u(x) \neq x \quad \text{and} \quad r_a(y) = -u(y) \neq y.$$

Therefore $f(a, x) \neq 0 \neq f(a, y)$. Hence if $f(x, x) = 0$ or $f(y, y) = 0$ W is nondegenerate. The same is true if $f(x, x) \neq 0$ and $f(y, y) \neq 0$ because, by (5.8), $f(x, y) = 0$. Hence we may assume that $V = W$ and consequently

$$\Delta = \Delta_0^{\epsilon_1}(\mu, \overline{\mu}) + \Delta_0^{\epsilon_2}(\mu, \overline{\mu}).$$

It follows from (5.9) and Lemma (5.1) that $\epsilon_2 \neq \epsilon_1$ and the proof is completed.

LEMMA (5.12). *Let g and h be a hermitian and a skew-hermitian form, respectively, on a right* **H**-*vector space V. If W is a subspace of V then the set*

$$S_W = \{x \in V \setminus W: g(x, x) = 0 \text{ and } h(x, x) \neq 0\}$$

is arcwise connected.

Proof. Given $a, b \in S_W$ we have to show that they can be joined by a continuous path lying in S_W. Clearly we may assume that a and b are linearly independent and that $g(a, b)$ is real. Let $p_1 = h(a, a)$, $p_2 = h(b, b)$, $q = h(a, b)$ and note that p_1 and p_2 are nonzero pure quaternions. If $\langle a, b\rangle \cap W \neq 0$ choose $\lambda \in \mathbf{H}$ such that $a + b\lambda \in W$. Choose a pure quaternion μ such that $\mu \notin \mathbf{R}\lambda$, and p_1 and $\bar{\mu}p_2\mu$ are **R**-linearly independent. By replacing μ with $-\mu$ (if necessary) we may also assume that $q\mu - \bar{\mu}\bar{q}$ is not of the form

$$(5.13) \quad \alpha p_1 + \beta\bar{\mu}p_2\mu, \ \alpha \leqq 0, \ \beta \leqq 0, \ \alpha + \beta < 0.$$

Now set

$$(5.14) \quad x(t) = a \cos t + b\mu \sin t, \quad 0 \leqq t \leqq \pi/2.$$

Since $\mu \notin \mathbf{R}\lambda$, we have $x(t) \notin W$ for all t. Since $g(a, a) = g(b, b) = 0$, $g(a, b) \in \mathbf{R}$, and $\mu + \bar{\mu} = 0$, we have

$$f(x(t), x(t)) = 0 \quad \text{for all } t.$$

A simple computation gives

$$h(x(t), x(t)) = p_1 \cos^2 t + \bar{\mu}p_2 \mu \sin^2 t + (q\mu - \bar{\mu}\bar{q}) \sin t \cos t.$$

Since p_1 and $\bar{\mu}p_2\mu$ are **R**-linearly independent and $q\mu - \bar{\mu}\bar{q}$ is not of the form (5.13), we have

$$h(x(t), x(t)) \neq 0 \quad \text{for } 0 \leqq t \leqq \pi/2.$$

Thus we have shown that the path (5.14) lies in S_W. Since $x(0) = a$ and $x(\pi/2) = b\mu$, the proof is completed.

LEMMA (5.15). *Let $u = SO^*(2n)$, $V = \mathbf{H}^n$, and let W be a subspace of V containing $E(u)$. Let T_W be the set consisting of all vectors $a \in V\setminus(1 - u)W$ such that $f(a, a) \neq 0$ and $u \overset{+}{\rightarrow} r_a u$. Then T_W is arcwise connected.*

Proof. Define the forms g and h on V by

$$g(x, y) = f(x, (u^{-1} - u)y), \ h(x, y) = f(x, (2 - u - u^{-1})y).$$

Then g is hermitian, h is skew-hermitian and by the previous lemma the set

$$S_W = \{x \in V\backslash W : g(x, x) = 0 \text{ and } h(x, x) \neq 0\}$$

is arcwise connected. Hence it suffices to show that $T_W = (1 - u)S_W$. Since $W \supset E(u) = \text{Ker}\,(1 - u)$, we have

$$x \in V\backslash W \Rightarrow (1 - u)x \notin (1 - u)W.$$

Now the equality $T_W = (1 - u)S_W$ follows from Lemma (1.2) and the observation that

$$g(x, u) = f(x, (1 - u)x) + \overline{f(x, (1 - u)x)},$$

$$h(x, x) = f(x, (1 - u)x) - \overline{f(x, (1 - u)x)}.$$

LEMMA (5.16). *Let* $\Delta \overset{+}{\rightarrow} \Delta'$ *where*

$$\Delta = m\Delta_1^{\epsilon}(1, 1) + \sum_{k=1}^{n} \Delta_0^{\epsilon}(\lambda_k, \bar{\lambda}_k) + \begin{cases} \Delta_0^{-\epsilon}(\mu, \bar{\mu}), \\ \Delta_1^{-\epsilon}(\mu, \bar{\mu}), \end{cases}$$

and $\text{Re}\,(\lambda_k - \mu) \geqq 0$ *for all* k*'s. Then*

(i) $\Delta \not\supset \Delta_1^{-\epsilon}(-1, -1)$, $\Delta \not\supset \Delta_1^{-\epsilon}(1, 1)$;

(ii) Δ' *contains only the indecomposable types* $\Delta_1^{\epsilon}(1, 1)$, $\Delta_0(1, 1)$, $\Delta_1^{-\epsilon}(-1, -1)$, $\Delta_0(-1, -1)$, *and* $\Delta_0^{\epsilon}(\lambda, \bar{\lambda})$ *for various* λ*'s.*

Proof. (i) Let $(V, f, u) \in \Delta$ and choose $x \in V$ such that $f(x, (1 - u)x) = i/2$, and with $a = (1 - u)x$, $(V, f, r_a u) \in \Delta'$. If $\Delta \supset \Delta_1^{-\epsilon}(1, 1)$ then necessarily $n = 0$ and $a \in E(u)$. This is a contradiction since $E(u)$ is totally isotropic and $f(a, a) = i$ by Lemma (1.2). If $\Delta \supset \Delta_1^{-\epsilon}(-1, -1)$ then let g be the hermitian form defined as in Lemma (5.1). Since

$$g(y, y) = 2\,\text{Re}\,f(y, (1 - u)y) \quad \text{for all } y \in V,$$

we have $g(x, x) = 0$. By Lemma (5.1) we have $g \geqq 0$ and

$$x \in \text{Ker}\,(u^2 - 1).$$

Thus $a \in E(-u)$ and since $E(-u)$ is totally isotropic we have again a contradiction.

(ii) If $-1 \in \text{eig}\,(\Delta')$ then the assertion follows from Lemma (5.6). From now on we assume that $-1 \notin \text{eig}\,(\Delta')$.

Let $(V, f, u) \in \Delta$, $W = \text{Ker}\,(u - 1) + \text{Ker}\,(u^2 - (\mu + \bar{\mu})u + 1)$ and let T be the set of all non-isotropic vectors $a \in V$ such that $u \overset{+}{\rightarrow} r_a u$. Note that $T\backslash(1 - u)W$ is the set T_W defined in Lemma (5.15). It follows from the proof of Lemma (5.6) that for $a \in T$ the condition $a \in T_W$ is equivalent to the condition $E(-r_a u) = 0$. For $a \in T_W$ let g_a be the hermitian form defined by

$$g_a(x, y) = \epsilon f(x, (u^{-1}r_a - r_a u)y), \quad x, y \in V.$$

The radical of g_a is the subspace

$$\begin{aligned} X_a &= \operatorname{Ker}(u^{-1}r_a - r_a u) = \operatorname{Ker}(1 - (r_a u)^2) \\ &= \operatorname{Ker}(1 - r_a u) = E(r_a u). \end{aligned}$$

Hence $\dim X_a = \dim E(r_a u) = \dim E(u) + 1$ is independent of $a \in T_W$. We claim that $g_a \geqq 0$ for all $a \in T_W$. Since T_W is arcwise connected and $\dim X_a$ is constant, it suffices to verify that $g_a \geqq 0$ for some $a \in T_W$. By Lemma (5.1) this is equivalent to the claim that there exists a type Δ'' such that $\Delta \overset{+}{\to} \Delta''$, $-1 \notin \operatorname{eig}(\Delta'')$, and Δ'' contains only the indecomposable types listed in (ii). In order to prove the latter assertion we may assume that Δ is one of the types

$$\Delta_1^{-\epsilon}(\mu, \bar{\mu}), \ \mu \neq \pm 1 \quad \text{or}$$
$$\Delta_0^{\epsilon}(\lambda, \bar{\lambda}) + \Delta_0^{-\epsilon}(\mu, \bar{\mu}) \quad \text{with } \operatorname{Re}(\lambda - \mu) \geqq 0.$$

By switching μ and $\bar{\mu}$, and λ and $\bar{\lambda}$ (if necessary) we may assume that $\epsilon \operatorname{Im} \lambda > 0$ and $\epsilon \operatorname{Im} \mu < 0$.

By [**5**, Lemmas (7.3) and (7.5)] we have

$$\Delta_1^{-\epsilon}(\mu) \to \begin{array}{ll} \Delta_0^{-\epsilon}(-\mu^2) + \Delta_0^{\epsilon}(1) & , \operatorname{Re} \mu > 0, \\ \Delta_1^{\epsilon}(1) & , \operatorname{Re} \mu = 0, \\ \Delta_0(-\mu^2) + \Delta_0^{-\epsilon}(1) & , \operatorname{Re} \mu < 0, \end{array}$$

and

$$\Delta_0^{+}(\lambda) + \Delta_0^{+}(\mu) \to \Delta_0^{+}(-\lambda\mu) + \Delta_0^{+}(1), \quad \operatorname{Re}(\lambda - \mu) \geqq 0,$$

where all these types are U-types. By quaternionizing and by using Table 1 we obtain

$$\Delta_1^{-\epsilon}(\mu, \bar{\mu}) \to \begin{cases} \Delta_1^{\epsilon}(1, 1) & , \operatorname{Re} \mu = 0, \\ \Delta_0^{\epsilon}(-\mu^2, -\bar{\mu}^2) + \Delta_0(1, 1) & , \operatorname{Re} \mu \neq 0 \end{cases}$$

and

$$\Delta_0^{\epsilon}(\lambda, \bar{\lambda}) + \Delta_0^{-\epsilon}(\mu, \bar{\mu}) \to \Delta_0(1, 1)$$

$$+ \begin{cases} \Delta_0(-1, -1) & , \operatorname{Re}(\lambda - \mu) = 0, \\ \Delta_0^{\epsilon}(-\lambda\mu, -\overline{\lambda\mu},) & , \operatorname{Re}(\lambda - \mu) > 0. \end{cases}$$

Thus our claim is proved.

Now choose $a \in T_W$ such that $(V, f, r_a u) \in \Delta'$. Since $g_a \geqq 0$, the assertion in (ii) follows from Lemma (5.1).

6. Proof of the main result.

Proof of Theorem (4.2). In a remark preceding Lemma (3.2) we have observed that $\delta = l(\Delta) - r(\Delta) = 0, 1$, or 2. The assertion $\delta = 0$ in case (i) follows from Lemmas (3.1) and (3.2), in case (ii) it is trivial, in case (iii) it follows from [**5**, Lemma (7.3)], and in case (iv) it follows from [**5**, Lemmas (7.2) and (7.10)].

We shall now prove that $\delta \leqq 1$ in the cases (v)-(viii). For that purpose we may assume that in cases (vi)-(viii) we have $\operatorname{Im} \lambda_k > 0$ for all k's and $\operatorname{Im} \mu > 0$. We shall need the following results about U-types. By [**5**, Lemma (7.9)] we have

$$\Delta_1^{\epsilon}(1) + \Delta_0^{+}(-1) \rightarrow \Delta_2^{+}(1).$$

By [**5**, Lemmas (4.6) and (4.7)] we have $\Gamma \rightarrow \Delta(\lambda, \bar{\lambda}^{-1})$ for some λ, if Γ is one of the types:

$$\Delta_1^{-\epsilon}(-1),\ \Delta_0^{\epsilon}(\lambda_1) + \Delta_0^{-\epsilon}(-1),\ \Delta_0^{\epsilon}(\lambda_1) + \Delta_0^{-\epsilon}(\bar{\lambda}_2),$$

$$\Delta_0^{\epsilon}(\lambda_1) + \Delta_0^{-\epsilon}(\mu),\ \Delta_1^{-\epsilon}(\mu).$$

By quaternionization we obtain that

$$(6.1) \qquad \Delta_1^{\epsilon}(1, 1) + \Delta_0(-1, -1) \rightarrow \Delta_2(1, 1),$$

and

$$(6.2) \qquad \Gamma' \rightarrow \Delta(\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1})$$

if Γ' is one of the types

$$(6.3) \quad \begin{cases} \Delta_1^{-\epsilon}(-1, -1),\ \Delta_0^{\epsilon}(\lambda_1, \bar{\lambda}_1) + \Delta_0(-1, -1),\ \Delta_0^{\epsilon}(\lambda_1, \bar{\lambda}_1) + \Delta_0^{\epsilon}(\lambda_2, \bar{\lambda}_2), \\ \qquad \Delta_0^{\epsilon}(\lambda_1, \bar{\lambda}_1) + \Delta_0^{-\epsilon}(\mu, \bar{\mu}),\ \Delta_1^{-\epsilon}(\mu, \bar{\mu}). \end{cases}$$

In case (v) it follows from (6.1) that $\Delta \xrightarrow{0} \Delta'$ where Δ_e' is as in (iii). Hence in that case

$$l(\Delta) \leqq l(\Delta') + 1 = r(\Delta') + 1 = r(\Delta) + 1.$$

In case (vi), if $p = n = 0$ it follows from (6.1) that we have

$$\Delta \xrightarrow{0} \Delta' \supset \Delta_0(-1, -1) + \Delta_2(1, 1).$$

By the assertion in case (i) we have then $l(\Delta') = r(\Delta')$ and so $\delta \leqq 1$. In case (vi) with $p + n > 0$ and in cases (vii) and (viii) Δ contains at least one of the types (6.3). Hence in these cases we have $\Delta \xrightarrow{0} \Delta'$ with Δ' loxodromic and so $\delta \leqq 1$.

It remains to prove that $\delta > 0$ in cases (v)-(viii) and $\delta > 1$ in cases (ix)-(xi). In case (vi) Lemma (5.2) implies that $\delta > 0$. Let us choose a sequence

$$\Delta = \Delta^{(0)} \rightarrow \Delta^{(1)} \rightarrow \dots \rightarrow \Delta^{(l)}$$

where $\Delta^{(l)}$ is the trivial type, $l = l(\Delta)$, and set $\Delta' = \Delta^{(1)}$.

(ix) In this case Lemma (1.1) implies that $r(\Delta') = r(\Delta) + 1$ and consequently $\delta > 1$.

(v) Assume that $\delta = 0$. Then we must have $\Delta^{(k)} \xrightarrow{+} \Delta^{(k+1)}$ for $0 \leqq k < l$. This is impossible by Lemmas (5.4), (5.5) and the case (ix).

(vii)-(viii) If $r(\Delta') \geqq r(\Delta)$ then

$$l(\Delta) = l(\Delta') + 1 \geqq r(\Delta') + 1 \geqq r(\Delta) + 1,$$

i.e., $\delta > 0$. Otherwise we have $\Delta \xrightarrow{+} \Delta'$. If $-1 \in \operatorname{eig}(\Delta')$ then by Lemma (5.2) (i) we are in case (viii), $\operatorname{Re}(\lambda_k - \mu) = 0$ for some k, say for $k = n$, and

$$\Delta'_e = m\Delta_1^\epsilon(1, 1) + \Delta_0(-1, -1) + \sum_{k=1}^{n-1} \Delta_0^\epsilon(\lambda_k, \bar{\lambda}_k).$$

Taking into account the restriction (c) in case (viii), we infer that $\Delta'_e \neq 0$. Then by Lemma (5.2) (ii) we have $l(\Delta') > r(\Delta')$ and so $\delta > 0$.

If $-1 \notin \operatorname{eig}(\Delta')$ then Lemma (5.16) implies that Δ' contains only the indecomposable types $\Delta_1^\epsilon(1, 1)$, $\Delta_0(1, 1)$, and $\Delta_0^\epsilon(\lambda, \bar{\lambda})$ for various λ's. Again by Lemma (5.2) (ii) we have $l(\Delta') > r(\Delta')$ and $\delta > 0$.

It remains to show that $\delta > 1$ in the cases (x) and (xi). If $r(\Delta') = r(\Delta) + 1$ then

$$l(\Delta) = l(\Delta') + 1 = r(\Delta) + 2$$

and so $\delta > 1$. Thus we may assume that $r(\Delta') \leqq r(\Delta)$, i.e., $\Delta \xrightarrow{0} \Delta'$ or $\Delta \xrightarrow{+} \Delta'$.

(x) First note that Lemmas (5.4) and (5.5) together with the case (ix) imply that $\delta > 0$. Assume that $\Delta \xrightarrow{0} \Delta'$. By Lemmas (5.4) and (5.5) we have the following facts: If

$$(6.4) \qquad \Delta_e = m\Delta_1^+(1, 1) + (n - 1)\Delta_1^-(1, 1) + \Delta_0(-1, -1)$$

then Δ'_e is one of the types

$$(m - 1)\Delta_1^+(1, 1) + n\Delta_1^-(1, 1) + \Delta_2(1, 1), \quad \text{or}$$

$$m\Delta_1^+(1, 1) + (n - 1)\Delta_1^-(1, 1) + \Delta_2(1, 1),$$

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while in case

(6.5) $\quad \Delta_e = m\overset{+}{\Delta}_1(1, 1) + n\overset{-}{\Delta}_1(1, 1) + \Delta_2(1, 1)$

we have

$$\Delta'_e = m\overset{+}{\Delta}_1(1, 1) + n\overset{-}{\Delta}_1(1, 1) + \Delta_1^{\epsilon}(1, 1) + \Delta_0(-1, -1).$$

Since $|m-n| > 1$, it follows from the case (v) and our opening note that $l(\Delta') > r(\Delta')$ and so

$$l(\Delta) = l(\Delta') + 1 > r(\Delta') + 1 = r(\Delta) + 1, \delta > 1.$$

Now assume that $\Delta \overset{+}{\rightarrow} \Delta'$. Lemmas (5.4) and (5.5) now give the following: If Δ_e is given by (6.4) then Δ'_e is one of the types

$$m\overset{+}{\Delta}_1(1, 1) + n\overset{-}{\Delta}_1(1, 1) \text{ or}$$

$$(m - 1)\overset{+}{\Delta}_1(1, 1) + (n - 1)\overset{-}{\Delta}_1(1, 1) + \Delta_2(1, 1),$$

while when Δ_e is given by (6.5) we have

$$\Delta'_e = m\overset{+}{\Delta}_1(1, 1) + n\overset{-}{\Delta}_1(1, 1) + \Delta_0(-1, -1).$$

If $\Delta'_e = m\overset{+}{\Delta}_1(1, 1) + n\overset{-}{\Delta}_1(1, 1)$ then by case (ix) we have

$$l(\Delta') = r(\Delta') + 2.$$

Otherwise Δ'_e is again of type (x) and by using induction on $r(\Delta)$ we obtain

$$l(\Delta') = r(\Delta') + 2.$$

Thus in both cases

$$l(\Delta) = l(\Delta') + 1 = r(\Delta') + 3 = r(\Delta) + 2, \delta = 2.$$

(xi) We shall prove that $\delta > 1$ by induction on $m + n$. If $\Delta \overset{+}{\rightarrow} \Delta'$ (this cannot happen when $m + n = 0$) then by Lemma (5.5) (ii) Δ'_e is of type (xi) and the induction hypothesis gives

$$l(\Delta') > r(\Delta') + 1.$$

Hence

$$l(\Delta) = l(\Delta') + 1 > r(\Delta') + 2 = r(\Delta) + 1, \delta > 1.$$

If $r(\Delta') \geqq r(\Delta)$ we obtain

$$l(\Delta) = l(\Delta') + 1 \geqq r(\Delta') + 1 > r(\Delta),$$

i.e., $\delta > 0$. Combining the two cases we see that $\delta > 0$. If $\Delta \overset{-}{\rightarrow} \Delta'$ then we have already observed that $\delta > 1$. If $\Delta \overset{0}{\rightarrow} \Delta'$ then Lemma (5.5) (ii) implies that Δ'_e is again of type (xi) and by the fact established above we have $l(\Delta') > r(\Delta')$. Hence

$$l(\Delta) = l(\Delta') + 1 > r(\Delta') + 1 = r(\Delta) + 1, \quad \delta > 1.$$

This completes the proof.

Proof of the Corollary (4.3). (i) Clearly if a type Δ contains one of the types a)-p) of Lemma (3.2) then by that lemma Δ is pseudo-loxodromic. Now assume that Δ is pseudo-loxodromic. By Lemma (3.1) we have $l(\Delta) = r(\Delta)$. By Theorem (4.2) one of the cases (i)-(iv) of Lemma (4.1) must hold. We have to show that the types specified in cases (ii)-(iv) are not pseudo-loxodromic. This is obvious in case (iii). In case (ii) this follows from Lemma (5.2) (i). In case (iv) this follows from Lemmas (5.4) and (5.5).

(ii) It follows from part (i) that the types a)-p) of Lemma (3.2) contain all minimal types. It remains to check that every type a)-p) in Lemma (3.2) is minimal. Since we know that these types are pseudo-loxodromic it suffices to prove the following: If Δ is one of the types a)-p) and $\Delta = \Delta' + \Delta''$ with Δ'' indecomposable then Δ' is not pseudo-loxodromic. Using (i) this amounts to showing that Δ' does not contain any of the types a)-p). The verification is straightforward and we omit the details.

(iii) This follows from part (i) and Lemma (3.3).

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