A THEOREM IN OPERATIONAL CALCULUS AND SOME INTEGRALS INVOLVING LEGENDRE, BESSEL AND E-FUNCTIONS

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1. In this paper we prove a theorem in Operational Calculus and use it to evaluate a few infinite integrals involving Legendre, Bessel and E-functions. We write

$$f(p) \doteqdot h(x)$$

when
$$f(p) = p \int_0^\infty e^{-px} h(x)\, dx \quad \text{.........(1)}$$

and
$$\phi(p) \overset{K}{\underset{K}{=}} h(x)$$

when
$$\phi(p) = \sqrt{(2/\pi)}p \int_0^\infty (px)^{\frac{1}{2}} K_\nu(px) h(x)\, dx. \quad \text{.........(2)}$$

(2) is a generalisation of (1) as given by Meijer [2] and it reduces to (1) when $\nu = \pm\frac{1}{2}$ by virtue of the relation

$$K_{\pm\frac{1}{2}}(z) = (\pi/2z)^{\frac{1}{2}} e^{-z}.$$

In (1), $f(p)$ is called the image of $h(x)$ which is known as the original. The following abbreviations will be used.

$$\Gamma_*(a \pm b) \equiv \Gamma(a+b)\Gamma(a-b),$$
$$\Gamma_*(a \pm b \pm c) \equiv \Gamma(a+b+c)\Gamma(a+b-c)\Gamma(a-b+c)\Gamma(a-b-c).$$

2. THEOREM.

If
$$f(p) \doteqdot h(x)$$

and
$$\phi(p) \overset{K}{\underset{K}{=}} x^{-\frac{1}{2}} h(x),$$

then
$$\int_0^\infty x^{-\nu-1}(x+z/x)^{-1} f(x+z/x)\, dx = \tfrac{1}{2}\sqrt{\pi}\, z^{-\frac{1}{2}\nu-\frac{3}{4}} \phi(2\sqrt{z}), \quad \text{.........(3)}$$

or,
$$\int_0^\infty \cosh \nu\theta \operatorname{sech} \theta\, f(p \cosh \theta)\, d\theta = (\pi/2p)^{\frac{1}{2}} \phi(p) \quad \text{.........(4)}$$

and
$$\int_0^\infty \cosh \nu\theta \operatorname{sech} \theta\, h(x \operatorname{sech} \theta)\, d\theta = (\pi/2)^{\frac{1}{2}} F(x), \quad \text{.........(5)}$$

when the integrals are convergent and $F(x) \doteqdot p^{-\frac{1}{2}}\phi(p)$.

Proof. We have

$$f(p) = p \int_0^\infty e^{-pt} h(t) dt.$$

Therefore,
$$\int_0^\infty x^{-\nu-1}(x+z/x)^{-1} f(x+z/x)\, dx = \int_0^\infty x^{-\nu-1} \left\{ \int_0^\infty e^{-(x+z/x)t} h(t)\, dt \right\} dx$$

$$=\int_0^\infty h(t)\left\{\int_0^\infty x^{-\nu-1}e^{-(x+z/x)t}\,dx\right\}dt$$

$$=2z^{-\frac{1}{2}\nu}\int_0^\infty h(t)K_\nu(2\sqrt{zt})\,dt$$

$$=\tfrac{1}{2}\sqrt{\pi}z^{-\frac{1}{2}\nu-\frac{1}{4}}\phi(2\sqrt{z}),\quad R(z)>0,$$

on changing the order of integration which we suppose to be permissible and using a well-known integral [**4**, p. 183].

(4) is obtained from (3) by the substitution $x=\sqrt{z}e^\theta$ and then replacing $2\sqrt{z}$ by p, and (5) follows from (4) on finding the originals of both sides.

3. (i) From the integral [**1**, ex. 14, p. 345]

$$\int_0^\infty e^{-px}K_{n+\frac{1}{2}}(x)x^{m-\frac{1}{2}}dx=\sqrt{(\tfrac{1}{2}\pi)}\,\Gamma(m+n+1)\Gamma(m-n)(p^2-1)^{-\frac{1}{2}m}P_n{}^m(p)$$

we find that

$$h(x)=x^{m-\frac{1}{2}}K_{n+\frac{1}{2}}(x)\doteqdot\sqrt{(\tfrac{1}{2}\pi)}\Gamma(m+n+1)\Gamma(m-n)\,p(p^2-1)^{-\frac{1}{2}m}P_n^{-m}(p) \quad\text{..................(6)}$$
$$=f(p),\quad R(m+n+1)>0,\quad R(m-n)>0,$$

and from [**1**, ex. 87, p. 367]

$$\int_0^\infty K_\nu(px)K_n(x)x^{l-1}\,dx$$
$$=\frac{\Gamma_*(\frac{1}{2}l\pm\frac{1}{2}\nu\pm\frac{1}{2}n)}{\Gamma(l)}2^{l-3}p^{-l-n}{}_2F_1(\tfrac{1}{2}l+\tfrac{1}{2}\nu+\tfrac{1}{2}n,\,\tfrac{1}{2}l-\tfrac{1}{2}\nu+\tfrac{1}{2}n\,;\;l\,;\;1-p^{-2})$$

we get

$$x^{-\frac{1}{2}}h(x)=x^{m-1}K_{n+\frac{1}{2}}(x)$$
$$\overset{K}{\underset{K}{=}}\frac{\Gamma_*(\frac{1}{2}m\pm\frac{1}{2}\nu+\frac{1}{2}n+\frac{1}{2})\Gamma_*(\frac{1}{2}m\pm\frac{1}{2}\nu-\frac{1}{2}n)}{\Gamma(\frac{1}{2})\Gamma(m+\frac{1}{2})}2^{m-2}p^{\frac{1}{2}-m-n}$$
$$\times{}_2F_1(\tfrac{1}{2}m+\tfrac{1}{2}\nu+\tfrac{1}{2}n+\tfrac{1}{2},\,\tfrac{1}{2}m-\tfrac{1}{2}\nu+\tfrac{1}{2}n+\tfrac{1}{2}\,;\;m+\tfrac{1}{2}\,;\;1-p^{-2}) \quad\text{..................(7)}$$
$$=\phi(p),\quad R(m\pm\nu+n+1)>0,\ R(m\pm\nu-n)>0.$$

Applying (3) and (4) we obtain

$$\int_0^\infty x^{-\nu-1}\{(x+z/x)^2-1\}^{-\frac{1}{2}m}P_n^{-m}(x+z/x)\,dx$$
$$=\frac{\Gamma_*(\frac{1}{2}m\pm\frac{1}{2}\nu+\frac{1}{2}n+\frac{1}{2})\Gamma_*(\frac{1}{2}m\pm\frac{1}{2}\nu-\frac{1}{2}n)}{\Gamma(\frac{1}{2})\Gamma(m+\frac{1}{2})\Gamma(m+n+1)\Gamma(m-n)}2^{-n-2}z^{-\frac{1}{2}(\nu+m+n+1)}$$
$$\times{}_2F_1\{\tfrac{1}{2}m+\tfrac{1}{2}\nu+\tfrac{1}{2}n+\tfrac{1}{2},\,\tfrac{1}{2}m-\tfrac{1}{2}\nu+\tfrac{1}{2}n+\tfrac{1}{2}\,;\;m+\tfrac{1}{2}\,;\;1-(4z)^{-1}\}, \quad\text{...............(8)}$$

$R(m\pm\nu+n+1)>0,\quad R(m\pm\nu-n)>0,\quad R(z)>0,\quad |1-(4z)^{-2}|<1.$

$$\int_0^\infty \cosh\nu\theta(p^2\cosh^2\theta-1)^{-\frac{1}{2}m}P_n^{-m}(p\cosh\theta)\,d\theta$$
$$=\frac{\Gamma_*(\frac{1}{2}m\pm\frac{1}{2}\nu+\frac{1}{2}n+\frac{1}{2})\Gamma_*(\frac{1}{2}m\pm\frac{1}{2}\nu-\frac{1}{2}n)}{\Gamma(\frac{1}{2})\Gamma(m+\frac{1}{2})\Gamma(m+n+1)\Gamma(m-n)}p^{-m-n-1}2^{m-2}$$
$$\times{}_2F_1(\tfrac{1}{2}m+\tfrac{1}{2}\nu+\tfrac{1}{2}n+\tfrac{1}{2},\,\tfrac{1}{2}m-\tfrac{1}{2}\nu+\tfrac{1}{2}n+\tfrac{1}{2}\,;\;m+\tfrac{1}{2}\,;\;1-p^{-2}), \quad\text{......................(9)}$$

$R(m \pm \nu + n + 1) > 0, \quad R(m \pm \nu - n) > 0, \quad R(p) > 0, \quad |1 - p^{-2}| < 1.$

If we take $\nu = \frac{1}{2}$ and use the relation [1, (24), p. 321]

$$P_n^{-m}(z) = \frac{(z^2-1)^{\frac{1}{2}m} 2^{-m}}{z^{m+n+1}\Gamma(m+1)} {}_2F_1(\tfrac{1}{2}m + \tfrac{1}{2}n + \tfrac{1}{2}, \tfrac{1}{2}m + \tfrac{1}{2}n + 1;\ m+1;\ 1 - z^{-2}), \quad \text{.......(10)}$$

we find that

$$\int_0^\infty \cosh(\tfrac{1}{2}\theta)(p^2\cosh^2\theta - 1)^{-\frac{1}{2}m} P_n^{-m}(p\cosh\theta)\, d\theta$$

$$= \sqrt{\frac{\pi}{2p}} \frac{\Gamma(m+n+\frac{1}{2})\Gamma(m-n-\frac{1}{2})}{\Gamma(m+n+1)\Gamma(m-n)} (p^2-1)^{\frac{1}{4}-\frac{1}{2}m} P_n^{\frac{1}{2}-m}(p), \quad \text{.....................(11)}$$

$R(m + n + \frac{1}{2}) > 0, \quad R(m - n - \frac{1}{2}) > 0, \quad R(p) > 1.$

When $p \to 1$, (9) gives

$$\int_0^\infty \cosh\nu\theta\,(\sinh\theta)^{-m} P_n{}^{-m}(\cosh\theta)\, d\theta = \frac{\Gamma_*(\frac{1}{2}m \pm \frac{1}{2}\nu + \frac{1}{2}n + \frac{1}{2})\,\Gamma_*(\frac{1}{2}m \pm \frac{1}{2}\nu - \frac{1}{2}n)}{\Gamma(\frac{1}{2})\Gamma(m+\frac{1}{2})\Gamma(m+n+1)\Gamma(m-n)} 2^{m-2}, \quad \text{.........(12)}$$

$R(m \pm \nu + n + 1) > 0, \quad R(m \pm \nu - n) > 0.$

Since

$${}_2F_1(a, b;\ c;\ z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b;\ a+b-c+1;\ 1-z)$$

$$+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} {}_2F_1(c-a, c-b;\ c-a-b+1;\ 1-z),$$

we have

$$p^{-\frac{1}{2}}\phi(p) = \frac{\Gamma_*(\frac{1}{2}m \pm \frac{1}{2}\nu + \frac{1}{2}n + \frac{1}{2})\Gamma_*(\frac{1}{2}m \pm \frac{1}{2}\nu - \frac{1}{2}n)}{\Gamma(\frac{1}{2})\Gamma(m+\frac{1}{2})} 2^{m-2} p^{-m-n}$$

$$\times {}_2F_1(\tfrac{1}{2}m + \tfrac{1}{2}\nu + \tfrac{1}{2}n + \tfrac{1}{2}, \tfrac{1}{2}m - \tfrac{1}{2}\nu + \tfrac{1}{2}n + \tfrac{1}{2};\ m + \tfrac{1}{2};\ 1 - p^{-2})$$

$$= \frac{2^{m-2}}{\Gamma(\frac{1}{2})} \Big[\Gamma_*(\tfrac{1}{2}m \pm \tfrac{1}{2}\nu + \tfrac{1}{2}n + \tfrac{1}{2})\Gamma(-n - \tfrac{1}{2}) p^{-m-n}$$

$$\times {}_2F_1(\tfrac{1}{2}m + \tfrac{1}{2}\nu + \tfrac{1}{2}n + \tfrac{1}{2}, \tfrac{1}{2}m - \tfrac{1}{2}\nu + \tfrac{1}{2}n + \tfrac{1}{2};\ n + \tfrac{3}{2};\ p^{-2})$$

$$+ \Gamma_*(\tfrac{1}{2}m \pm \tfrac{1}{2}\nu - \tfrac{1}{2}n)\,\Gamma(n + \tfrac{1}{2}) p^{n-m+1}$$

$$\times {}_2F_1(\tfrac{1}{2}m + \tfrac{1}{2}\nu - \tfrac{1}{2}n, \tfrac{1}{2}m - \tfrac{1}{2}\nu - \tfrac{1}{2}n;\ \tfrac{1}{2} - n;\ p^{-2}) \Big]$$

$$\doteqdot \frac{2^{m-2}}{\Gamma(\frac{1}{2})} \Big[\frac{\Gamma_*(\frac{1}{2}m \pm \frac{1}{2}\nu + \frac{1}{2}n + \frac{1}{2})\Gamma(-n-\frac{1}{2})}{\Gamma(m+n+1)} x^{m+n}$$

$$\times {}_2F_3(\tfrac{1}{2}m + \tfrac{1}{2}\nu + \tfrac{1}{2}n + \tfrac{1}{2}, \tfrac{1}{2}m - \tfrac{1}{2}\nu + \tfrac{1}{2}n + \tfrac{1}{2};\ \tfrac{1}{2}m + \tfrac{1}{2}n + \tfrac{1}{2}, \tfrac{1}{2}m + \tfrac{1}{2}n + 1, n + \tfrac{3}{2};\ \tfrac{1}{4}x^2)$$

$$+ \frac{\Gamma_*(\frac{1}{2}m \pm \frac{1}{2}\nu - \frac{1}{2}n)\Gamma(n+\frac{1}{2})}{\Gamma(m-n)} x^{m-n-1}$$

$$\times {}_2F_3(\tfrac{1}{2}m + \tfrac{1}{2}\nu - \tfrac{1}{2}n, \tfrac{1}{2}m - \tfrac{1}{2}\nu - \tfrac{1}{2}n;\ \tfrac{1}{2}m - \tfrac{1}{2}n, \tfrac{1}{2}m - \tfrac{1}{2}n + \tfrac{1}{2}, \tfrac{1}{2} - n;\ \tfrac{1}{4}x^2) \Big]$$

$$= F(x).$$

Hence applying (5) we have

$$\int_0^\infty \cosh \nu\theta\, (\operatorname{sech} \theta)^{m+\frac{1}{2}} K_{n+\frac{1}{2}}(x \operatorname{sech} \theta)\, d\theta$$

$$= \frac{\Gamma_*(\frac{1}{2}m \pm \frac{1}{2}\nu + \frac{1}{2}n + \frac{1}{2})\Gamma(-n-\frac{1}{2})}{\Gamma(m+n+1)} x^{n+\frac{1}{2}} 2^{m-\frac{5}{2}}$$

$$\times {}_2F_3(\tfrac{1}{2}m + \tfrac{1}{2}\nu + \tfrac{1}{2}n + \tfrac{1}{2}, \tfrac{1}{2}m - \tfrac{1}{2}\nu + \tfrac{1}{2}n + \tfrac{1}{2}\,;\, \tfrac{1}{2}m + \tfrac{1}{2}n + \tfrac{1}{2}, \tfrac{1}{2}m + \tfrac{1}{2}n + 1, n + \tfrac{3}{2}\,;\, \tfrac{1}{4}x^2)$$

$$+ \frac{\Gamma_*(\frac{1}{2}m \pm \frac{1}{2}\nu - \frac{1}{2}n)\Gamma(n+\frac{1}{2})}{\Gamma(m-n)} x^{n+\frac{1}{2}} 2^{m-\frac{5}{2}}$$

$$\times {}_2F_3(\tfrac{1}{2}m + \tfrac{1}{2}\nu - \tfrac{1}{2}n, \tfrac{1}{2}m - \tfrac{1}{2}\nu - \tfrac{1}{2}n\,;\, \tfrac{1}{2}m - \tfrac{1}{2}n, \tfrac{1}{2}m - \tfrac{1}{2}n + \tfrac{1}{2}, \tfrac{1}{2} - n\,;\, \tfrac{1}{4}x^2), \quad \text{.......(13)}$$

$R(m \pm \nu + n + 1) > 0, \quad R(m \pm \nu - n) > 0, \quad R(x) > 0.$

(ii) From the formula [**1**, (89), p. 342]

$$\int_0^\infty e^{-px} I_{n+\frac{1}{2}}(x)\, x^{m-\frac{1}{2}} dx = \surd(2/\pi)(p^2-1)^{-\frac{1}{2}m} Q_n^m(p),$$

we have

$$h(x) = x^{m-\frac{1}{2}} I_{n+\frac{1}{2}}(x)$$

$$\doteqdot \surd(2/\pi) p (p^2-1)^{-\frac{1}{2}m} Q_n^m(p) \quad \text{.......................................(14)}$$

$$= f(p), \quad R(m+n+1) > 0, \quad R(p) > 1,$$

and from [**1**, ex. 88, p. 367]

$$\int_0^\infty K_\nu(px) I_n(x) x^{l-1}\, dx = \frac{\Gamma_*(\frac{1}{2}l \pm \frac{1}{2}\nu + \frac{1}{2}n)}{\Gamma(n+1)} 2^{l-2} p^{-l-n} {}_2F_1(\tfrac{1}{2}l + \tfrac{1}{2}\nu + \tfrac{1}{2}n, \tfrac{1}{2}l - \tfrac{1}{2}\nu + \tfrac{1}{2}n;\; n+1\,;\, p^{-2}),$$

we get

$$x^{-\frac{1}{2}} h(x) = x^{m-1} I_{n+\frac{1}{2}}(x)$$

$$\overset{K}{\underset{K}{=}} \frac{\Gamma_*(\frac{1}{2}m \pm \frac{1}{2}\nu + \frac{1}{2}n + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+\frac{3}{2})} 2^{m-1} p^{\frac{1}{2}-m-n}$$

$$\times {}_2F_1(\tfrac{1}{2}m + \tfrac{1}{2}\nu + \tfrac{1}{2}n + \tfrac{1}{2}, \tfrac{1}{2}m - \tfrac{1}{2}\nu + \tfrac{1}{2}n + \tfrac{1}{2}\,;\, n + \tfrac{3}{2}\,;\, p^{-2}) \quad \text{.........(15)}$$

$$= \phi(p), \quad R(m \pm \nu + n + 1) > 0, \quad R(p) > 1.$$

Hence (3) and (4) give

$$\int_0^\infty x^{-\nu-1} \{(x + z/x)^2 - 1\}^{-\frac{1}{2}m} Q_n^m(x + z/x)\, dx$$

$$= \frac{\Gamma(\frac{1}{2})\Gamma_*(\frac{1}{2}m \pm \frac{1}{2}\nu + \frac{1}{2}n + \frac{1}{2})}{\Gamma(n+\frac{3}{2})} 2^{-n-2} z^{-\frac{1}{2}(\nu+m+n+1)}$$

$$\times {}_2F_1(\tfrac{1}{2}m + \tfrac{1}{2}\nu + \tfrac{1}{2}n + \tfrac{1}{2}, \tfrac{1}{2}m - \tfrac{1}{2}\nu + \tfrac{1}{2}n + \tfrac{1}{2}\,;\, n + \tfrac{3}{2}\,;\, 1/4z), \quad \text{......(16)}$$

$R(m \pm \nu + n + 1) > 0, \quad R(z) > \frac{1}{4}.$

$$\int_0^\infty \cosh \nu\theta \{(p \cosh \theta)^2 - 1\}^{-\frac{1}{2}m} Q_n^m(p \cosh \theta)\, d\theta$$

$$= \frac{\Gamma(\frac{1}{2})\Gamma_*(\frac{1}{2}m \pm \frac{1}{2}\nu + \frac{1}{2}n + \frac{1}{2})}{\Gamma(n+\frac{3}{2})} 2^{m-2} p^{-m-n-1}$$

$$\times {}_2F_1(\tfrac{1}{2}m + \tfrac{1}{2}\nu + \tfrac{1}{2}n + \tfrac{1}{2}, \tfrac{1}{2}m - \tfrac{1}{2}\nu + \tfrac{1}{2}n + \tfrac{1}{2}\,;\, n + \tfrac{3}{2}\,;\, p^{-2}), \quad \text{.........(17)}$$

$R(m \pm \nu + n + 1) > 0, \quad R(p) > 1.$

When $p \to 1$, we have

$$\int_0^\infty \cosh \nu\theta \, (\sinh \theta)^{-m} \, Q_n^m(\cosh \theta) \, d\theta = \frac{\Gamma_*(\frac{1}{2}m \pm \frac{1}{2}\nu + \frac{1}{2}n + \frac{1}{2}) \Gamma(\frac{1}{2} - m) \Gamma(\frac{1}{2})}{\Gamma_*(\frac{1}{2}n \pm \frac{1}{2}\nu - \frac{1}{2}m + 1)} 2^{m-2}, \quad \text{.................(18)}$$

$R(m \pm \nu + n + 1) > 0, \quad R(m) < \frac{1}{2}$.

If we take $\nu = \frac{1}{2}$ in (17) and use the relation [1, (9), p. 316]

$$Q_n^m(z) = \frac{\Gamma(\frac{1}{2}) \Gamma(n+m+1) (z^2-1)^{\frac{1}{2}m}}{2^{n+1} \Gamma(n+\frac{3}{2}) z^{n+m+1}} \, {}_2F_1(\tfrac{1}{2}n + \tfrac{1}{2}m + \tfrac{1}{2}, \tfrac{1}{2}n + \tfrac{1}{2}m + 1 \,;\, n + \tfrac{3}{2} \,;\, z^{-2}), \quad \text{.................(19)}$$

we get

$$\int_0^\infty \cosh(\tfrac{1}{2}\theta) \{(p \cosh \theta)^2 - 1\}^{-\frac{1}{2}m} Q_n^m(p \cosh \theta) \, d\theta = \sqrt{(\pi/2)} (p^2-1)^{\frac{1}{4}-\frac{1}{2}m} Q_n^{m-\frac{1}{2}}(p), \quad \text{..........(20)}$$

$R(m + n + \frac{1}{2}) > 0, \quad R(p) > 1$.

Also,

$$p^{-\frac{1}{2}} \phi(p) = \frac{\Gamma_*(\frac{1}{2}m \pm \frac{1}{2}\nu + \frac{1}{2}n + \frac{1}{2})}{\Gamma(\frac{1}{2}) \Gamma(n + \frac{3}{2})} 2^{m-1} p^{-m-n}$$
$$\times \, {}_2F_1(\tfrac{1}{2}m + \tfrac{1}{2}\nu + \tfrac{1}{2}n + \tfrac{1}{2}, \tfrac{1}{2}m - \tfrac{1}{2}\nu + \tfrac{1}{2}n + \tfrac{1}{2} \,;\, n + \tfrac{3}{2} \,;\, p^{-2})$$
$$\doteqdot \frac{\Gamma_*(\frac{1}{2}m \pm \frac{1}{2}\nu + \frac{1}{2}n + \frac{1}{2})}{\Gamma(\frac{1}{2}) \Gamma(n + \frac{3}{2}) \Gamma(m+n+1)} 2^{m-1} x^{m+n}$$
$$\times \, {}_2F_3(\tfrac{1}{2}m + \tfrac{1}{2}\nu + \tfrac{1}{2}n + \tfrac{1}{2}, \tfrac{1}{2}m - \tfrac{1}{2}\nu + \tfrac{1}{2}n + \tfrac{1}{2} \,;\, \tfrac{1}{2}m + \tfrac{1}{2}n + \tfrac{1}{2}, \tfrac{1}{2}m + \tfrac{1}{2}n + 1, n + \tfrac{3}{2} \,;\, \tfrac{1}{4}x^2)$$
$$= F(x).$$

Hence (5) gives

$$\int_0^\infty \cosh \nu\theta \, (\operatorname{sech} \theta)^{m+\frac{1}{2}} I_{n+\frac{1}{2}}(x \operatorname{sech} \theta) \, d\theta$$
$$= \frac{\Gamma_*(\frac{1}{2}m \pm \frac{1}{2}\nu + \frac{1}{2}n + \frac{1}{2})}{\Gamma(n + \frac{3}{2}) \Gamma(m+n+1)} 2^{m-\frac{3}{2}} x^{n+\frac{1}{2}}$$
$$\times \, {}_2F_3(\tfrac{1}{2}m + \tfrac{1}{2}\nu + \tfrac{1}{2}n + \tfrac{1}{2}, \tfrac{1}{2}m - \tfrac{1}{2}\nu + \tfrac{1}{2}n + \tfrac{1}{2} \,;\, \tfrac{1}{2}m + \tfrac{1}{2}n + \tfrac{1}{2}, \tfrac{1}{2}m + \tfrac{1}{2}n + 1, n + \tfrac{3}{2} \,;\, \tfrac{1}{4}x^2), \quad \text{...(21)}$$

$R(m \pm \nu + n + 1) > 0, \; R(x) > 0$.

(iii) Writing the integral [1, ex. 8, p. 344]

$$\int_0^\infty e^{-\lambda x} K_m\{\lambda \sqrt{x^2 - 1}\} \lambda^n \, d\lambda = \Gamma(n+m+1) \, Q_n^{-m}(x)$$

in the form

$$\int_0^\infty e^{-pt} t^n K_m(t) \, dt = \Gamma(n-m+1) (p^2-1)^{-\frac{1}{2}n-\frac{1}{2}} Q_n^m\{p/\sqrt{(p^2-1)}\},$$

we find that

$$h(x) = x^n K_m(x)$$
$$\doteqdot \Gamma(n-m+1) p (p^2-1)^{-\frac{1}{2}n-\frac{1}{2}} Q_n^m\{p/\sqrt{(p^2-1)}\} \quad \text{.......................(22)}$$
$$= f(p), \quad R(n \pm m + 1) > 0, \quad R(p) > 1,$$

and from (7) we have

$$x^{-\frac{1}{2}}h(x) = x^{n-\frac{1}{2}}K_m(x)$$
$$\underset{K}{\overset{K}{=}} \frac{\Gamma_*(\frac{1}{2}n \pm \frac{1}{2}\nu \pm \frac{1}{2}m + \frac{1}{2})}{\Gamma(n+1)\Gamma(\frac{1}{2})} 2^{n-\frac{3}{2}} p^{\frac{1}{2}-n-m}$$
$$\times {}_2F_1(\tfrac{1}{2}n + \tfrac{1}{2}\nu + \tfrac{1}{2}m + \tfrac{1}{2}, \tfrac{1}{2}n - \tfrac{1}{2}\nu + \tfrac{1}{2}n + \tfrac{1}{2}\ ;\ n+1\ ;\ 1-p^{-2}) \quad \ldots\ldots(23)$$
$$= \phi(p), \quad R(n \pm \nu \pm m + 1) > 0.$$

Applying (3) and (4) we get

$$\int_0^\infty x^{-\nu-1}\{(x+z/x)^2 - 1\}^{-\frac{1}{2}n-\frac{1}{2}} Q_n^m[(x+z/x)/\surd\{(x+z/x)^2-1\}]\,dx$$
$$= \frac{\Gamma_*(\frac{1}{2}n \pm \frac{1}{2}\nu \pm \frac{1}{2}m + \frac{1}{2})}{\Gamma(n+1)\Gamma(n-m+1)} z^{-\frac{1}{2}(\nu+m+n+1)} 2^{-m-2}$$
$$\times {}_2F_1\{\tfrac{1}{2}n + \tfrac{1}{2}\nu + \tfrac{1}{2}m + \tfrac{1}{2}, \tfrac{1}{2}n - \tfrac{1}{2}\nu + \tfrac{1}{2}m + \tfrac{1}{2}\ ;\ n+1\ ;\ 1-(4z)^{-1}\}, \quad \ldots\ldots(24)$$

$R(n \pm \nu \pm m + 1) > 0$, $R(z) > \frac{1}{8}$, and

$$\int_0^\infty \cosh \nu\theta\, (p^2 \cosh^2\theta - 1)^{-\frac{1}{2}n-\frac{1}{2}} Q_n^m\{p \cosh\theta/\surd(p^2\cosh^2\theta - 1\}\,d\theta$$
$$= \frac{\Gamma_*(\frac{1}{2}n \pm \frac{1}{2}\nu \pm \frac{1}{2}m + \frac{1}{2})}{\Gamma(n-m+1)\Gamma(n+1)} 2^{n-2} p^{-n-m-1}$$
$$\times {}_2F_1(\tfrac{1}{2}n + \tfrac{1}{2}\nu + \tfrac{1}{2}m + \tfrac{1}{2}, \tfrac{1}{2}n - \tfrac{1}{2}\nu + \tfrac{1}{2}m + \tfrac{1}{2}\ ;\ n+1\ ;\ 1-p^{-2}), \quad \ldots\ldots(25)$$

$R(n \pm \nu \pm m + 1) > 0$, $|\,p^2 - 1\,| < |\,p^2\,|$.

If we take $\nu = \frac{1}{2}$ and use (19) we have

$$\int_0^\infty \cosh(\tfrac{1}{2}\theta)(p^2\cosh^2\theta - 1)^{-\frac{1}{2}n-\frac{1}{2}} Q_n^m\{p\cosh\theta/\surd(p^2\cosh^2\theta - 1)\}\,d\theta$$
$$= \sqrt{\frac{\pi}{2p}}\, \frac{\Gamma(n-m+\frac{1}{2})}{\Gamma(n-m+1)} (p^2-1)^{-\frac{n}{2}-\frac{1}{4}} Q_{n-\frac{1}{2}}^m\{p/\surd(p^2-1)\}, \quad \ldots\ldots(26)$$

$R(n \pm m + \frac{1}{2}) > 0$, $|\,p^2 - 1\,| < |\,p^2\,|$.

When $p \to 1$, (25) yields

$$\int_0^\infty \cosh\nu\theta\,(\sinh\theta)^{-n-1} Q_n^m(\coth\theta)\,d\theta = \frac{\Gamma_*(\frac{1}{2}n \pm \frac{1}{2}\nu \pm \frac{1}{2}m + \frac{1}{2})}{\Gamma(n-m+1)\Gamma(n+1)} 2^{n-2}, \quad \ldots\ldots\ldots\ldots(27)$$

$R(n \pm \nu \pm m + 1) > 0$.

Note.—Results (24) to (27) may also be derived from (8) to (12) by applying Whipple's formula

$$P_n^{-m}(p) = \sqrt{\frac{2}{\pi}}\, \frac{(p^2-1)^{-\frac{1}{4}}}{\Gamma(m+n+1)} Q_{m-\frac{1}{2}}^{n+\frac{1}{2}}\{p/\surd(p^2-1)\}.$$

(iv) From the integral [3, p. 119]

$$\int_0^\infty K_\nu(x)x^{\gamma-1}E(l\ ;\ \alpha_r : m\ ;\ \rho_s : z/x^{2n})\,dx = (2\pi)^{1-n}2^{\gamma-2}n^{\gamma-1}E\{l+2n\ ;\ \alpha_r : m\ ;\ \rho_s : z/(2n)^{2n}\}$$

we have

$$x^{-\frac{1}{2}}h(x) = x^{\gamma-\frac{3}{2}}E(l\ ;\ \alpha_r : m\ ;\ \rho_s : 1/x^{2n})$$

$$\frac{K}{\overline{K}}\sqrt{(2/\pi)}\,(2\pi)^{1-n}2^{\gamma-2n}n^{\gamma-1}p^{\frac{3}{2}-\gamma}\,E\{l+2n\ ;\ \alpha_r : m\ ;\ \rho_s : (p/2n)^{2n}\} \quad\ldots\ldots\ldots\ldots\ldots(28)$$

$$=\phi(p),$$

$R(\gamma\pm\nu)>0,\quad \alpha_{l+k+1} = (\gamma+\nu+2k)/2n,$

$$\alpha_{l+n+k+1} = (\gamma-\nu+2k)/2n, \quad k=0,\ 1,\ \ldots,\ (n-1).$$

Taking $\nu=\frac{1}{2}$ in the above and replacing γ by $\gamma+\frac{1}{2}$, we have

$$h(x) = x^{\gamma-1}E(l\ ;\ \alpha_r : m\ ;\ \rho_s : 1/x^{2n})$$

$$\doteqdot \sqrt{(2/\pi)}\,(2\pi)^{1-n}2^{\gamma-\frac{3}{2}}n^{\gamma-\frac{1}{2}}p^{1-\gamma}E\{l+2n\ ;\ \alpha_r^* : m\ ;\ \rho_s : (p/2n)^{2n}\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(29)$$

$$=f(p),$$

$R(\gamma)>0,\ \alpha_q^* = \alpha_q,\ q=1,\ 2,\ \ldots,\ l\ ;\ \alpha^*_{l+k+1} = (\gamma+1+2k)/2n,\ \alpha^*_{l+n+k+1} = (\gamma+2k)/2n,\ k=0,\ 1,\ \ldots,\ (n-1).$

Applying (3) and (4) we get

$$\int_0^\infty x^{-\nu-1}(x+z/x)^{-\gamma}E[l+2n\ ;\ \alpha_r^* : m\ ;\ \rho_s : \{(x+z/x)/2n\}^{2n}]\,dx$$

$$=\sqrt{(\pi/n)}2^{-\gamma}z^{-\frac{1}{2}\nu-\frac{1}{2}\gamma}E\{l+2n\ ;\ \alpha_r : m\ ;\ \rho_s : (z/n^2)^n\}, \quad\ldots\ldots\ldots\ldots\ldots\ldots(30)$$

$R(\gamma\pm\nu)>0,\quad R(z)>0.$

$$\int_0^\infty \cosh\nu\theta\,(\cosh\theta)^{-\gamma}E\{l+2n\ ;\ \alpha_r^* : m\ ;\ \rho_s : (p\cosh\theta/2n)^{2n}\}\,d\theta$$

$$=\tfrac{1}{2}\sqrt{(\pi/n)}\,E\{l+2n\ ;\ \alpha_r : m\ ;\ \rho_s : (p/2n)^{2n}\},\ldots\ldots\ldots\ldots\ldots\ldots\ldots(31)$$

$R(\gamma\pm\nu)>0,\quad R(p)>0.$

Results (30) and (31) may be put in a more compact form thus:

$$\int_0^\infty x^{-\nu-1}(x+z/x)^{-\gamma}E[l\ ;\ \alpha_r : m\ ;\ \rho_s : \{(x+z/x)/2n\}^{2n}]\,dx$$

$$=\sqrt{(\pi/n)}2^{-\gamma}z^{-\frac{1}{2}\nu-\frac{1}{2}\gamma}E\{l+2n\ ;\ \alpha_r : m+2n\ ;\ \rho_s : (z/n^2)^n\}, \quad\ldots\ldots\ldots\ldots\ldots(32)$$

$R(\gamma\pm\nu)>0,\ R(z)>0.$

$$\int_0^\infty \cosh\nu\theta\,(\cosh\theta)^{-\gamma}E\{l\ ;\ \alpha_r : m\ ;\ \rho_s : \lambda(\cosh\theta)^{2n}\}\,d\theta$$

$$=\tfrac{1}{2}\sqrt{(\pi/n)}\,E(l+2n\ ;\ \alpha_r : m+2n\ ;\ \rho_s : \lambda), \quad\ldots\ldots\ldots\ldots\ldots\ldots(33)$$

$R(\gamma\pm\nu)>0,\ R(\lambda)>0,\ \alpha_{l+k+1} = (\gamma+\nu+2k)/2n,\ \alpha_{l+n+k+1} = (\gamma-\nu+2k)/2n,\ \rho_{m+k+1} = (\gamma+1+2k)/2n$
$\rho_{m+n+k+1} = (\gamma+2k)/2n,\ k=0,\ 1,\ \ldots,\ (n-1).$

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