SUBSPACES OF A GENERALIZED METRIC SPACE

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Introduction. In a paper published in 1956, Rund **(4)** developed the differential geometry of a hypersurface of $n - 1$ dimensions imbedded in a Finsler space of n dimensions, considered as locally Minkowskian.

The purpose of the present paper is to provide an extension of the results of **(4)** and thus develop a theory for the case of m-dimensional subspaces imbedded in a generalized (Finsler) metric space.

We consider an n-dimensional differentiable manifold X_n and we restrict our attention to a suitably chosen co-ordinate neighbourhood of X_n in which a co-ordinate system x^i $(i = 1, 2, \ldots, n)$, is defined. A system of equations of the type $x^i = x^i(t)$ defines a curve C of X_n, the tangent vector dx^i/dt of which is denoted by $\dot{x}^i$. We say that the manifold X_n is endowed with a locally Minkowskian (Finsler) metric, if the length of an arc of the curve C between two points P_1 and P_2 of C, corresponding to parameter values t_1 and t_2, is defined by an integral of the type

$$\int_{t_1}^{t_2} F(x^i, \dot{x}^i)dt,$$

where the function $F(x^i, \dot{x}^i)$ is continuous and continuously differentiable up to any required order in all its arguments, and also positively homogeneous of the first degree in the $\dot{x}^i$.

Defining the metric tensor of X_n by

$$g_{ij}(x, \dot{x}) = \tfrac{1}{2} \frac{\partial^2 F^2(x, \dot{x})}{\partial \dot{x}^i \partial \dot{x}^j}, \qquad g^{ik}(x, \dot{x}) g_{ih}(x, \dot{x}) = \delta^k_h,$$

we can put

$$F^2(x, \dot{x}) = g_{ij}(x, \dot{x}) \dot{x}^i \dot{x}^j;$$

F must satisfy a third condition,

$$g_{ij}(x, \dot{x}) \xi^i \xi^j > 0,$$

for all $\dot{x}^i$ and all ξ^i, provided not all ξ^i are equal to zero.

From Euler's theorem on homogeneous functions we have

$$\frac{\partial g_{ij}(x, \dot{x})}{\partial \dot{x}^k} \dot{x}^i = 0, \qquad \frac{\partial^2 g_{ij}(x, \dot{x})}{\partial x^h \partial \dot{x}^k} \dot{x}^i = 0.$$

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We also define the generalized Christoffel symbols of the first and second kind by the relations

$$\begin{Bmatrix} i \\ hk \end{Bmatrix}_{(x,\dot{x})} = g^{ij}(x, \dot{x})[hk, j]_{(x,\dot{x})},$$

$$[hk, j]_{(x,\dot{x})} = \tfrac{1}{2}\left(\frac{\partial g_{kj}(x, \dot{x})}{\partial x^h} + \frac{\partial g_{hj}(x, \dot{x})}{\partial x^k} - \frac{\partial g_{hk}(x, \dot{x})}{\partial x^j}\right).$$

Let C be a continuous and continuously differentiable curve. At each point P of C, with co-ordinates x^k, a Minkowskian tangent space $T_n(P)$ is defined by $F(x^k, \dot{x}^k)$. We consider an arbitrary vector field $X^i(x^k)$ along C such that in each $T_n(P)$ a vector X^i is defined. Let Q be a neighbouring point with co-ordinates $x^k + dx^k$ on C, such that the arc length $PQ = ds$. The covariant differential DX^i of X^i at P for the transition from P to Q is then defined by

$$\text{(A.3)} \qquad DX^i = \left(\frac{\partial X^i}{\partial x^k} + P^i_{hk}(x, x')X^h\right)dx^k,$$

where

$$P^i_{hk}(x, x') = \begin{Bmatrix} i \\ hk \end{Bmatrix}_{(x,x')} - \tfrac{1}{2}\, g^{im}(x, x')\, \frac{\partial g_{hm}(x, x')}{\partial x'^l} \begin{Bmatrix} l \\ pk \end{Bmatrix}_{(x,x')} x'^p,$$

and $x'^i = dx/ds$.

We note that (A.3) depends only on the vector X^i and the displacement PQ for which it has been defined, and not on the curve C passing through P and Q. On the other hand, the covariant derivative of X^i with respect to x^k is given by

$$\text{(A.4)} \qquad X^i_{,k} = \frac{\partial X^i}{\partial x^k} + P^{*i}_{hk}(x, x')x^h,$$

where **(5)**

$$P^*_{ij,k} = g_{hk}P^{*h}_{ij} \equiv [ij, k] - \tfrac{1}{2}\left(\frac{\partial g_{hj}}{\partial x'^l} P^l_{ik} + \frac{\partial g_{hi}}{\partial x'^k} P^l_{jk} - \frac{\partial g_{ij}}{\partial x'^h} P^l_{hk}\right) x'^k.$$

Consider a continuous curve C of X_n, which lies in some two-dimensional subspace X_2 of X_n, and let the parameters of X_2 be u and v. The parametric curves u = const. and v = const. may cut C in an arbitrary manner. Two directions

$$\xi^k = \frac{\partial x^k}{\partial u}, \qquad \eta^k = \frac{\partial x^k}{\partial v}$$

are defined at each point of C, and they represent the directions of the tangents to the co-ordinate curves. Then, for a vector field $X^i(x^k)$, we have in the X_2,

$$\frac{DX^i}{Du} = X^i_{,k}\xi^k, \qquad \frac{DX^i}{Dv} = X^i_{,k}\eta^k,$$

and thus, we obtain the commutation formula **(6)**,

$$\text{(A.6)} \qquad \frac{D^2X^i}{DvDu} - \frac{D^2X^i}{DuDv} = (X^i_{;nm} - X^i_{;mn})\xi^n\xi^m + X^i_{;n}(\xi^n_{;m}\eta^m - \eta^n_{;m}\xi^m).$$

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If we use the relation

$$\frac{\partial \xi^k}{\partial v} = \frac{\partial \eta^k}{\partial u},$$

we reduce (A.6) to

$$\frac{D^2X^i}{DvDu} - \frac{D^2X^i}{DuDv} = (X^i_{;nm} - X^i_{;mn})\xi^n\eta^m.$$

Introducing the expression

$$\text{(A.7)} \qquad K^i_{.hmn}(x, x') = \frac{\partial P^{*i}_{hn}}{\partial x^m} - \frac{\partial P^{*i}_{hm}}{\partial x^n} + P^{*i}_{sm}P^{*s}_{hn} - P^{*i}_{sn}P^{*s}_{hm} + \left(\frac{\partial P^{*i}_{hn}}{\partial x'^l}\frac{\partial x'^l}{\partial x^m} - \frac{\partial P^{*i}_{hm}}{\partial x'^l}\frac{\partial x'^l}{\partial x^n}\right)_{(x,x')},$$

which we call the relative curvature tensor in view of the derivative $\partial x'^l/\partial x^m$ which appears in it, we may obtain the commutation relation

$$X^i_{;nm} - X^i_{;mn} = K^i_{.hmn}X^h.$$

We also define a covariant curvature tensor from the relation

$$K_{ihmn}(x, x') = g_{ij}(x, x')\, K^j_{.hmn}(x, x');$$

then, if $Y_i(x^k)$ are the covariant components of the vector field, we may obtain the relation

$$\text{(A.8)} \qquad Y_{i,mn} - Y_{i,nm} = -K^h_{.imn}Y_h.$$

1. Generalities. Consider a differentiable subspace of m dimensions F_m, imbedded in a locally Minkowskian (Finsler) space F_n, where $m < n$. Let

$$\text{(1.1)} \qquad x^i = x^i(u^\alpha), \qquad (i = 1 \ldots n, \alpha = 1 \ldots m),$$

be the equations defining F_m. We assume that the Jacobian matrix

$$(X^i_\alpha) = \left(\frac{\partial x^i}{\partial u^\alpha}\right)$$

is of rank m.

If the co-ordinate curves are regarded as curves of the F_m, then their tangents are given by

$$X^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$$

and at each point P of F_m we have m independent vectors $\partial x^i/\partial u^\alpha$, which will span an m-dimensional plane $T_m(P) \subset T_n(P)$, where by $T_m(P)$ we mean the m-dimensional linear space tangent to F_m at P.

A vector X^i lies in F_m if $X^i \in T_m(P)$, which implies that it is of the form

$$\text{(1.2)} \qquad X^i = U^\alpha \frac{\partial x^i}{\partial u^\alpha}.$$

F_m will be endowed with an induced metric

$$ds^2 = g_{\alpha\beta}(u, u')du^\alpha du^\beta$$

with fundamental tensor given by

$$g_{\alpha\beta}(u, u') = g_{ij}(x, x') \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta}, \tag{1.3}$$

where the tangent u'^α to F_m, satisfies the relation

$$x'^i = X^i_\alpha u'^\alpha. \tag{1.4}$$

In general, we have to consider two sets of normals to F_m at a given point P of F_m. The first set is defined by the solutions n^i of the equations

$$n_i X^i_\alpha \equiv g_{ij}(x, n)n^j X^i_\alpha = 0. \tag{1.5}$$

These solutions are normalized by means of the relation

$$F(x, n) = 1 \quad \text{or} \quad g_{ij}(x, n)n^i n^j = 1. \tag{1.6}$$

Since the matrix $(X_\alpha{}^i)$ is of rank m, we have $n - m$ independent solutions and, therefore, $n - m$ independent normal vectors. They span a vector space at P, and any vector of this space will be a linear combination of the independent vectors spanning the space.

We may define a different set of normals in the following way. Let x'^i be an arbitrary but fixed direction tangential to F_m at P. A second set of normals can be defined by the solutions $n^*(x, x')$ of the equations

$$g_{ij}(x, x')n^{*j}(x, x')X^i_\alpha = 0. \tag{1.7}$$

The matrix $(X_\alpha{}^i)$ being of rank m, the system (1.7) admits $n - m$ independent solutions of the direction considered. We may write

$$n^{*i}_{(\mu)} = n^{*i}_{(\mu)}(x, x'), \qquad (\mu = 1 \ldots n - m).$$

To each direction x' tangent to F_m at P corresponds a set of vectors $n^{*i}{}_{(\mu)}$ (x, x'), and the totality of these sets, for the different x' at P, defines $n - m$ cones which are the normal cones of the subspace F_m at a given point. We must emphasize that the generators of the normal cones do not necessarily lie in the space spanned by the normals n at the same point. The concept of the normal cones for subspaces is an extension of the idea of a normal cone of a hypersurface F_{n-1} (4).

We assume as in the case of the $n(x)$, that $n^*(x, x')$ are normalized according to the relation

$$F(x, n^*(x, x')) = g_{ij}(x, n^*(x, x'))n^{*i}(x, x)n^{*j}(x, x') = 1. \tag{1.8}$$

We may also define $n - m$ tensors, independent of direction,

$$\gamma_{(\mu)\alpha\beta}(u) = g_{ij}(x, n_{(\mu)})X^i_\alpha X^j_\beta,$$

for the $n - m$ normals at P. Then we define the following sets of inverse projection parameters corresponding to $X_\beta{}^j$:

(1.9)
$$X_i^\alpha(x, x') = g_{ij}(x, x')g^{\alpha\beta}(u, u')X_\beta^j,$$
$$Y_{(\mu)i}^\alpha(x) = g_{ij}(x, n_{(\mu)})\gamma_{(\mu)}^{\alpha\beta}(u)X_\beta^j,$$

so that, in view of the equations (1.5), (1.7), we have

(1.10)
$$\overset{*}{n}_{(\mu)}X_i^\alpha = 0, \qquad Y_{(\mu)i}^\alpha n_{(\mu)}^i = 0,$$

and also

(1.10a)
$$X_i^\alpha X_\beta^i = \delta_\beta^\alpha, \qquad Y_{(\mu)i}^\alpha X_\beta^i = \delta_\beta^\alpha.$$

It is always possible to choose a set of $n - m$ orthogonal independent vectors $n^*(x, x')$. Indeed, for any vector of the space spanned by the $n^{*i}{}_{(\mu)}$ we have

$$N^{*i}(x, x') = \sum_{(\mu)} \lambda_{(\mu)}\overset{*}{n}{}_{(\mu)}^i(x, x), \qquad (\mu = 1 \ldots n - m).$$

Let us consider a set of $n - m$ such vectors; we can write down the $n - m$ relations

$$\overset{*}{N}{}_{(\nu)}^i(x, x') = \sum_{(\mu)} \lambda_{(\nu)(\mu)}\overset{*}{n}{}_{(\mu)}^i(x, x'), \qquad (\nu, \mu = 1 \ldots n - m).$$

In order that $N^{*i}{}_{(\nu)}$ should be orthogonal (with respect to $g_{ij}(x, x')$) the functions $\lambda_{(\nu)(\mu)}$ must satisfy the relations

(1.11)
$$g_{ij}(x, x')\overset{*}{N}{}_{(\nu)}^i\overset{*}{N}{}_{(\sigma)}^j = \sum_{(\mu)}\sum_{(\kappa)} g_{ij}(x, x')\lambda_{(\nu)(\mu)}\lambda_{(\sigma)(\kappa)}\overset{*}{n}{}_{(\mu)}^i(x, x')\overset{*}{n}{}_{(\kappa)}^j(x, x') = \delta(\nu)(\sigma).$$

If we put

(1.12)
$$T_{(\mu)(\kappa)}(x, x') = g_{ij}(x, x')\overset{*}{n}{}_{(\mu)}^i\overset{*}{n}{}_{(\kappa)}^j, \qquad (\mu, \kappa = 1 \ldots n - m),$$

the equation (1.11) can be written

(1.13)
$$\sum_{(\mu)}\sum_{(\kappa)} T_{(\mu)(\kappa)}\lambda_{(\nu)(\mu)}\lambda_{(\sigma)(\kappa)} = 0, \qquad \text{for } \nu \neq \sigma.$$

Our problem reduces to finding $n - m$ sets of functions $\lambda_{(\nu)(\mu)}$ satisfying the equations (1.13).

It is known that, if in a projective $(n - 1)$-dimensional space we introduce homogeneous co-ordinates, the equation of a hyperquadric has the form

(1.13a)
$$a_{kl}z_kz_l = 0,$$

and the co-ordinates x_k, y_l of two points harmonically conjugate with respect to (1.13a) satisfy the relation

$$a_{kl}x_ky_l = 0,$$

(see **(1)** for the 2-dimensional case). The problem of finding sets of functions $\lambda_{(\mu)(\nu)}$ satisfying (1.13), is equivalent to the problem of finding the vertices of polyhedra self-polar with respect to

$$\sum_{(\mu)(\kappa)} T_{(\mu)(\kappa)}\lambda_{(\mu)}\lambda_{(\kappa)} = 0.$$

One vertex P_1 of such a polyhedron can be chosen arbitrarily in the space, but not on the quadric; a second vertex P_2, arbitrarily in the polar hyperplane of P_1, but not on the quadric; a third vertex P_3, arbitrarily on the intersection of the polar planes of P_1, P_2, but not on the quadric, P_4 on the intersection of the polar planes of P_1, P_2, P_3, and so on. The last one will be on the intersection of the polar hyperplanes of all the previous points. Since $P_1, P_2, \ldots, P_{n-1}$ can be chosen with $n - m - 1, n - m - 2, \ldots, 1$ degrees of freedom respectively, there are

$$(n - m - 1) + (n - m - 2) + \ldots + 1 = \tfrac{1}{2}(n - m)(n - m - 1)$$

degrees of freedom in choosing the $n - m$ sets of functions λ.

The induced covariant derivative of the vector X^i can be defined just as for a hypersurface **(4)**. Let $x^i = x^i(s)$ be a curve C of F_m so that x'^i is tangent to F_m. We consider a continuous and continuously differentiable vector field tangent to F_m:

$$X^i(x^\kappa) = X^i_\alpha U^\alpha(u^\beta). \tag{1.16}$$

The induced covariant derivative of the vector field along C in the space F_m, that is, the tensor defined by

$$U^\beta_{,\gamma}(u, u') = \frac{\partial U^\beta}{\partial u^\gamma} + P^{*\beta}_{\delta\gamma}(u, u')U^\delta \tag{1.17}$$

is given by projection onto F_m of the covariant derivative $X_{,k}{}^i$ of X^i with respect to F_n,

$$g_{ij}(x, x')X^j_\gamma X^k_\alpha X^i_{,\kappa} = g_{\beta\gamma}(u, u')U^\beta_{,\alpha}, \tag{1.18}$$

where

$$X^i_{,k} = \frac{\partial X^i}{\partial x^k} + P^{*i}_{hk}(x, x')X^h. \tag{1.19}$$

One can prove easily that

$$g_{ij}(x, x')X^j_\gamma \left(\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + P^{*i}_{hk}X^k_\alpha X^h_\beta\right) = P^*_{\beta\alpha,\gamma}(u, u'),$$

with

$$P^*_{\beta\alpha,\gamma}(u, u') = g_{\epsilon\gamma}P^\epsilon_{\beta\alpha}.$$

It is obvious that $P^{*\gamma}{}_{\beta\alpha}$ are symmetric in the lower indices, because $P^{*i}{}_{hk}$ are symmetric.

It is very easy to show that the quantities (1.17) *form the components of a tensor, in the sense indicated by their indices, under a transformation of the co-ordinates u^α of F_m,* Eliopoulos **(3)**.

Since the subspace F_m is endowed with a metric tensor $g_{\alpha\beta}(u, u')$, we can write immediately the Euler-Lagrange equations for the geodesics of that space

$$\frac{d^2u}{ds^2} + \left\{ {\alpha \atop \beta\gamma} \right\}_{(u,u')} \frac{du^\beta}{ds}\frac{du^\gamma}{ds} = 0,$$

where

$$\left\{\begin{matrix}\alpha \\ \beta\gamma\end{matrix}\right\}_{(u,u')}$$

are the intrinsic Christoffel symbols. We may also write

$$g_{\alpha\delta}\frac{du'^{\alpha}}{ds} + [\beta\gamma, \delta]_{(u,u')}\frac{du^{\beta}}{ds}\frac{du^{\gamma}}{ds} = 0,$$

or

$$g_{\alpha\delta}\frac{du'^{\alpha}}{ds} + P^{*\delta}_{\beta\delta}u'^{\alpha}u'^{\beta} = 0.$$

We immediately see that

$$\frac{\delta u'^{\alpha}}{\delta s} = 0$$

along a geodesic, that is, the geodesics are autoparallel curves.

2. Normal curvatures of F_m. We consider a curve C of F_m, $x_1 = x_1(s)$, passing through a given point P. We take the parameter s to be the arc-length, and the unit tangent vector to C at P will be denoted by x'^i. Let us assume, for the moment, that the vector field U^{α} of equations (1.16) coincides with the tangent vectors u'^{α} of C. If we denote covariant differentiations in F_m by δ, we obtain

$$\text{(2.1)} \qquad \frac{\delta u'^{\alpha}}{\delta s} = \frac{du'^{\alpha}}{ds} + P^{*\alpha}_{\beta\gamma}(u, u')u'^{\beta}u'^{\gamma} = \frac{du'^{\alpha}}{ds} + \left\{\begin{matrix}\alpha \\ \beta\gamma\end{matrix}\right\}u'^{\beta}u'^{\gamma}.$$

By using the expression of Dx'^i/Ds and differentiating $x' = X_{\alpha}{}^{i}u'^{\alpha}$ we find

$$\frac{Dx'^i}{Ds} = \frac{\partial^2 x^i}{\partial u^{\alpha}\partial u^{\beta}}u'^{\alpha}u'^{\beta} + X^i_{\alpha}\frac{\delta u'^{\alpha}}{\delta s} - X^i_{\alpha}P^{*\alpha}_{\beta\gamma}u'^{\beta}u'^{\gamma} + P^{*\,i}_{hk}x'^{h}x'^{k}.$$

If we put

$$\text{(2.3)} \qquad X^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^{\alpha}\partial u^{\beta}} - X^i_{\gamma}P^{*\gamma}_{\alpha\beta} + P^{*\,i}_{hk}X^h_{\alpha}X^k_{\beta},$$

we may write

$$\text{(2.4)} \qquad \frac{Dx'^i}{Ds} = X^i_{\alpha\beta}u'^{\alpha}u'^{\beta} + X^i_{\alpha}\frac{\delta u'^{\alpha}}{\delta s}.$$

The expressions $X_{\alpha\beta}{}^{i}$ which are the components of a tensor, may be considered as the generalized covariant derivatives of the $X_{\alpha}{}^{i}$ with respect to u^{β}, in the sense used in **(4, 7)**. We note that $X_{\alpha\beta}{}^{i}$ are symmetric with respect to the lower indices.

The $X_{\alpha\beta}{}^{i}$ can be given the following geometric interpretation: We consider the geodesic $\bar{C}$, of the space F_n through the point P, tangent to the given direction x'^i. Let $\bar{x}^i = \bar{x}^i(s)$ be the equations of $\bar{C}$. We also consider a geodesic $\tilde{C}$ of the space F_m through the same point P, and tangent to x'^i. Let $\tilde{x}^i = \tilde{x}^i(s)$ be its equations. We choose two points one on $\bar{C}$ and the other on $\tilde{C}$ corresponding to the same value of s, and in the neighbourhood of P. The coordinates of these points can be expended in Taylor series, for small values of s, so that

$$\bar{x}^i(s) = \bar{x}^i_P + \bar{x}'^i_P s + \tfrac{1}{2}\bar{x}''^i_P s^2 + \dots$$
$$\tilde{x}^i(s) = \tilde{x}^i_P + \tilde{x}'^i_P s + \tfrac{1}{2}\tilde{x}''^i_P s^2 + \dots,$$

where by $\bar{x}_P$, $\tilde{x}_P$, etc., we mean the values of these functions at the point P. Then

$$\xi^i = \tilde{x}^i - \bar{x}^i = \tfrac{1}{2}(\tilde{x}''^i - \bar{x}''^i)s^2 + O(s^2)$$

because $\tilde{x}_P{}^i = \bar{x}_P{}^i$ and $\tilde{x}_P{}'^i = \bar{x}_P{}'^i$. From the equations of geodesics we have for $\bar{C}$

$$\frac{d\bar{x}'^i}{ds} = -\left\{ {i \atop hk} \right\}_{(c)} \bar{x}'^h \bar{x}'^k.$$

Also for $\tilde{C}$, we have

$$\frac{D\tilde{x}'^i}{Ds} = \frac{d\tilde{x}'^i}{ds} + \left\{ {i \atop hk} \right\}_{(c)} \tilde{x}'^h \tilde{x}'^k,$$

and therefore

$$\xi^i = \tfrac{1}{2}\frac{D\tilde{x}'^i}{Ds} s^2 + O(s^3).$$

In view of (2.4) applied to a geodesic of F_m we obtain

$$X^i_{\alpha\beta}(u, u')u'^\alpha u'^\beta = \lim_{s\to 0} \frac{2\xi^i}{s^2}.$$

We consider the formulae (1.5) and (1.7). Since $n_{(\mu)i}$ and $n^*_{(\nu)i}$ are solutions of the same linear equations, we may write

$$n_{(\mu)i} = \sum_{(\nu)} p_{\mu\nu} \overset{*}{n}_{(\nu)i}; \tag{2.8}$$

multiplying the above equations by $n_{(\mu)}{}^i$, and since $n_{(\mu)}{}^i$ are unit vectors, we obtain

$$\sum_{(\nu)} p_{\mu\nu} \overset{*}{n}_{(\nu)i} n^i_{(\mu)} = 1.$$

The equations (2.8) can also be written as

$$g_{ij}(x, n_{(\mu)}) n^j_{(\mu)} = \sum_{(\nu)} p_{\mu\nu} g_{ij}(x, x') \overset{*}{n}{}^j_{(\nu)}$$

and if we multiply by $n^*{}_{(\lambda)}{}^i$ we find

$$g_{ij}(x, n_{(\mu)}) n^j_{(\mu)} \overset{*}{n}{}^i_{(\lambda)} = \sum_{(\nu)} p_{\mu\nu} g_{ij}(x, x') \overset{*}{n}{}^j_{(\nu)} \overset{*}{n}{}^i_{(\lambda)} = p_{\mu\lambda}\psi_2,$$

since

$$g_{ij}(x, x') \overset{*}{n}{}^j_{(\nu)} \overset{*}{n}{}^i_{(\lambda)} = \delta^\nu_\lambda \psi_\lambda$$

(no summation over λ involved). The above relation may be written

$$\cos(n_{(\mu)}, \overset{*}{n}_{(\lambda)}) = p_{\mu\lambda}\psi_\lambda,$$

or

$$p_{\mu\lambda} = \frac{\cos\,(n_{(\mu)}, \overset{*}{n}_{(\lambda)})}{\psi_\lambda}, \tag{2.9}$$

since the cosine of the angle of two vectors $n_{(\mu)}$, $n^*_{(\lambda)}$ is defined by

$$\cos\,(n_{(\mu)}, \overset{*}{n}_{(\lambda)}) = \frac{g(x, n_{(\mu)})n^j_{(\mu)}\overset{*}{n}{}^i_{(\lambda)}}{[g_{ij}(x, n_{(\mu)})n^i_{(\mu)}n^j_{(\mu)}]^{\frac{1}{2}}[g_{ij}(x, \overset{*}{n}_{(\lambda)})\overset{*}{n}{}^i_{(\lambda)}\overset{*}{n}{}^j_{(\lambda)}]^{\frac{1}{2}}},$$

and $n_{(\mu)}$, $n^*_{(\lambda)}$ are unit vectors.

We now prove the following theorems.

THEOREM I. *The principal normal of a geodesic G of F_m lies in the space spanned by the secondary normals n^*.*

Proof. We multiply the relation (1.18) by u'^α, obtaining

$$g_{ij}X^j_\gamma \frac{Dx'^i}{Ds} = g_{\alpha\gamma}\frac{\delta u'^\alpha}{\delta s},$$

which is satisfied by the tangent vector u'^α to any curve C in F_m. For a geodesic G we have $\delta u'^\alpha/\delta s = 0$, hence

$$g_{ij}(x, x')X^j_\gamma\left(\frac{Dx'^i}{Ds}\right)_{(G)} = 0.$$

Since the vector Dx'^i/Ds, which defines the principal normal to the geodesics G, satisfies the equation (1.7), it belongs in the space spanned by $n^*_{(\mu)}$, therefore

$$\left(\frac{Dx'^i}{Ds}\right)_{(G)} = \sum_{(\mu)} \lambda_{(\mu)}\overset{*}{n}{}^i_{(\mu)}, \tag{2.10}$$

where $n^{*i}_{(\mu)}$ is a set of $n - m$ orthogonal independent vectors of that space.

THEOREM II. *The tensor $X_{\alpha\beta}{}^i$ considered as a function of a given line element (x^i, x'^i) lies in the space spanned by the secondary normals n^*.*

Proof. We consider the equations

$$X^i = X^i_\alpha U^\alpha, \qquad g_{ij}X^j_\gamma X^k_\alpha X^i_{.k} = g_{\beta\gamma}U^\beta_{.\alpha},$$

then we can write

$$g_{ij}X^j_\gamma X^i_\delta P^{*\delta}_{\alpha\beta} = g_{ij}X^j_\gamma\left(\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + P^{*\,i}_{hk}X^h_\alpha X^k_\beta\right)$$

and because of (2.3) we obtain

$$g_{ij}(x, x')X^j_\gamma X^i_{\alpha\beta}(u, u') = 0, \tag{2.11}$$

which proves the theorem.

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The vector $X_{\alpha\beta}{}^i$ (in i) will be a linear combination of the n^* and therefore

$$X^i_{\alpha\beta} = \sum_{(\mu)} \Omega^*_{(\mu)\alpha\beta}(u, u')n^{*i}_{(\mu)}; \tag{2.12}$$

multiplying the relation (2.12) by $n_{(\nu)}{}^i$ and putting

$$\sum_{(\mu)} \Omega^*_{(\mu)\alpha\beta} \cos(n_{(\nu)}, n^*_{(\mu)}) = \Omega_{(\nu)\alpha\beta},$$

we find

$$n_{(\nu)i}X^j_{\alpha\beta} = \Omega_{(\nu)\alpha\beta}. \tag{2.13}$$

It is obvious that $\Omega_{(\nu)\alpha\beta}$ are tensors symmetric in α, β.

The relations (2.12) and (2.13) are fundamental for the whole theory of subspaces of a Finsler space.

We consider the relation (2.13) and we multiply by $u'^\alpha u'^\beta$, then

$$\Omega_{(\nu)\alpha\beta}u'^\alpha u'^\beta = n_{(\nu)i}\left[\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} u'^\alpha u'^\beta + P^{*i}_{hk}x'^h x'^k\right]; \tag{2.14}$$

but

$$n_{(\nu)}\frac{dx'^i}{ds} = n_{(\nu)i}\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} u'^\alpha u'^\beta. \tag{2.15}$$

Therefore, combining the above equation with (2.14), we obtain

$$\Omega_{(\nu)\alpha\beta}u'^\alpha u'^\beta = n_{(\nu)i}\left[\frac{dx'^i}{ds} + P^*_{hk}x'^h x'^k\right] = n_{(\nu)i}\frac{Dx'^i}{Ds}. \tag{2.16}$$

We can easily see that this is the same for all curves of F_m with tangent vector x'^i, but depends on the choice of (x, x'), as in classical differential geometry. Indeed, differentiating the relation

$$n_i x'^i = 0$$

we find

$$\frac{Dn_i}{Ds}x'^i = -n_i\frac{Dx'^i}{Ds},$$

and since $Dn_i/Ds\, x'^i$ depends on x, x' only, so does the right-hand side.

From the identity

$$n_{(\nu)i}\frac{Dx'^i}{Ds} = \left|\frac{Dx'^i}{Ds}\right| \cos\left(n_{(\nu)}, \frac{Dx'}{Ds}\right)$$

we obtain

$$\left|\frac{Dx'^i}{Ds}\right| \equiv \frac{1}{\rho_c} = \frac{\Omega_{(\nu)\alpha\beta}u'^\alpha u'^\beta}{\cos(n_{(\nu)}, Dx'/Ds)}, \tag{2.17}$$

where ρ_c is the radius of curvature of the curve regarded as a curve of F_n. The relation (2.17) may also be written

$$\frac{\cos(n_{(\nu)}, Dx'/Ds)}{\rho_c} = \Omega_{(\nu)\alpha\beta}u'^\alpha u'^\beta, \tag{2.18}$$

and since

$$\Omega_{(\nu)\alpha\beta}u'^{\alpha}u'^{\beta}$$

is the same for all curves of F_m tangent to x'^i, we obtain Meusnier's theorem of classical differential geometry. *We may therefore regard*

$$\Omega_{(\nu)\alpha\beta}u'^{\alpha}u'^{\beta} = \frac{1}{R_{(\nu)}}$$

as the normal curvature corresponding to the normal $n_{(\nu)}{}^i$. It is obvious from (2.17) that the ratio

$$\frac{\Omega_{(\nu)\alpha\beta}u'^{\alpha}u'^{\beta}}{\cos\,(n_{(\nu)}, Dx'/Ds)}$$

is independent of the choice of $n_{(\nu)}{}^i$.

The concept of the principal direction of a hypersurface F_{n-1} can be extended to any subspace F_m. Indeed, we have shown that to each direction at a point P of F_m correspond $n - m$ normal curvatures

$$(R_{(\nu)}(u, u'))^{-1} = \frac{\Omega_{(\nu)\alpha\beta}(u, u')du^{\alpha}du^{\beta}}{g_{\alpha\beta}(u, u')du^{\alpha}du^{\beta}}$$

associated with the given direction u'.

If we put

$$\Omega_{(\nu)\alpha\beta}(u, u')du^{\alpha}du^{\beta} = 1, \tag{2.19}$$

we obtain a number of $n - m$ loci, of $m - 1$ dimensions each, on the hyperplane spanned by $X_\alpha{}^i$, in the Minkowskian tangent space to F_m, at the given point. The principal directions will be given by the extreme values of $g_{\alpha\beta}(u, u')u'^{\alpha}u'^{\beta}$ subject to the conditions (2.19), where u^α is kept fixed. In other words, principal directions are directions for which the normal curvatures assume extreme values. According to the multiplier rule we must seek solutions of the equations

$$\frac{\partial}{\partial u'^{\gamma}}\,[g_{\alpha\beta}(u, u')u'^{\alpha}u'^{\beta} + \lambda(\Omega_{(\nu)\alpha\beta}(u, u')u'^{\alpha}u'^{\beta} - 1)] = 0,$$

which, after performing the differentiations and using Euler's theorem for homogeneous functions, may be written

$$2\,g_{\alpha\gamma}(u, u')u'^{\alpha} + 2\lambda\Omega_{(\nu)}(u, u')u'^{\alpha} + \lambda\,\frac{\partial\Omega_{(\nu)}}{\partial u'^{\gamma}}\,u'^{\alpha}u'^{\beta} = 0. \tag{2.20}$$

The equations (2.20) are of the same type as the corresponding equations for the principal directions of a hypersurface F_{n-1} **(4)**. Applying the same algebraic algorithm, we obtain the following eigenvalue equations:

$$g_{\alpha\gamma}(u, u')u'^{\alpha} = R_{(\nu)}(u, u')\Omega_{(\nu)\alpha\gamma}(u, u')u'^{\alpha}, \tag{2.21}$$

where $(R_{(\nu)}(u, u'))^{-1}$ is the normal curvature corresponding to a solution of (2.21). This is a non-linear eigenvalue problem with eigenvalue $R_\nu{}^{-1}$ and little can be said about the number of possible solutions.

Let us assume that at least two independent solutions $u_{(\mu)1}{}'^{\alpha}$, $u_{(\nu)2}{}'^{\alpha}$ corresponding to two distinct normal curvatures $1/R_{(\mu)1}$, $1/R_{(\nu)2}$ exist. Then, from (2.21) we obtain

$$g_{\alpha\gamma}(u, u'_{(\mu)1})u'^{\alpha}_{(\mu)1}u'^{\gamma}_{(\nu)2} = R_{(\mu)1}\Omega_{(\mu)\alpha\gamma}(u, u'_{(\mu)1})u'^{\alpha}_{(\mu)1}u'^{\gamma}_{(\nu)2}$$
$$g_{\alpha\gamma}(u, u'_{(\nu)2})u'^{\alpha}_{(\nu)2}u'^{\gamma}_{(\mu)1} = R_{(\nu)2}\Omega_{(\nu)\alpha\gamma}(u, u'_{(\nu)2})u'^{\alpha}_{(\nu)2}u'^{\gamma}_{(\mu)1},$$

subtracting, we find

$$(2.22)\quad \frac{\cos(u'_{(\mu)1}, u'_{(\nu)2}) - \cos(u'_{(\mu)2}, u'_{(\nu)1})}{R_{(\mu)1}R_{(\nu)2}} = \left[\frac{\Omega_{(\mu)\alpha\gamma}(u, u'_{(\mu)1})u'^{\alpha}_{(\mu)1}u'^{\gamma}_{(\nu)2}}{R_{(\nu)2}}\right] - \left[\frac{\Omega_{(\nu)\alpha\gamma}(u, u_{(\nu)2})u'^{\alpha}_{(\nu)2}u'^{\gamma}_{(\mu)1}}{R_{(\mu)1}}\right].$$

When we refer to the same normal $n_{(\nu)}{}^{i}$, the above formula becomes

$$(2.23)\quad \frac{\cos(u'_{(1)}, u'_{(2)}) - \cos(u'_{(2)}, u'_{(1)})}{R_{(\nu)1}R_{(\nu)2}} = \left[\frac{\Omega_{(\nu)\alpha\gamma}(u, u'_{(1)})}{R_{(\nu)1}} - \frac{\Omega_{(\nu)\alpha\gamma}(u, u'_{(2)})}{R_{(\nu)2}}\right]u'^{\alpha}_{1}u'^{\gamma}_{2}.$$

The equation (2.23) is a generalization of the orthogonality relation between principal directions of surfaces in classical differential geometry. Indeed, in a locally Euclidean space, the cosine of the angle of two directions is a symmetric function of them. Therefore the left-hand side of (2.23) vanishes and we obtain

$$(2.24)\quad \frac{\Omega_{(\nu)\alpha\gamma}(u, u'_{(1)})u'^{\alpha}_{1}u'^{\gamma}_{2}}{R_{(\nu)1}} - \frac{\Omega_{(\nu)\alpha\gamma}(u, u'_{(2)})u'^{\alpha}_{1}u'^{\gamma}_{2}}{R_{(\nu)2}} = 0.$$

But (2.21) provides

$$\frac{g_{\alpha\gamma}(u)u'^{\alpha}_{1}u'^{\gamma}_{2}}{R_{(\nu)1}} = \Omega_{(\nu)\alpha\gamma}(u, u')u'^{\alpha}_{1}u'^{\gamma}_{2},$$

and thus equation (2.24) becomes

$$\cos(u'_1, u'_2)\left(\frac{1}{R^2_{(\nu)1}} - \frac{1}{R^2_{(\nu)2}}\right) = 0.$$

Since $R_{(\nu)1} \neq R_{(\nu)2}$ we obtain $\cos(u_1', u_2') = 0$, which demonstrates the orthogonality of u_1', u_2'.

We can also define a secondary normal curvature associated to a line element x, x' and depending on Ω^*. For that purpose we consider the relation

$$(2.25)\quad \frac{Dx'^{i}}{Ds} = \sum \lambda_{(\mu)}n^{*i}_{(\mu)} + X^{i}_{\alpha}\frac{\delta u'^{\alpha}}{\delta s},$$

for an arbitrary curve of F_m and we multiply it by $n_{(\nu)i}$, then

$$n_{(\nu)i}\frac{Dx'^{i}}{Ds} = \sum \lambda_{(\mu)}\cos(n_{(\nu)}, n^{*}_{(\mu)})$$

and

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$$\cos\left(n_{(\nu)}, \frac{Dx'}{Ds}\right) = \frac{n_{(\nu)i}\, Dx'^i/Ds}{|Dx^i/Ds|} = \rho_c \sum_{\mu} \lambda_{(\mu)} \cos\,(n_{(\nu)}, \overset{*}{n}_{(\mu)}).$$

Or, because of (2.17)

$$\Omega_{(\nu)\alpha\beta} u'^\alpha u'^\beta = \sum_{(\mu)} \lambda_{(\mu)} \cos\,(n_{(\nu)}, \overset{*}{n}_{(\mu)}),$$

and hence, in view of (2.12a),

$$\lambda_{(\mu)} = \overset{*}{\Omega}_{(\mu)\alpha\beta} u'^\alpha u'^\beta.$$

From (2.25) we obtain

$$\frac{Dx'^i}{Ds} = \sum_{(\mu)} (\overset{*}{\Omega}_{(\mu)\alpha\beta} u'^\alpha u'^\beta) \overset{*}{n}{}^i_{(\mu)} + X^i_\alpha \frac{\delta u'^\alpha}{\delta s},$$

and for a geodesic

$$\frac{Dx'^i}{Ds} = \sum_{(\mu)} (\overset{*}{\Omega}_{(\mu)\alpha\beta} u'^\alpha u'^\beta) \overset{*}{n}{}^i_{(\mu)}.$$

We define the secondary normal curvature to be

$$\frac{1}{R^{*2}} = g_{ij}(x, x') \frac{Dx'^i}{Ds} \frac{Dx'^j}{Ds} = \sum_{(\mu)} \psi_{(\mu)}(x, x') \overset{*}{\Omega}_{(\mu)\alpha\beta} \overset{*}{\Omega}_{(\mu)\gamma\delta} u'^\alpha u'^\beta u'^\gamma u'^\delta.$$

In the way $1/R^*$ is defined we see that it is independent of the particular set of normals $n^*_{(\mu)}$.

Let us consider the biquadratic form in the differentials,

$$\phi = \sum_{(\mu)} \psi_{(\mu)}(x, x') \overset{*}{\Omega}_{(\mu)\alpha\beta} \overset{*}{\Omega}_{(\mu)\gamma\delta} du^\alpha du^\beta du^\gamma du^\delta,$$

we may call it the secondary second fundamental form of F_m. Generalizing the concepts of conjugate and asymptotic directions of a surface in classical differential geometry, we may say that two directions at a point defined by du^α and δu^α are conjugate when

$$\sum_{(\mu)} \psi_{(\mu)} \overset{*}{\Omega}_{(\mu)\alpha\beta} \overset{*}{\Omega}_{(\mu)\gamma\delta} du^\alpha \delta u^\beta du^\gamma \delta u^\delta = 0,$$

and asymptotic or self-conjugate when

$$\sum_{(\mu)} \psi_{(\mu)} \overset{*}{\Omega}_{(\mu)\alpha\beta} \overset{*}{\Omega}_{(\mu)\gamma\delta} du^\alpha du^\beta du^\gamma du^\delta = 0.$$

From the above relation and the one defining the secondary normal curvature we conclude that *the secondary normal curvature in an asymptotic direction is always equal to zero* as in Riemannian geometry **(2)**.

3. Covariant derivatives of the normal vectors n^*, n. We define the tensor $n^{*i}{}_{(\mu),\beta}$, covariant derivative of the vector $n^{*i}{}_{(\mu)}$, by projecting $n^{*i}{}_{(\mu),k}$ onto F_m:

$$\textbf{(3.1)} \qquad \overset{*}{n}{}^i_{(\mu),\beta} = \overset{*}{n}{}^i_{(\mu),k} X^k_\beta.$$

Obviously

$$n^{*i}_{(\mu),\beta} = \frac{\partial n^{*i}_{(\mu)}}{\partial u^\beta} + P^{*i}_{hk}(x, x')n^{*h}_{(\mu)}X^k_\beta. \tag{3.2}$$

The $n^{*i}{}_{(\mu),\beta}$ are not tangential to F_m, in contrast to Riemannian geometry, and this is the source of much of the difficulty in the derivation of the Gauss-Codazzi equations.

By differentiating the relation (1.7) with respect to u^β and combining the result with (3.2), we obtain

$$\frac{\partial g_{ij}}{\partial u^\beta} n^{*j}_{(\mu)}X^i_\alpha + g_{ij}(x, x')X^i_\alpha(n^{*j}_{(\mu),\beta} - P^{*j}_{hk}(x, x')n^{*h}_{(\mu)}X^k_\beta) + g_{ij}n^{*j}_{(\mu)} \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} = 0,$$

or, after rearranging the terms,

$$g_{ij}(x, x')X^i_\alpha n^{*j}_{(\mu),\beta} + g_{ij}n^{*j}_{(\mu)} \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + n^{*j}_{(\mu)}X^i_\alpha\left(\frac{\partial g_{ij}}{\partial u^\beta} - g_{hi}P^{*h}_{jk}X^k_\beta\right) = 0.$$

We add and subtract $g_{lj}P^{*l}{}_{hk}X_\alpha{}^h X_\beta{}^k$ in the left-hand side of the above relation, thus obtaining

$$\begin{aligned} g_{ij}(x, x')X^i_\alpha n^{*j}_{(\mu),\beta} &+ g_{ij}n^{*j}_{(\mu)}\left(\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + P^{*i}_{hk}X^h_\alpha X^k_\beta\right) \\ &+ n^{*j}_{(\mu)}X^h_\alpha\left(\frac{\partial g_{hj}}{\partial u^\beta} - g_{lh}P^{*l}_{jk}X^k_\beta - g_{lj}P^{*l}_{hk}X^k_\beta\right) = 0. \end{aligned} \tag{3.3}$$

The term in the last bracket of (3.3) represents the covariant derivative of g_{ij} (x, x') with respect to x^k, multiplied by $X_\beta{}^k$; if we put

$$C^*_{hj,k}(x, x') = g_{hj,k}(x, x'),$$

we may write for (3.3),

$$g_{ij}(x, x')X^i_\alpha n^{*j}_{,\beta} + \psi_{(\mu)}\Omega^*_{(\mu)\alpha\beta} + C^*_{ij,k}X^k_\beta X^i_\alpha n^{*j}_{(\mu)} = 0. \tag{3.4}$$

We decompose $n^{*j}{}_{,\beta}$, which is not tangential to F_m, as follows:

$$n^{*j}_{(\mu),\beta} = B^\delta_{(\mu)\beta}X^j_\delta + \sum_{(\kappa)} N^{(\mu)}_{(\kappa)}n^{*j}_{(k)}. \tag{3.5}$$

In order to find $B_{(\mu)\beta}{}^\delta$, we multiply (3.5) by $g_{ij}X_\alpha{}^i$, in view of (1.7) and (3.4), we have

$$B^\epsilon_{(\mu)\beta} = -\psi_{(\mu)}\Omega^*_{(\mu)\alpha\beta}g^{\alpha\epsilon} - C^*_{ihk}(x, x')g^{\alpha\epsilon}X^k_\beta X^i_\alpha n^{*h}_{(\mu)}. \tag{3.6}$$

To obtain the N's we multiply (3.5) by $n^*{}_{(\lambda)j}$, then

$$n^{*i}_{(\mu),\beta}n^*_{(\lambda)j} = N^{(\mu)}_{(\lambda)\beta}\psi_{(\lambda)}. \tag{3.7}$$

The N's are not independent since they satisfy some symmetry conditions which we obtain in the following way. We consider the relations

$$g_{ij}(x, x)n^{*i}_{(\mu)}n^{*j}_{(\lambda)} = \psi_{(\mu)}\delta^\lambda_\mu \qquad \text{(no summation is involved in } \mu),$$

and we differentiate them with respect to u_β; between the relation which we find and the equation (3.2) we eliminate

$$\frac{\partial n^{*i}_{(\mu)}}{\partial u^\beta},$$

thus

(3.8) $$\frac{\partial g_{ij}(x, x')}{\partial u^\beta} n^{*i}_{(\mu)} n^{*j}_{(\lambda)} + g_{ij}(x, x')[n^{*i}_{(\mu),\beta} n^{*j}_{(\lambda)} + n^{*j}_{(\lambda),\beta} n^{*i}_{(\mu)}$$
$$- P^{*i}_{hk}(x, x') n^{*h}_{(\mu)} n^{*j}_{(\lambda)} X^k_\beta - P^{*j}_{hk}(x, x') n^{*h}_{(\lambda)} n^{*i}_{(\mu)} X^k_\beta] = \frac{\partial \psi_{(\mu)}}{\partial u^\beta} \delta^\lambda_\mu.$$

If we use the relation (3.7), we find

(3.9) $$\psi_{(\lambda)} N^{(\mu)}_{(\lambda)\beta} + \psi_{(\mu)} N^{(\lambda)}_{(\mu)\beta} = \frac{\partial \psi_{(\mu)}}{\partial u^\beta} \delta^\lambda_\mu - C^*_{ijk} n^{*j}_{(\lambda)} n^{*i}_{(\mu)} X^k_\beta.$$

In conclusion, we have the covariant derivative of $n^*{}_{(\mu)}$ *given by*

(3.10) $$n^{*j}_{(\mu),\beta} = \psi_{(\mu)} X^j_\delta g^{\delta\alpha} \Omega_{(\mu)\alpha\beta} - C^*_{ihk} g^{ij} X^k_\beta n^{*h}_{(\mu)} + \sum_{(\lambda)} N^{(\mu)}_{(\lambda)\beta} \psi_{(\lambda)},$$

where the quantities $N_{(\lambda)\beta}{}^{(\mu)}$ *(vectors in* β*) satisfy the symmetry conditions* (3.9). In the case of a hypersurface F_{n-1}, the equation (3.10) becomes identical with (4.9) **(4)**, the relation (3.9) giving

$$N_\beta = \frac{1}{\psi} \frac{\partial \psi}{\partial u^\beta} - C_{ijk} n^{*i} n^{*j} X^k_\beta.$$

The equations (3.10) suffer from the disadvantage that the terms $\psi_{(\mu),\beta}$ in the right-hand side involve the derivatives of the tangent x'^i to the curve along which we are differentiating, so that (3.10) depends on the curve under consideration.

As in the case of the $n^{*i}{}_{,\beta}$, we define the covariant derivative $n_{(\mu),\beta}{}^i$ by projecting $n_{(\mu),\kappa}{}^i$ onto F_m,

(3.11) $$n^i_{(\mu),\beta} = n^i_{(\mu),k} X^k_\beta = \frac{\partial n^i{}_{(\mu)}}{\partial u^\beta} + P^{*i}_{hk}(x, x') n^h_{(\mu)} X^k_\beta,$$

where x'^i is some direction tangential to F_m at the given point. Here we obtain

(3.12) $$\Omega_{(\mu)\alpha\beta} = - C_{(\mu)ijk} X^k_\beta X^i_\alpha n^j_{(\mu)} - g_{ij}(x, n_{(\mu)}) X^i_\alpha n^j_{(\mu),\beta},$$

where

(3.13) $$C_{(\mu)ijk} = g_{ij.k}(x, n_{(\mu)}).$$

We decompose the tensor $n_{(\mu),\beta}{}^j$ as follows:

(3.14) $$n^i_{(\mu),\beta} = A^\delta_{(\mu)\beta} X^i_\delta + \sum_{(\kappa)} \nu^{(\mu)}_{(\kappa)\beta} n^i_{(\kappa)};$$

multiplying (3.14) by $g_{ij}(x, n_{(\mu)}) X_\alpha{}^i$ we find that

(3.15) $$A^\delta_{(\mu)\beta} = - \gamma^{\alpha\delta}_{(\mu)} \Omega_{(\mu)\alpha\delta} - \gamma^{\alpha\delta}_{(\mu)} C_{(\mu)ijk} X^k_\beta X^i_\alpha n^j_{(\mu)},$$

and therefore

(3.16) $$n^i_{(\mu),\beta} = - \gamma^{\alpha\delta}_{(\mu)} \Omega_{(\mu)\alpha\beta} X^j_\delta - \gamma^{\alpha\delta}_{(\mu)} C_{(\mu)ihk} X^j_\delta X^k_\beta X^i_\alpha n^h_{(\mu)} + \sum_{(\kappa)} \nu^{(\mu)}_{(\kappa)\beta} n^j_{(\kappa)}.$$

In order to obtain more information on the ν's we multiply (3.14) by $n_{(\lambda)j}$, then

$$n^j_{(\kappa),\beta} n_{(\lambda)j} = \sum_{(\kappa)} \nu^{(\mu)}_{(\kappa)\beta} n^j_{(\kappa)} n_{(\lambda)j} = \sum_{(\kappa)} \nu^{(\mu)}_{(\kappa)\beta} a_{(\kappa)(\lambda)} \tag{3.17}$$

where

$$a_{(k)(\lambda)} = n^j_{(k)} n_{(\lambda)j} = \cos(n_{(\lambda)}, n_{(k)}).$$

We note that in general $a_{(\kappa)(\lambda)} \neq a_{(\lambda)(\kappa)}$. Assuming that the determinant $|a_{(\kappa)(\lambda)}|$ is different from zero, we can solve the system (3.17) with respect to the values of $\nu_{(\kappa),\beta}{}^{(\mu)}$ and we obtain the ν's as linear combinations of the expressions $n_{(\mu),\beta}{}^j n_{(\lambda)j}$, that is,

$$\nu^{(\mu)}_{(\kappa)\beta} = \sum_{(\lambda)} \frac{A^{(\kappa)(\lambda)}}{A} (n^j_{(\mu),\beta} n_{(\lambda)j}); \tag{3.18}$$

where $A^{(\kappa)(\lambda)}$ is the cofactor of the determinant $|a_{(\kappa)(\lambda)}|$ corresponding to the element $a_{(\kappa)(\lambda)}$ and A is the value of that determinant.

As an application of the above theory we may obtain Rodrigues' formula of classical differential geometry.

If we consider the relation (3.16) we may write

$$\frac{Dn^j_{(\mu)}}{Ds} = -\gamma^{\alpha\delta}_{(\mu)} \Omega_{(\mu)\alpha\beta} u'^\beta X^j_\delta - \gamma^{\alpha\delta}_{(\mu)} C_{(\mu)\,lhk} X^j_\delta X^l_\alpha X'^k n^h_{(\mu)} + \sum_{(\kappa)} \nu^{(\mu)}_{(\kappa)\beta} n^j_{(k)} u'^\beta, \tag{3.19}$$

since $n_{(\mu)i} = g_{ij}(x, n_{(\mu)}) n_{(\mu)}{}^j$, we obtain by differentiation

$$\frac{Dn_{(\mu)i}}{Ds} = C_{(\mu)ijk} n^j_{(\mu)} + g_{ij}(x, n_{(\mu)}) \frac{Dn^j_{(\mu)}}{Ds}, \tag{3.20}$$

and substituting (3.19) in (3.20), we find

$$\frac{Dn_{(\mu)i}}{Ds} = -g_{ij}(x, n_{(\mu)}) \gamma^{\epsilon\delta}_{(\mu)} \Omega_{(\mu)\epsilon\beta} u'^\beta X^j_\delta + \sum_{(k)} \nu^{(\mu)}_{(\kappa)\beta} u'^\beta g_{ij}(x, n_{(\mu)}) n^j_{(\kappa)}.$$

Multiplying the above equation by $X_\alpha{}^i$, we obtain

$$X^i_\alpha \frac{Dn_{(\mu)i}}{Ds} = -\Omega_{(\mu)\alpha\beta} u'^\beta + \sum_{(k)} \nu^{(\mu)}_{(\kappa)\beta} u'^\beta g_{ij}(x, n_{(\mu)}) n^i_{(\kappa)} X^i_\alpha. \tag{3.21}$$

If $R_{(\mu)}{}^{-1}(x, x')$ is the normal curvature corresponding to a principal direction x'^i of F_m and to a normal $n_{(\mu)}$, we have from equation (2.21)

$$g_{\alpha\beta}(u, u') u'^\beta = R_{(\mu)}(x, x') \Omega_{(\mu)\alpha\beta} u'^\beta .$$

Combining the above relation with (3.21), we write

$$X^i_\alpha \frac{Dn^i_{(\mu)}}{Ds} = -[R_{(\mu)}(x, x')]^{-1} g_{\alpha\beta}(u, u') u'^\beta + \sum_{(k)} \nu^{(\mu)}_{(\kappa)(\beta)} u'^\beta g_{ij}(x, n_{(\mu)}) n^j_{(\kappa)} X^i_\alpha. \tag{3.22}$$

For $m = n - 1$ (hypersurface F_{n-1}), the second term in the right-hand side becomes zero. Indeed in that case the hypersurface has a unique normal n, and therefore, the sum

$$g_{ij}(x, n_{(\mu)})n^j_{(\kappa)}X^i_\alpha$$

is reduced to $g_{ij}\,(x, n)n^jX_\alpha{}^i$ which is identically equal to zero. But in the case of any subspace F_m, the second term does not vanish, unless we choose a particular set of normals $n_{(\mu)}$ such that

$$\sum_{(\kappa)} \nu^{(\mu)}_{(\kappa)\beta}u'^\beta g_{ij}(x, n_{(\mu)})n^j_{(\kappa)}X^i_\alpha = 0,$$

then

$$X^i_\alpha \frac{Dn^i_{(\mu)}}{Ds} = -\,[R_{(\mu)}(x, x')]^{-1}g_{\alpha\beta}(u, u')u'^\beta,$$

putting $g_{\alpha\beta}(u, u')u'^\beta = y_\alpha$, that is, introducing the covariant component in F_m of u'^β, we find

$$X^i_\alpha \frac{Dn^i_{(\mu)}}{Ds} = -\,(R_{(\mu)}(x, x'))^{-1}y_\alpha. \tag{3.22}$$

The above formula is analogous to Rodrigues' formula and it is similar to the one for a hypersurface **(4)**.

4. The Gauss and Codazzi equations for an F_m. We may obtain relations connecting the curvature tensor of the space F_n with the curvature tensor of F_m and the coefficients Ω^*. To do so, we first consider the covariant derivative of $X_\alpha{}^i$ with respect to u^β (metric in F_m),

$$X^i_{\alpha,\beta} = \frac{\partial X^i_\alpha}{\partial u^\beta} - P^{*\delta}_{\alpha\beta}X^i_\delta.$$

Combining the above relation with relation (2.3), we obtain

$$X^i_{\alpha,\beta} = X^i_{\alpha\beta} - P^*_{hk}X^h_\alpha X^k_\beta \tag{4.1}$$

or, because of (2.12),

$$X^i_{\alpha,\beta} = \sum_{(\mu)} \Omega^*_{(\mu)\alpha\beta}\overset{*}{n}_{(\mu)} - P^{*\,i}_{hk}X^h_\alpha X^k_\beta. \tag{4.2}$$

We know that

$$X^i_{\alpha,\beta\gamma} - X^i_{\alpha,\gamma\beta} = R^\delta_{.\alpha\beta\gamma}X^i_\delta,$$

where $R^\delta{}_{.\alpha\beta\gamma}$ is the curvature tensor of F_m corresponding to the induced connection coefficients $P_{\beta\gamma}{}^{*\alpha}$. By using the expression (4.2), we can write

$$\begin{aligned} R^\delta_{.\alpha\beta\gamma}X^i_\delta = X^h_\alpha\Bigg[&\left(\frac{\partial P^{*\,i}_{hk}}{\partial u^\gamma} + \frac{\partial P^{*\,i}_{hk}}{\partial x'^j}\frac{\partial x'^j}{\partial u^\gamma}\right)X^k_\beta - \left(\frac{\partial P^*_{hk}}{\partial u^\beta} + \frac{\partial P^{*\,i}_{hk}}{\partial x'^j}\frac{\partial x'^j}{\partial u^\beta}\right)X^k_\gamma\Bigg] \\ &- P^{*\,i}_{hk}(X^h_{\alpha,\gamma}X^k_\beta - X^k_{\alpha,\beta}X^h_\gamma) + \sum_{(\mu)} (\Omega^*_{(\mu)\alpha\beta}n^{*\,i}_{(\mu),\gamma} - \Omega^*_{(\mu)\alpha\gamma}n^{*\,i}_{(\mu),\beta}) \\ &+ \sum_{(\mu)} n^{*\,i}_{(\mu)}(\Omega^*_{(\mu)\alpha\beta,\gamma} - \Omega^*_{(\mu)\alpha\gamma,\beta}) \\ &\qquad + \sum_{(\mu)} P^{*\,i}_{hk}n^{*\,h}_{(\mu)}(\Omega^*_{(\mu)\alpha\gamma}X^k_\beta - \Omega^*_{(\mu)\alpha\beta}X^k_\gamma)\,. \end{aligned} \tag{4.3}$$

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With the help of (4.2) the second term of (4.3) can be written

$$X_\alpha^h X_\beta^k X_\gamma^l (P^{*i}_{kj} P^{*j}_{hl} - P^{*i}_{lj} P^{*j}_{hk}) - \sum P^{*i}_{hk} n^{*h}_{(\mu)} (\Omega^*_{(\mu)\alpha\gamma} X^k_\beta - \Omega^*_{(\mu)\alpha\beta} X^k_\gamma),$$

therefore, (4.3) becomes

$$(4.4)\quad R^{\delta}_{.\alpha\beta\gamma} X^i_\delta = X^i_\alpha \left[\left(\frac{\partial P^{*i}_{hk}}{\partial u^\gamma} + \frac{\partial P^{*i}_{hk}}{\partial x'^j}\frac{\partial x'^j}{\partial u^\gamma}\right) X^k_\beta - \left(\frac{\partial P^{*i}_{hk}}{\partial u^\beta} + \frac{\partial P^{*i}_{hk}}{\partial x'^j}\frac{\partial x'^j}{\partial u^\beta}\right) X^k_\beta\right]$$
$$+ X^h_\alpha X^k_\beta X^l_\gamma (P^{*i}_{kj} P^{*j}_{kl} - P^{*i}_{lj} P^{*j}_{hk}) + \sum_{(\mu)} (\Omega^*_{(\mu)\alpha\beta} n^{*i}_{(\mu),\gamma} - \Omega^*_{(\mu)\alpha\gamma} n^{*i}_{(\mu),\beta})$$
$$+ \sum_{(\mu)} n^{*i}_{(\mu)} (\Omega^*_{(\mu)\alpha\beta,\gamma} - \Omega^*_{(\mu)\alpha\gamma,\beta});$$

the first and the second term in the above equation may be substituted by

$$R^{\,i}_{.hkl}(x, x') X^h_\alpha X^k_\beta X^l_\gamma$$

according to (A.7), where $R_{hkl}{}^i$ is the curvature tensor of the space F_n. The equation (4.4) then becomes

$$(4.5)\quad X^i_\delta R^\delta_{.\alpha\beta\gamma}(u, u') = R^i_{hkl}(x, x') X^h_\alpha X^k_\beta X^l_\gamma + \sum_{(\mu)} (\Omega^*_{(\mu)\alpha\beta} n^{*i}_{(\mu),\gamma} - \Omega^*_{(\mu)\alpha\gamma} n^{*i}_{(\mu),\beta})$$
$$+ \sum_{(\mu)} n^{*i}_{(\mu)} (\Omega^*_{(\mu)\alpha\beta,\gamma} - \Omega^*_{(\mu)\alpha\gamma,\beta}).$$

If we use (3.12), we may eliminate the derivatives of $n^*_{(\mu)}$ from (4.5) and thus we obtain

$$(4.5\text{a})\quad X^i_\delta R^\delta_{.\alpha\beta\gamma}(u, u') = R^{\,i}_{.hkl}(x, x') X^h_\alpha X^k_\beta X^l_\gamma$$
$$+ \sum_{(\mu)} \psi_{(\mu)} (\Omega^*_{(\mu)\alpha\beta} \Omega^*_{(\mu)\epsilon\gamma} - \Omega^*_{(\mu)\alpha\gamma} \Omega^*_{(\mu)\epsilon\gamma}) X^i_\delta g^{\delta\epsilon}$$
$$- C^*_{jhk} g^{ij} \sum_{(\mu)} (\Omega^*_{(\mu)\alpha\beta} X^k_\gamma - \Omega^*_{(\mu)\alpha\gamma} X^k_\beta) n^{*h}_{(\mu)} + \sum_{(\mu)} \sum_{(\lambda)} (N^{(\mu)}_{(\lambda)\beta} \Omega^*_{(\mu)\alpha\beta}$$
$$- N^{(\mu)}_{(\lambda)\gamma} \Omega^*_{(\mu)\alpha\gamma}) n^{*i}_{(\lambda)} + \sum_{(\mu)} (\Omega^*_{(\mu)\alpha\beta,\gamma} - \Omega^*_{(\mu)\alpha\gamma,\beta}) n^{*i}_{(\mu)}.$$

Multiplying the above equation by $g_{ij}\,(x, x')\, X_\lambda{}^j$, we find

$$(4.6)\quad g_{\delta\lambda} R^\delta_{.\alpha\beta\gamma}(u, u') - \sum_{(\mu)} \psi_{(\mu)} (\Omega^*_{(\mu)\alpha\beta} \Omega^*_{(\mu)\lambda\gamma} - \Omega^*_{(\mu)\alpha\gamma} \Omega^*_{(\mu)\lambda\beta})$$
$$= g_{ij}(x, x') R^{\,i}_{.hkl}(x, x') X^h_\alpha X^k_\beta X^l_\gamma X^j_\lambda - X^j_\lambda C^*_{jhk} \sum (\Omega^*_{(\mu)\alpha\beta} X^k_\gamma - \Omega^*_{(\mu)\alpha\gamma} X^k_\beta) n^{*h}_{(\mu)}$$

and multiplying the same equation by $g_{ij}\,(x, x')\, n^{*j}{}_{(\mu)}$, we get

$$(4.7)\quad g_{ij}(x, x') n^{*j}_{(\nu)} R^i_{hkl}(x, x') X^h_\alpha X^k_\beta X^l_\gamma - C^*_{jhk} \sum_{(\mu)} (\Omega^*_{(\mu)\alpha\beta} X^k_\gamma - \Omega^*_{(\mu)\alpha\gamma} X^k_\beta) n^{*h}_{(\mu)} n^{*j}_{(\nu)}$$
$$+ \sum_{(\mu)} \psi_{(\mu)} (N^{(\mu)}_{(\nu)\beta} \Omega^*_{(\mu)\alpha\beta} - N^{(\mu)}_{(\nu)\gamma} \Omega^*_{(\mu)\alpha\gamma}) + \psi_{(\nu)} (\Omega^*_{(\nu)\alpha\beta,\gamma} - \Omega^*_{(\nu)\alpha\gamma,\beta}) = 0.$$

The equations (4.6) and (4.7) represent a generalization of the Gauss-Codazzi equations of Riemannian geometry.

It is obvious that different forms of Gauss-Codazzi equations are obtained when one considers the fundamental forms $\Omega_{(\nu)\alpha\beta} du^\alpha du^\beta$ together with the normals $n(x)$. For that purpose, we decompose the vector $X_{\alpha\beta}{}^i$ (considered as a vector with respect to the upper index i) into components along the normals n and the tangent plane at the considered point. We put

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(4.8) $$X_{\alpha\beta} = \sum_{(\mu)} A_{(\mu)\alpha\beta} n^i_{(\mu)} + W^i_{\alpha\beta}$$

where $W_{\alpha\beta}{}^i$ satisfies the condition

(4.9) $$W^i_{\alpha\beta} n_{(\nu)i} = 0,$$

and by multiplying (4.8) by $n_{(\nu)i}$, we obtain

(4.10) $$\Omega_{(\nu)\alpha\beta} = n_{(\nu)i} X^i_{\alpha\beta} \sum_{(\mu)} A_{(\mu)\alpha\beta} \cos (n_{(\nu)}, n_{(\mu)}),$$

hence $W_{\alpha\beta}{}^i$ is given by the relation

(4.10a) $$W^i_{\alpha\beta} = X^i_{\alpha\beta} - \sum_{(\mu)} A_{(\mu)\alpha\beta} n^i_{(\mu)} = \sum_{(\mu)} \overset{*}{\Omega}_{(\mu)\alpha\beta} \overset{*}{n}{}^i_{(\mu)} - \sum_{(\mu)} A_{(\mu)\alpha\beta} n^i_{(\mu)}.$$

Since the vectors n^i are in general different from the vectors n^{*i} and they do not belong in the space spanned by n^{*i}, we look for a decomposition of the n^i's along the n^{*i} and the vectors defining the tangent space to F_m. We decompose the vector n^i in the form

(4.11) $$n^i_{(\mu)} = \sum_{(\lambda)} T^{(\mu)}_{(\lambda)} \overset{*}{n}{}^i_{(\lambda)} - M^\alpha_{(\mu)} X^i_\alpha;$$

multiplication by $n_{(\nu)}{}^i$ provides

(4.11a) $$n^i_{(\mu)} n_{(\nu)i} = \sum_{(\lambda)} T^{(\mu)}_{(\lambda)} \cos (n_{(\nu)}, \overset{*}{n}_{(\lambda)});$$

from (4.11) we also obtain

(4.12) $$M^\delta_{(\mu)} = n^i_{(\mu)} X^\delta_i.$$

Combining (4.11) with (4.10a) and also (4.11a), (4.10), we may write

(4.12a) $$W^i_{\alpha\beta} = \sum_{(\mu)} \overset{*}{\Omega}_{(\mu)\alpha\beta} \overset{*}{n}_{(\mu)} - \sum_{(\lambda)} \sum_{(\mu)} (A_{(\mu)\alpha\beta} T^{(\mu)}_{(\lambda)}) \overset{*}{n}_{(\lambda)} + X^i_\delta \sum A_{(\mu)\alpha\beta} M^\delta_{(\mu)},$$

(4.13) $$\Omega_{(\nu)\alpha\beta} = \sum_{(\lambda)} \Big(\sum_{(\mu)} A_{(\mu)\alpha\beta} T^{(\mu)}_{(\lambda)} \Big) \cos (n_{(\nu)}, \overset{*}{n}_{(\lambda)}).$$

If we compare the equations (4.13) awith (2.21a), we see that

(4.14) $$\sum_{(\mu)} A_{(\mu)\alpha\beta} T^{(\mu)}_{(\lambda)} = \overset{*}{\Omega}_{(\lambda)\alpha\beta}.$$

In view of the equation (4.14), the relation (4.12a) becomes

(4.15) $$W^i_{\alpha\beta} = X^i_\delta \sum_{(\mu)} A_{(\mu)\alpha\beta} M^\delta_{(\mu)},$$

thus, $M_{(\mu)}{}^\delta$ is given by (4.12), $A_{(\mu)\alpha\beta}$ by (4.10) and $W_{\alpha\beta}{}^i$ by (4.15).

Using the equations (4.1) and (A.8) again, we write

(4.1a) $$X^i_{\alpha,\beta} = \sum_{(\mu)} A_{\alpha\beta} n^i_{(\mu)} + W^i_{\alpha\beta} - \overset{*}{P}{}^i_{hk} X^h_\alpha X^k_\beta;$$

differentiating with respect to the metric of F_m and because of (3.11) we obtain

$$(4.16)\quad X^i_{\alpha,\beta\gamma} = \sum_{(\mu)} A_{(\mu)\alpha\beta} n^i_{(\mu),\gamma} + W^i_{\alpha\beta} - \sum_{(\mu)} P^{*\,i}_{hk} A_{(\mu)\alpha\beta} n^h_{(\mu)} X^k_\gamma$$
$$- \left(\frac{\partial P^{*\,i}_{hk}}{\partial u^\gamma} + \frac{\partial P^{*\,i}_{hk}}{\partial x'^j}\frac{\partial x'^j}{\partial u^\gamma}\right) X^h_\alpha X^k_\beta - P^{*\,i}_{hk} X^h_{\alpha,\beta} X^k_\beta - P^{*\,i}_{hk} X^h_\alpha X^k_{\beta,\gamma} + \sum_{(\mu)} A_{(\mu)\alpha\beta,\gamma} n^i_{(\mu)},$$

or

$$(4.17)\quad \sum (A_{(\mu)\alpha\beta,\gamma} - A_{(\mu)\alpha\gamma,\beta}) n^i_{(\mu)} + \sum_{(\mu)} (A_{(\mu)\alpha\beta} n^i_{(\mu),\gamma} - A_{(\mu)\alpha\gamma} n^i_{(\mu),\beta})$$
$$+ X^h_\alpha X^k_\beta X^l_\gamma R^i_{.hkl}(x, x') + W^i_{\alpha\beta,\gamma} - W^i_{\alpha\gamma,\beta} - P^{*\,i}_{hk}(W^h_{\alpha\gamma} X^k_\beta - W^h_{\alpha\beta} X^k_\gamma) = R^\delta_{\alpha\beta\gamma} X^i_\delta.$$

Using the expression for the generalized covariant derivative of $W_{\alpha\beta}{}^i$ with respect to u^γ we find

$$(4.18)\quad X^i_\delta R^\delta_{\alpha\beta\gamma} = \sum_{(\mu)} (A_{(\mu)\alpha\beta,\gamma} - A_{(\mu)\alpha\gamma,\beta}) n^i_{(\mu)} + \sum_{(\mu)} (A_{(\mu)\alpha\beta} n^i_{(\mu),\gamma} - A_{(\mu)\alpha\gamma} n^i_{(\mu),\beta})$$
$$+ X^h_\alpha X^k_\beta X^l_\gamma R^i_{.hkl}(x, x') + W^i_{\alpha\beta\gamma} - W^i_{\alpha\gamma\beta},$$

which, with the help of (3.16), can be written

$$(4.19)\quad X^i_\delta[R^\delta_{\alpha\beta\gamma} - \sum_{(\mu)} \gamma^{\epsilon\delta}_{(\mu)}(\Omega_{(\mu)\epsilon\beta} A_{(\mu)\alpha\gamma} - \Omega_{(\mu)\epsilon\gamma} A_{(\mu)\alpha\beta})] = R^i_{hkl}(x, x') X^h_\alpha X^k_\beta X^l_\gamma$$
$$- \sum_{(\mu)} g^{si}_{(\mu)} C_{(\mu)shk} n^h_{(\mu)} (A_{(\mu)\alpha\beta} X^k_\gamma - A_{(\mu)\alpha\gamma} X^k_\beta) + \sum_{(\mu)} (A_{(\mu)\alpha\beta,\gamma} - A_{(\mu)\alpha\gamma,\beta}) n^i_{(\mu)}$$
$$+ \sum_{(\mu)} \sum_{(\lambda)} n^i_{(\lambda)} (A_{(\mu)\alpha\beta} \nu^{(\mu)}_{(\lambda)\gamma} - A_{(\mu)\alpha\gamma} \nu^{(\lambda)}_{(\mu)\beta}) + W^i_{\alpha\beta\gamma} - W^i_{\alpha\gamma\beta}.$$

The relation (4.19) is important because it provides the Gauss and Codazzi formulae. Indeed, multiplying (4.19) by $g_{ij}(x, n_{(\nu)}) n_{(\nu)}{}^j$ and putting

$$D_{(\mu)(\nu)jhk} = g^{si}(x, n_{(\mu)}) g_{ij}(x, n_{(\nu)}) C_{(\mu)shk},$$
$$m_{(\mu)(\nu)\gamma} = \sum_{(\lambda)} \nu^{(\mu)}_{(\lambda)\gamma} \cos(n_{(\nu)}, n_{(\lambda)}),$$

we obtain the final equation

$$(4.20)\quad \sum_{(\mu)} a_{(\mu)(\nu)} (A_{(\mu)\alpha\beta,\gamma} - A_{(\mu)\alpha\gamma,\beta}) = \sum_{(\mu)} D_{(\mu)(\nu)hkj} n^j_{(\nu)} n^h_{(\mu)} (A_{(\mu)\alpha\beta} X^k_\gamma - A_{(\mu)\alpha\gamma} X^k_\beta)$$
$$- \sum_{(\mu)} (m_{(\nu)(\mu)\gamma} A_{(\mu)\alpha\beta} - m_{(\mu)(\nu)\beta} A_{(\mu)\alpha\gamma}) - R^i_{hkl} X^h_\alpha X^k_\beta X^l_\gamma n_{(\nu)i}$$
$$- (W^i_{\alpha\beta\gamma} - W^i_{\alpha\gamma\beta}) g_{ij}(x, n_{(\nu)}) n^j_{(\nu)}.$$

It is possible to remove the terms involving $W_{\alpha\beta\gamma}{}^i$ and replace them by expressions depending on $A_{\alpha\beta}$ or $\Omega_{\alpha\beta}$.

Indeed,

$$(4.21)\quad W^i_{\alpha\beta\gamma} = X^i_\delta\left(\sum_{(\mu)} A_{(\mu)\alpha\beta\gamma} M^\delta_{(\mu)} + \sum_{(\mu)} A_{(\mu)\alpha\beta} M^\delta_{(\mu),\gamma}\right)$$
$$+ \left(\sum_{(\mu)} A_{(\mu)\alpha\beta} M^\delta_{(\mu)}\right) X^i_{\delta\gamma},$$

and since $X_{\delta\gamma}{}^i = \Sigma_{(\mu)} A_{\delta\gamma} n_{(\mu)}{}^i + W_{\delta\gamma}{}^i$ and $n_{(\nu)i} W_{\delta\gamma}{}^i = 0$, we obtain instead of (4.20),

$$\sum_{(\kappa)} a_{(\mu)(\nu)}(A_{(\mu)\alpha\beta,\gamma} - A_{(\mu)\alpha\gamma,\beta}) = \sum_{(\mu)} D_{(\mu)(\nu)\,ihk} n^j_{(\mu)} n^h_{(\mu)} (A_{(\mu)\alpha\beta} X^k_\gamma - A_{(\mu)\alpha\gamma} X^k_\beta)$$

$$(4.22) \qquad - \sum_{(\mu)} (m_{(\nu)(\mu)\gamma} A_{(\mu)\alpha\beta} - m_{(\nu)(\mu)\beta} A_{(\mu)\alpha\gamma})$$

$$- \sum_{(\mu)} \sum_{(\lambda)} M^\delta_{(\mu)} a_{(\nu)(\lambda)} (A_{(\mu)\alpha\beta} A_{(\lambda)\delta\gamma} - A_{(\mu)\alpha\gamma} A_{(\lambda)\delta\beta})$$

$$- R^i_{hkl} X^h_\alpha X^k_\beta X^l_\gamma n^i_{(\nu)}.$$

We consider again the equation (4.19). Multiplying by $g_{ij}(x, n_{(\nu)}) X_\xi{}^j$ we obtain

$$(4.23) \quad \gamma_{(\nu)\delta\xi}\Big[R^\delta_{\alpha\beta\gamma} - \sum_{(\mu)} \gamma^{\epsilon\delta}_{(\mu)} (\Omega_{(\mu)\epsilon\beta} A_{(\mu)\alpha\gamma} - \Omega_{(\mu)\epsilon\gamma} A_{(\mu)\alpha\beta})\Big]$$

$$= R^i_{hkl}(x, x') X^h_\alpha X^k_\beta X^l_\gamma X^j_\xi g_{ij}(x, n_{(\nu)})$$

$$- \sum_{(\mu)} g^{si}(x, n_{(\mu)}) g_{ij}(x, n_{(\nu)}) C_{(\mu)shk} n^h_{(\mu)} (A_{(\mu)\alpha\beta} X^k_\gamma - A_{(\mu)\alpha\gamma} X^k_\beta) X^j_\xi$$

$$+ \sum_{(\mu)} (A_{(\mu)\alpha\beta,\gamma} - A_{(\mu)\alpha\gamma,\beta}) g_{ij}(x, n_{(\nu)}) X^j_\xi n^i_{(\mu)}$$

$$+ \sum_{(\mu)} \sum_{(\lambda)} g_{ij}(x, n_{(\nu)}) X^j_\xi n^i_{(\lambda)} (A_{(\mu)\alpha\beta} \nu_{(\lambda)(\mu)\gamma} - A_{(\mu)\alpha\gamma} \nu_{(\lambda)(\mu)\beta})$$

$$+ (W^i_{\alpha\beta\gamma} - W^i_{\alpha\gamma\beta}) g_{ij}(x, n_{(\nu)}) X^j_\xi;$$

by eliminating the derivatives $W_{\alpha\beta\gamma}{}^i$ we find a relation

$$(4.24) \quad g_{ij}(x, n_{(\nu)}) X^j_\xi (W^i_{\alpha\beta\gamma} - W^i_{\alpha\gamma\beta}) = \gamma_{(\nu)\xi\delta}\Big[\sum_{(\mu)} (A_{(\mu)\alpha\beta,\gamma} - A_{\alpha\gamma,\beta}) M^\delta_{(\mu)}$$

$$+ \sum_{(\mu)} A_{(\mu)\alpha\beta} M^\delta_{,\gamma} - A_{(\mu)\alpha\gamma} M^\delta_{,\beta}\Big] + g_{ij}(x, n_{(\nu)}) X^j_\xi \Big[\sum_{(\mu)} \sum_{(\lambda)} (A_{(\mu)\alpha\beta} A_{(\lambda)\delta\gamma}$$

$$- A_{(\mu)\alpha\gamma} A_{(\lambda)\delta\beta}) \,.\, M^\delta_{(\mu)} n^i_{(\lambda)} + X^i_\sigma \sum_{(\mu)} \sum_{(\lambda)} (A_{(\mu)\alpha\beta} A_{(\lambda)\delta\gamma} - A_{(\mu)\alpha\gamma} A_{(\lambda)\delta\beta}) M^\delta_{(\mu)} M^\sigma_{(\lambda)}\Big].$$

for the last term of (4.23).

The relations (4.22) *and* (4.23) *thus represent alternative forms of the generalized Gauss and Codazzi equations.*

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