1

Preliminaries

1.1 Integration

We presuppose basic knowledge of the theory of Lebesgue integration on measurable subsets of $\mathbf{R}^n$. The few results listed below are often used subsequently and are given simply for the convenience of the reader. Proofs may be found in [54] and [157], for example.

By Ω we shall usually mean a measurable (often open) subset of $\mathbf{R}^n$; its Lebesgue n-measure will be denoted by $|\Omega|_n$, or even by $|\Omega|$ if no ambiguity is possible. All functions mentioned in this subsection are assumed to be extended real-valued; given any such function f on Ω, we set

$$f^+ = \max\{f, 0\}, f^- = -\min\{f, 0\}.$$

A measurable function f on Ω is said to be integrable over Ω if both $\int_\Omega f^+(x)\,dx$ and $\int_\Omega f^-(x)\,dx$ are finite.

Theorem 1.1 (The monotone convergence theorem) *Let $\{f_k\}_{k\in\mathbb{N}}$ be a non-decreasing sequence of measurable functions on Ω such that for some $k \in \mathbb{N}$, $\int_\Omega f_k^-(x)\,dx < \infty$. Then*

$$\lim_{l\to\infty} \int_\Omega f_l(x)\,dx = \int_\Omega \lim_{l\to\infty} f_l(x)\,dx.$$

Theorem 1.2 (Fatou's lemma) *Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of non-negative measurable functions on Ω. Then*

$$\int_\Omega \liminf_{k\to\infty} f_k(x)\,dx \leq \liminf_{k\to\infty} \int_\Omega f_k(x)\,dx.$$

Theorem 1.3 (Lebesgue's dominated convergence theorem) *Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of measurable functions on Ω such that for almost all $x \in \Omega$, $\lim_{k\to\infty} f_k(x) = f(x)$. Moreover, suppose that there is a function g, integrable over Ω, such that*

$$|f_k(x)| \leq g(x) \textit{ for all } k \in \mathbb{N} \textit{ and almost all } x \in \Omega.$$

Then f *and each* f_k *are integrable over* Ω *and*

$$\int_\Omega f(x)\,dx = \lim_{k\to\infty} \int_\Omega f_k(x)\,dx.$$

Theorem 1.4 (Fubini's theorem) *For* $i = 1, 2$, *let* Ω_i *be a measurable subset of* $\mathbb{R}^{n_i}$; *put* $\Omega = \Omega_1 \times \Omega_2$ *and suppose that* $f\colon \Omega \to \mathbb{R}$ *is such that* $\int_\Omega f(x, y)\,dx\,dy$ *is finite. Then* $\int_{\Omega_1} f(x, y)\,dx$ *exists for almost all* $y \in \Omega_2$, $\int_{\Omega_2} f(x, y)\,dy$ *exists for almost all* $x \in \Omega_1$, *and*

$$\int_\Omega f(x, y)\,dx\,dy = \int_{\Omega_1} \left(\int_{\Omega_2} f(x, y)\,dy \right) dx = \int_{\Omega_2} \left(\int_{\Omega 1} f(x, y)\,dx \right) dy.$$

To apply this theorem we need to know that the function f is integrable over Ω. This difficulty is overcome by Tonelli's theorem, which leads to the conclusion that if one of the iterated integrals

$$\int_{\Omega_1} \left(\int_{\Omega_2} |f(x, y)|\,dy \right) dx, \int_{\Omega_2} \left(\int_{\Omega_1} |f(x, y)|\,dx \right) dy$$

is finite, then f is integrable over Ω and the conclusion of Theorem 1.4 holds. For details of this, see [54], p. 194 and [164], pp. 353–354.

The next result gives connections between various types of convergence of functions.

Theorem 1.5 *Let* $p \in [1, \infty)$, *let* Ω *be a measurable subset of* $\mathbb{R}^n$ *and suppose that* f, f_k $(k \in \mathbb{N})$ *are functions on* Ω *such that*

$$\int_\Omega |f(x)|^p\,dx < \infty, \int_\Omega |f_k(x)|^p\,dx < \infty \quad (k \in \mathbb{N})$$

and

$$\int_\Omega |f(x) - f_k(x)|^p\,dx \to 0 \textit{ as } k \to \infty.$$

Then:

(i) There is a subsequence of $\{f_k\}$ *that converges pointwise a.e. to* f.
(ii) The sequence $\{f_k\}$ ***converges in measure*** *to* f*: that is, given any* $\varepsilon > 0$,

$$\lim_{k\to\infty} |\{x \in \Omega\colon\ |f_k(x) - f(x)| > \varepsilon\}| = 0.$$

We shall occasionally need to deal with integration over σ-finite measure spaces: details of this, which follow similar lines to that just detailed, may be found in Chapter 1 of [146].

1.2 Banach Spaces

It is assumed that the reader is familiar with the fundamental concepts concerning normed linear spaces: our purpose here is to place on record the notation and some basic facts. More details and proofs of the various assertions may be found in standard texts on functional analysis, such as [54].

Given a normed linear space X over the real or complex field, its norm will be denoted by $\|\cdot|X\|$ or $\|\cdot\|_X$, depending on the size of the expression X; if there is no ambiguity, we shall simply write $\|\cdot\|$. The closed ball in X with centre x and radius r is represented by $B(x, r)$, abbreviated to B_X if $x = 0$ and $r = 1$; by S_X will be meant the unit sphere $\{x \in X\colon \ \|x\| = 1\}$. A *Banach space* X is a normed linear space that is complete in the sense that every Cauchy sequence in X converges to a point in X. Let X, Y be Banach spaces over the same field of scalars and let $T\colon \ X \to Y$ be linear. Then T is continuous if and only if

$$\|T\| := \sup\{\|Tx\|_Y : \ \|x\|_X \le 1\} < \infty;$$

$B(X, Y)$ stands for the set of all continuous linear maps from X to Y, abbreviated to $B(X)$ if $X = Y$. The map $T \longmapsto \|T\|$ is a norm on this space endowed with which $B(X, Y)$ is a Banach space. The *dual* X^* of X is the space $B(X, \Phi)$, where Φ is the underlying scalar field. Given $x \in X$ and $x^* \in X^*$, we shall often denote $x^*(x)$ by $\langle x, x^*\rangle_X$, or even $\langle x, x^*\rangle$ if the context is clear. A sequence $\left\{x_j\right\}_{j\in\mathbb{N}}$ in X converges *strongly* to $x \in X$, written $x_j \to x$, if and only if $\left\|x - x_j\right\| \to 0$; it converges *weakly* to x, written $x_j \rightharpoonup x$, if and only if $\left\langle x_j - x, x^*\right\rangle \to 0$ for all $x^* \in X^*$. The *adjoint* of a map $T \in B(X, Y)$ is the map $T^*\colon \ Y^* \to X^*$ defined by

$$\left\langle x, T^* y^*\right\rangle_X = \left\langle Tx, y^*\right\rangle_Y \text{ for all } x \in X \text{ and all } y^* \in Y^*.$$

It emerges that $T^* \in B(Y^*, X^*)$ and $\|T^*\| = \|T\|$. A linear map $T\colon X \to Y$ is said to be *compact* if, for every bounded set $B \subset X$, the closure $\overline{T(B)}$ is compact in Y; equivalently, given any bounded sequence $\{x_n\}$ in X, $\{Tx_n\}$ has a subsequence that converges in Y. A compact linear map is necessarily bounded; the family $K(X, Y)$ of all compact linear maps from X to Y is closed in $B(X, Y)$.

A map $T \in B(X, Y)$ is said to be *strictly singular* if there is no infinite dimensional closed subspace Z of X such that the restriction $T|_Z$ of T to Z is an isomorphism of Z onto $T(Z)$. Equivalently, for each infinite-dimensional closed subspace Z of X,

$$\inf\{\|Tx\|_Y : \ \|x\|_X = 1, x \in Z\} = 0.$$

If instead T has the property that given any $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that if E is a subspace of X with $\dim E \ge N(\varepsilon)$, then there exists $x \in E$, with

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$\|x\|_X = 1$, such that $\|Tx\|_Y \le \varepsilon$, then T is said to be *finitely strictly singular.* This second definition can be expressed in terms of the Bernstein numbers $b_k(T)$ of T. We recall that these are given, for each $k \in \mathbb{N}$, by

$$b_k(T) = \sup_{E \subset X, \dim E = k} \inf_{x \in E, \|x\|_X = 1} \|Tx\|_X .$$

Then T is finitely strictly singular if and only if

$$b_k(T) \to 0 \text{ as } k \to \infty.$$

The relations between these notions and that of compactness of T are illustrated by the following diagram:

$$T \text{ compact} \implies T \text{ finitely strictly singular} \implies T \text{ strictly singular}$$

and each reverse implication is false in general. For further details and general background information concerning these matters, together with particular examples, we refer to [1], [117], [118], [119] and [147].

We write $X \hookrightarrow Y$ to signify that X can be identified with a subset of Y and that the natural embedding map from X to Y is continuous; if this map is compact we write $X \hookrightarrow\hookrightarrow Y$. The dual X^* of a Banach space X is also a Banach space, the dual of which is denoted by X^{**}. There is a natural mapping $\kappa\colon X \to X^{**}$ defined by

$$\left\langle x^*, \kappa x \right\rangle_{X^*} = \left\langle x, x^* \right\rangle_X \text{ for all } x \in X \text{ and all } x^* \in X^*;$$

κ is an isometric isomorphism of X onto $\kappa(X)$. If $\kappa(X) = X^{**}$ the space X is said to be *reflexive*. An important property of any reflexive space X is that every bounded sequence in X has a subsequence that is weakly convergent to some point of X.

The *modulus of convexity* of a Banach space X (with $\dim X \ge 2$) is the map $\delta_X\colon (0, 2] \to [0, 1]$ defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x + y\| : x, y \in B_X, \|x - y\| \ge \varepsilon\right\};$$

the space X is said to be *uniformly convex* if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$; every uniformly convex space is reflexive. Morover, if $\left\{x_j\right\}_{j\in\mathbb{N}}$ is a sequence in a uniformly convex space X such that $x_j \rightharpoonup x \in X$ and $\left\|x_j\right\| \to \|x\|$, then $x_j \to x$. For details of the companion notion of uniform smoothness see [61], Chapter 1.

A Banach space X is said to have the *approximation property* (AP) if, given any compact subset K of X and any $\varepsilon > 0$, there exists $T \in B(X)$ with finite rank such that $\|Tx - x\| < \varepsilon$ for all $x \in K$. Every Banach space X with a basis has the AP: we recall that a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X is a (Schauder) *basis* of X if, given any $x \in X$, there is a unique sequence $\{a_n\}_{n\in\mathbb{N}}$ of scalars such that $x = \sum_{n=1}^{\infty} a_n x_n$. We refer to Chapter 1 of [61] and the references given there for proofs of these assertions, together with that made below, and further background.

A map $\mu : [0, \infty) \to [0, \infty)$ that is continuous, strictly increasing and has the properties that $\mu(0) = 0$ and $\lim_{t\to\infty} \mu(t) = \infty$ is called a *gauge function*. Given a gauge function μ and a Banach space X with uniformly convex dual X^*, there is a map $\kappa : X \to X^*$ such that for each $x \in X$, $\kappa x = x^*$, where

$$\langle x, x^* \rangle = \|x\| \left\|x^*\right\| \text{ and } \left\|x^*\right\| = \mu\left(\|x\|\right).$$

That this map is well defined is a consequence of the uniform convexity of X^*: indeed this is the case under weaker assumptions on X^*. The map κ is called the *duality map* on X (with gauge function μ) and is continuous on X: $\kappa x_k \to \kappa x$ in X^* whenever $x_k \to x$ in X. For proofs of these assertions and further details we refer to [61], Chapter 1.

A particularly important class of Banach spaces is that of Hilbert spaces, which we now briefly recall. An *inner product* on a linear space X over a scalar field Φ is a map $(\cdot, \cdot) : X \times X \to \Phi$ such that

(i) $(\alpha x_1 + \beta x_2, y) = \alpha\,(x_1, y) + \beta\,(x_2, y)$ for all $\alpha, \beta \in \Phi$ and all $x_1, x_2, y \in X$;
(ii) $(x, y) = \overline{(y, x)}$ for all $x, y \in X$;
(iii) $(x, x) > 0$ if $x \in X\backslash\{0\}$.

A linear space X equipped with an inner product is called an *inner product space*; the map $x \mapsto (x, x)^{1/2}$ is a norm on X; and if the resulting normed linear space is complete, it is said to be a *Hilbert space*. Every Hilbert space is uniformly convex.

To conclude this section we give various examples of Banach spaces.

(i) $\mathbb{R}^n$ and $\mathbb{C}^n$ with norm given by $\|x\| = \left(\sum_{j=1}^{n} \left|x_j\right|^2\right)^{1/2}$, $x = (x_1, \ldots, x_n)$; these are Hilbert spaces, with the natural definition of the inner product.
(ii) l_p, the space of all sequences $x = \left\{x_j\right\}_{j\in\mathbb{N}}$ of scalars such that

$$\|x\|_p := \left(\sum_{j=1}^{\infty} \left|x_j\right|^p\right)^{1/p} < \infty\,(1 \le p < \infty),$$

and

$$\|x\|_\infty := \sup_{j\in\mathbb{N}} \left|x_j\right| < \infty.$$

(iii) $L_p(\Omega)$, the linear space of all (Lebesgue) measurable functions on a measurable subset Ω of $\mathbb{R}^n$, functions equal almost everywhere being identified, such that

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$$\|f\|_{p,\Omega} := \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p} < \infty \; (1 \le p < \infty) \, ,$$

and

$$\|f\|_{\infty,\Omega} := \operatorname*{ess\,sup}_{\Omega} |f(x)|.$$

When $1 < p < \infty$, the spaces l_p and $L_p(\Omega)$ are uniformly convex and have the AP; they are Hilbert spaces, with natural definitions of the inner product, if $p = 2$. The duality map κ on $L_p(\Omega)$ $(1 < p < \infty)$ with gauge function $t \longmapsto t^{p-1}$ is given by $\kappa f = |f|^{p-2} f \; \big(f \in L_p(\Omega)\big)$.

1.3 Function Spaces

1.3.1 Spaces of Continuous Functions

Throughout, Ω will stand for a non-empty open subset of $\mathbb{R}^n$ with boundary $\partial\Omega$ and closure $\overline{\Omega}$; a *domain* is a connected open set. Points of $\mathbb{R}^n$ will be denoted by $x = (x_i) = (x_1, \ldots, x_n)$ and we write $|x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$ and $(x, y) = \sum_{i=1}^n x_i y_i$; given $r > 0$, we put $B(x, r) = \{y \in \mathbb{R}^n \colon \; |x - y| < r\}$, abbreviating this to B_r if $x = 0$. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we write

$$\alpha! = \prod_{j=1}^n \alpha_j!, \;\; |\alpha| = \sum_{j=1}^n \alpha_j, \;\; x^\alpha = \prod_{j=1}^n x_j^{\alpha_j} \;\; (x \in \mathbb{R}^n)$$

and

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} := \prod_{j=1}^n D_j^{\alpha_j}, \;\; \text{where } D_j = \partial/\partial x_j;$$

it is to be understood that if some α_j is zero, then the corresponding term is to be omitted; if all α_j are zero, so that $\alpha = 0$, then $D^\alpha u = u$ for any appropriate function u.

Given any $k \in \mathbb{N}_0$, by $C^k(\Omega)$ is meant the linear space of all real- or complex-valued functions u on Ω such that for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le k$, the function $D^\alpha u$ exists and is continuous on Ω. The subspace of $C^k(\Omega)$ consisting of all those functions with compact support contained in Ω is denoted by $C_0^k(\Omega)$, and $C_0^\infty(\Omega) := \cap_{k=1}^\infty C_0^k(\Omega)$; recall that the *support* of a function u, supp u, is the closure of $\{x \in \Omega \colon \; u(x) \ne 0\}$. The function ϕ defined on $\mathbb{R}^n$ by

$$\phi(x) = \begin{cases} \exp\left(\frac{-1}{1-|x|^2}\right), & |x| < 1, \\ 0, & |x| \ge 1 \end{cases}$$

belongs to $C_0^\infty(\mathbb{R}^n)$, with $\operatorname{supp} \phi = \overline{B(0,1)}$ and $\int_{\mathbb{R}^n} \phi(x) \, dx > 0$, so that $\psi := \phi / \int_{\mathbb{R}^n} \phi(x) \, dx$ has the useful properties that $\psi \in C_0^\infty(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \psi(x) \, dx = 1$.

We define $C^k(\overline{\Omega})$ to be the linear space of all bounded functions u in $C^k(\Omega)$ such that u and all its derivatives $D^\alpha u$ with $|\alpha| \le k$ have bounded, continuous extensions to $\overline{\Omega}$: a norm $||| \cdot |||_{k,\Omega}$ is defined on this space by

$$||| u |||_{k,\Omega} := \max_{|\alpha|\le k} \sup_{x\in\Omega} |D^\alpha u(x)|,$$

and $C^k(\overline{\Omega})$ becomes a Banach space when given this norm. For a discussion of the advantages and disadvantages of this notation see [65], p. 10.

Let $k \in \mathbb{N}_0$, $\lambda \in (0, 1]$. We shall need various spaces of Hölder-continuous functions. First, $C^{0,\lambda}(\Omega)$ (often written as $C^\lambda(\Omega)$) will stand for the linear space of all continuous functions on Ω which satisfy a local Hölder condition on Ω; that is, given any compact subset K of Ω, there is a constant $C > 0$ such that

$$|u(x) - u(y)| \le C\,|x - y|^\lambda \text{ for all } x, y \in K.$$

We also put

$$C^{k,\lambda}(\Omega) := \left\{u \in C^k(\Omega) \colon D^\alpha u \in C^\lambda(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| = k\right\}.$$

These spaces are not given norms. However,

$$C^{k,\lambda}(\overline{\Omega}) := \Big\{u \in C^k(\overline{\Omega}) \colon \text{given any } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| = k, \text{ there exists } C > 0 \text{ such that for all } x, y \in \Omega,\ |D^\alpha u(x) - D^\alpha u(y)| \le C\,|x - y|^\lambda\Big\}$$

becomes a Banach space when provided with the norm

$$||| u |||_{k,\lambda,\Omega} := ||| u |||_{k,\Omega} + |u|_{k,\lambda,\Omega},$$

where

$$|u|_{k,\lambda,\Omega} = \max_{|\alpha|=k} \sup_{x,y\in\Omega, x\ne y} |D^\alpha u(x) - D^\alpha u(y)| \,/\, |x - y|^\lambda.$$

For convenience, when $\lambda \in (0, 1)$, we write $C^\lambda(\overline{\Omega}) = C^{0,\lambda}(\overline{\Omega})$ and $||| \cdot |||_{\lambda,\Omega}$, $| \cdot |_{\lambda,\Omega}$ instead of $||| \cdot |||_{0,\lambda,\Omega}$, $| \cdot |_{0,\lambda,\Omega}$ respectively. By $C_0^{k,\lambda}(\Omega)$ will be meant the linear subspace of $C^{k,\lambda}(\Omega)$ consisting of all those functions with compact support contained in Ω. Note that if $u, v \in C^\lambda(\overline{\Omega})$, then

$$|uv|_{\lambda,\Omega} \le ||| u |||_{0,\Omega} |v|_{\lambda,\Omega} + ||| v |||_{0,\Omega} |u|_{\lambda,\Omega};$$

and that if $u \in C^{\lambda_1}(\overline{\Omega})$, $v \in C^{\lambda_2}(\overline{\Omega})$ and Ω is bounded, then $uv \in C^\gamma(\overline{\Omega})$, where $\gamma = \min(\lambda_1, \lambda_2)$, and

$$||| uv |||_{\gamma,\Omega} \le \max\left\{1, |\text{diam } \Omega|^{\lambda_1+\lambda_2-2\gamma}\right\} ||| u |||_{\lambda_1,\Omega} ||| v |||_{\lambda_2,\Omega}.$$

Useful properties relating these spaces of functions are given in the following theorem.

Theorem 1.6 *Let $k \in \mathbb{N}_0$, $0 < \nu < \lambda \leq 1$ and suppose that Ω is an open subset of $\mathbb{R}^n$. Then*

$$(i)\ C^{k+1}\left(\overline{\Omega}\right) \hookrightarrow C^k\left(\overline{\Omega}\right)$$

and

$$(ii)\ C^{k,\lambda}\left(\overline{\Omega}\right) \hookrightarrow C^{k,\nu}\left(\overline{\Omega}\right) \hookrightarrow C^k\left(\overline{\Omega}\right).$$

If Ω is bounded, both the embeddings in (ii) are compact. If Ω is convex, then

$$(iii)\ C^{k+1}\left(\overline{\Omega}\right) \hookrightarrow C^{k,1}\left(\overline{\Omega}\right)$$

and

$$(iv)\ C^{k+1}\left(\overline{\Omega}\right) \hookrightarrow C^{k,\nu}\left(\overline{\Omega}\right).$$

If Ω is bounded and convex, then the embeddings in (i) and (iv) are compact.

Finally we turn to conditions useful in the extension of functions. Let Ω be an open subset of $\mathbb{R}^n$ ($n \geq 2$) with non-empty boundary $\partial\Omega$, let $k \in \mathbb{N}_0$ and suppose that $\gamma \in [0, 1]$. Given $x_0 \in \partial\Omega$, $r > 0$, $\beta > 0$, local Cartesian co-ordinates $y = (y_1, ..., y_n) = (y', y_n)$ (where $y' = (y_1, ..., y_{n-1})$), with $y = 0$ at $x = x_0$, and a real continuous function $h\colon y' \longmapsto h(y')$ ($\left|y'\right| < r$), we define a neighbourhood $U_{r,\beta,h}(x_0)$ of x_0 (an open subset of $\mathbb{R}^n$ containing x_0) by

$$U = U_{r,\beta,h}(x_0) = \left\{y \in \mathbb{R}^n\colon h(y') - \beta < y_n < h(y') + \beta, \left|y'\right| < r\right\}.$$

Then Ω is said to have boundary $\partial\Omega$ of class $C^{k,\gamma}$ if for each $x_0 \in \partial\Omega$ there are a local co-ordinate system, positive constants r and β and a function $h \in C^{k,\gamma}\left(B_r'\right)$ (where $B_r' = \left\{y' \in \mathbb{R}^{n-1}\colon \left|y'\right| < r\right\}$) such that

$$U_{r,\beta,h}(x_0) \cap \partial\Omega = \left\{y \in \mathbb{R}^n\colon y_n = h(y'), \left|y'\right| < r\right\}$$

and

$$U_{r,\beta,h}(x_0) \cap \Omega = \left\{y \in \mathbb{R}^n\colon h(y') - \beta < y_n < h(y'), \left|y'\right| < r\right\}.$$

In general, the constants r, β and the function h depend on x_0. However, if in addition Ω is bounded, there are points $x_1, ..., x_m \in \partial\Omega$, positive numbers r and β (independent of the x_j) and functions $h_1, ..., h_m$ such that the neighbourhoods $U_j = U_{r,\beta,h_j}(x_j)$ ($j = 1, ..., m$) cover $\partial\Omega$. When $\gamma = 0$ we simply write $\partial\Omega \in C^k$ (or $\partial\Omega \in C$ if $k = 0$). If $\partial\Omega \in C^{0,1}$ we shall say that the boundary is of Lipschitz class: if Ω is convex its boundary is of this class.

1.3.2 Morrey and Campanato Spaces

These are defined by means of some kind of mean oscillation property imposed on their elements. All we need in this book is that certain Campanato spaces are isomorphic to spaces of Hölder-continuous functions, but their great importance in the past coupled with considerable current research activity lead us to give some basic definitions and results, together with references in which more details are provided.

Throughout this subsection we shall suppose that Ω is a bounded open subset of $\mathbb{R}^n$ with the property that there exists $A > 0$ such that

$$|B(x, r) \cap \Omega| \geq Ar^n \text{ for all } x \in \Omega \text{ and all } r \leq \operatorname{diam} \Omega. \tag{1.3.1}$$

This condition means that $\partial\Omega$ cannot have sharp outward cusps; Lipschitz boundaries are allowed.

Definition 1.7 Let $p \in [1, \infty)$ and $\lambda \geq 0$. The Morrey space $M^{p,\lambda}(\Omega)$ is the space of all $u \in L_p(\Omega)$ such that

$$\left\| u|M^{p,\lambda}(\Omega) \right\|^p := \sup_{x_0 \in \Omega, 0<r<\operatorname{diam}\Omega} r^{-\lambda} \int_{\Omega\cap B(x_0,r)} |u(x)|^p \, dx < \infty.$$

When endowed with the norm $\left\| \cdot|M^{p,\lambda}(\Omega) \right\|$ it becomes a Banach space. The Campanato space $\mathcal{L}^{p,\lambda}(\Omega)$ is the space of all $u \in L_p(\Omega)$ such that

$$\{u\}_{p,\lambda}^p := \sup_{x_0 \in \Omega, 0<r<\operatorname{diam}\Omega} r^{-\lambda} \int_{\Omega\cap B(x_0,r)} \left| u(x) - u_{x_0,r} \right|^p dx < \infty,$$

where

$$u_{x_0,r} = |\Omega \cap B(x_0, r)|^{-1} \int_{\Omega\cap B(x_0,r)} u(x) \, dx.$$

Furnished with the norm

$$\left\| u|\mathcal{L}^{p,\lambda}(\Omega) \right\| := \|u\|_{p,\Omega} + \{u\}_{p,\lambda},$$

it is a Banach space. The space $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ is defined analogously.

By way of background we list some of the main properties of these spaces.

(i) For all $p \in (1, \infty)$, both $M^{p,0}(\Omega)$ and $\mathcal{L}^{p,0}(\Omega)$ are isomorphic to $L_p(\Omega)$; $M^{p,n}(\Omega)$ is isomorphic to $L_\infty(\Omega)$.

(ii) If $1 \leq p \leq q < \infty$ and λ, ν are non-negative numbers such that $(\lambda - n)/p \leq (\nu - n)/q$, then

$$M^{q,\nu}(\Omega) \hookrightarrow M^{p,\lambda}(\Omega) \text{ and } \mathcal{L}^{q,\nu}(\Omega) \hookrightarrow \mathcal{L}^{p,\lambda}(\Omega).$$

(iii) Suppose that $p \in [1, \infty)$. Then

$$\mathcal{L}^{p,\lambda}(\Omega) \text{ is isomorphic to } M^{p,\lambda}(\Omega) \text{ if } \lambda \in [0, n),$$

$$M^{p,\lambda}(\Omega) = \{0\} \text{ if } \lambda > n,$$

$$\mathcal{L}^{p,\lambda}(\Omega) \text{ is isomorphic to } C^{(\lambda-n)/p}\left(\overline{\Omega}\right) \text{ if } \lambda \in (n, n+p],$$

and

$$\mathcal{L}^{p,\lambda}(\mathbb{R}^n) \text{ is isomorphic to } C^{(\lambda-n)/p}(\mathbb{R}^n) \text{ if } \lambda \in (n, n+p].$$

For proofs of these assertions and further details, we refer to [88], [106] (especially for the claim concerning $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$), [146] and [152].

1.3.3 Banach Function Spaces

To explain what these are we begin with the notion of the non-increasing rearrangement of a measurable function and refer to [21], [60] or [146] for further details and proofs. Let (R, μ) be a σ-finite measure space and set

$$\mathcal{M}(R, \mu) = \left\{f : f \text{ is a measurable function on } R \text{ with values in } [-\infty, \infty]\right\},$$
$$\mathcal{M}_0(R, \mu) = \left\{f \in \mathcal{M}(R, \mu) : f \text{ is finite } \mu\text{-a.e. on } R\right\}$$

and

$$\mathcal{M}_+(R, \mu) = \{f \in \mathcal{M}_0(R, \mu) : f \geq 0\}.$$

When R is a Lebesgue-measurable subset Ω of $\mathbb{R}^n$ and μ is a Lebesgue n-measure these objects are denoted by $\mathcal{M}(\Omega)$, etc. The *non-increasing rearrangement* $f^*: [0, \infty) \to [0, \infty]$ of a function $f \in \mathcal{M}(R, \mu)$ is defined by

$$f^*(t) = \inf\{\lambda \in (0, \infty) : \mu(\{s \in R : |f(s)| > \lambda\}) \leq t\}, \, t \in [0, \infty).$$

The *maximal non-increasing rearrangement* $f^{**}: (0, \infty) \to [0, \infty]$ of a function $f \in \mathcal{M}(R, \mu)$ is given by

$$f^{**}(t) = t^{-1} \int_0^t f^*(s) ds, \, t \in (0, \infty).$$

If $|f| \leq |g|$ μ-a.e. in R, then $f^* \leq g^*$; however, the map $f \longmapsto f^*$ does not preserve sums or products of functions and is not subadditive. By way of compensation it turns out (see Chapter 2, (3.10) of [21]) that for all $t \in (0, \infty)$ and all $f, g \in \mathcal{M}_0(R, \mu)$,

$$\int_0^t (f+g)^*(s)\, ds \leq \int_0^t f^*(s)\, ds + \int_0^t g^*(s)\, ds,$$

so that

$$(f+g)^{**} \le f^{**} + g^{**}.$$

Moreover, the *Hardy lemma* (see Chapter 2, Proposition 3.6 of [21]) asserts that if f, g are non-negative measurable functions on $(0, \infty)$ such that

$$\int_0^t f(s)\, ds \le \int_0^t g(s)\, ds$$

for all $t \in (0, \infty)$ and $h\colon (0,\infty) \to [0,\infty)$ is non-increasing, then

$$\int_0^\infty f(s)h(s)\, ds \le \int_0^\infty g(s)h(s)\, ds.$$

The *Hardy–Littlewood inequality* (see Chapter 2, Theorem 2.2 of [21]) states that for all $f, g \in \mathcal{M}_0(R,\mu)$,

$$\int_R |fg|\, d\mu \le \int_0^\infty f^*(t)g^*(t)\, dt.$$

If (R,μ) and (S,ν) are σ-finite measure spaces, functions $f \in \mathcal{M}_0(R,\mu)$ and $g \in \mathcal{M}_0(S,\nu)$ are said to be *equimeasurable,* and we write $f \sim g$ if $f^* = g^*$ on $(0,\infty)$.

After these preliminaries we introduce the notion of a *Banach function norm,* by which is meant a functional $\rho\colon\ \mathcal{M}_0(R,\mu) \to [0,\infty]$ such that, for all f, g and $\{f_j\}_{j\in\mathbb{N}}$ and all $\lambda \ge 0$, the following conditions are satisfied:

(P1) $\rho(f) = 0$ if and only if $f = 0$; $\rho(\lambda f) = \lambda\rho(f)$; $\rho(f+g) \le \rho(f) + \rho(g)$ (the *norm axiom*);

(P2) $f \le g$ μ–a.e. implies $\rho(f) \le \rho(g)$ (the *lattice axiom*);

(P3) $f_j \uparrow f$ μ–a.e. implies $\rho\left(f_j\right) \uparrow \rho(f)$ (the *Fatou axiom*);

(P4) $\rho(\chi_E) < \infty$ for every $E \subset R$ with finite measure (the *non-triviality axiom*);

(P5) if $E \subset R$ with $\mu(E) < \infty$, then there is a constant $C_E \in (0,\infty)$, depending only on E and ρ, such that for all $f \in \mathcal{M}_+(R,\mu)$,

$$\int_E f d\mu \le C_E\rho(f)$$

(the *local embedding in* L_1);

if, in addition, ρ satisfies

(P6) $\rho(f) = \rho(g)$ whenever $f^* = g^*$ $(f,\ g \in \mathcal{M}_+(R,\mu))$ (the *rearrangement-invariance axiom*),

then we say that ρ is a *rearrangement-invariant (r.i.) norm.*

If ρ satifies conditions (P1)–(P5), the space

$$X = X(\rho) := \{f \in \mathcal{M}(R,\mu) : \rho(|f|) < \infty\}$$

is said to be a *Banach function space*. With the natural linear space operations it is easy to check that X is a linear space and $\|\cdot\|_X$, where

$$\|f\|_X := \rho(|f|),$$

is a norm on it that makes it into a Banach space. If this Banach space X also satisfies condition (P6), it is called a *rearrangement-invariant space,* written r.i. Note that $\|f\|_X$ is defined for every $f \in \mathcal{M}(R,\mu)$ and that $\|f\|_X < \infty$ if and only if $f \in X$.

With any r.i. function norm ρ is associated another functional, ρ', defined for all $g \in \mathcal{M}_+(R,\mu)$ by

$$\rho'(g) = \sup\left\{\int_R fg d\mu : f \in \mathcal{M}_+(R,\mu),\, \rho(f) \le 1\right\}.$$

This functional is also an r.i. norm and is called the *associate norm* of ρ. For every r.i. norm ρ and every $f \in \mathcal{M}_+(R,\mu)$, it turns out that (see Chapter 1, Theorem 2.9 of [21])

$$\rho(f) = \sup\left\{\int_R fg d\mu : g \in \mathcal{M}_+(R,\mu),\, \rho(g) \le 1\right\}.$$

If ρ is an r.i. norm, $X = X(\rho)$ is the r.i. space determined by ρ and ρ' is the associate norm of ρ, then the function space $X(\rho')$ determined by ρ' is called the *associate space* of X and is denoted by X'. It emerges that $\left(X'\right)' = X$; and the Hölder inequality

$$\int_R fg d\mu \le \|f\|_X \|g\|_{X'}$$

holds for all $f, g \in \mathcal{M}(R,\mu)$.

From the Hardy lemma follows (see Chapter 2, Theorem 4.6 of [21]) the *Hardy–Littlewood principle,* which asserts that if functions f and g satisfy the *Hardy–Littlewood–Pólya relation,* defined by

$$\int_0^t f^*(s)\, ds \le \int_0^t g^*(s)\, ds,\ t \in (0,\infty),$$

sometimes denoted by $f \prec g$ in the literature, then $\|f\|_X \le \|g\|_X$ if the underlying measure space is resonant: this condition means that either (R,μ) is nonatomic or it is completely atomic, all atoms having equal measure. From now on we shall suppose that (R,μ) is nonatomic.

Given any r.i. space X over a measure space (R,μ), the Luxemburg representation theorem (see Chapter 2, Theorem 4.10 of [21]) implies that there is a unique r.i. space $X(0,\mu(R))$ over the interval $(0,\mu(R))$ endowed with the one-dimensional Lebesgue measure such that $\|f\|_X = \|f^*\|_{X(0,\mu(R))}$. This space is called the *representation space* of X and is often denoted by $\overline{X}(0,\mu(R))$ or

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even $\overline{X}$. When $R = (0, \infty)$ and μ is Lebesgue measure, every r.i. space X over (R, μ) coincides with its representation space.

If $X = X(\rho)$ is the r.i. space determined by an r.i. norm ρ, its *fundamental function* ϕ_X is defined by

$$\phi_X(t) = \rho(\chi_E), t \in [0, \mu(R)),$$

where $E \subset R$ is such that $\mu(E) = t$. That ϕ_X is well defined is a consequence of the properties of r.i. norms and the nonatomicity of (R, μ). A useful property of the fundamental function is that

$$\phi_X(t)\phi_{X'}(t) = t,\ \ t \in [0, \mu(R)).$$

When $p \in [1, \infty)$ and $X = L_p(R, \mu)$ it is easy to see that $\phi_X(t) = t^{1/p}$ $(t \in [0, \mu(R))$.

Basic examples of r.i. spaces are provided by the usual L_p spaces. With an eye to later applications, we confine ourselves for the moment to the case in which R is an open subset Ω of $\mathbb{R}^n$ and μ is Lebesgue n-measure. For $p \in [1, \infty]$ define the functional ρ_p by

$$\rho_p(f) = \|f\|_{p,\Omega} = \|f\|_p = \begin{cases} \left(\int_\Omega |f|^p\, dx\right)^{1/p}, & \text{if } 1 \le p < \infty, \\ \text{ess sup}_{\,\Omega} |f|, & \text{if } p = \infty \end{cases}$$

for $f \in \mathcal{M}(\Omega, \mu)$. This is an r.i. norm with corresponding r.i. space $L_p(\Omega)$. More generally, given $p, q \in [1, \infty]$, define $\rho_{p,q}$ by

$$\rho_{p,q}(f) = \|f\|_{p,q,\Omega} = \|f\|_{p,q} = \left\| s^{1/p-1/q} f^*(s) \right\|_{q,(0,|\Omega|)}$$

for $f \in \mathcal{M}(\Omega, \mu)$. The set $L_{p,q}(\Omega)$, defined to be the family of all $f \in \mathcal{M}(\Omega, \mu)$ such that $\rho_{p,q}(f) < \infty$, is called a *Lorentz space*. If either $1 < p < \infty$ and $1 \le q \le \infty$, or $p = q = 1$, or $p = q = \infty$, then $\rho_{p,q}$ is equivalent to an r.i. norm in the sense that there are an r.i. norm σ and a constant $C \in (0, \infty)$, depending on p and q but independent of f, such that for all $f \in L_{p,q}(\Omega)$,

$$C^{-1}\sigma(f) \le \rho_{p,q}(f) \le C\sigma(f).$$

Accordingly $L_{p,q}(\Omega)$ is considered to be an r.i. space for these values of p and q: see [21], Chapter 4. If $p = 1$ and $q > 1$, then $L_{p,q}(\Omega)$ is a quasi-normed space; if $p = \infty$ and $q < \infty$, then $L_{p,q}(\Omega) = \{0\}$.

For all $p \in [1, \infty]$, $L_{p,p}(\Omega) = L_p(\Omega)$. The dependence of the Lorentz spaces on the first index is given by

$$L_{r,s}(\Omega) \hookrightarrow L_{p,q}(\Omega) \text{ if } 1 \le p < r \le \infty \text{ and } q, s \in [1, \infty];$$

as for the second index we have, when $|\Omega| < \infty$,

$$L_{p,q}(\Omega) \hookrightarrow L_{p,r}(\Omega) \text{ if } p \in [1, \infty] \text{ and } 1 \le q < r \le \infty.$$

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For further details and proofs see [60] and [146].

Given any sequence $\{E_j\}$ of measurable subsets of R, we write $E_j \to \emptyset$ a.e. if $\chi_{E_j} \to 0$ $\mu-$a.e. If every f in the Banach function space X has the property that $\left\| f\chi_{E_j} \right\|_X \to 0$ whenever $E_j \to \emptyset$ a.e., X is said to have *absolutely continuous norm*; it turns out that to verify this property it is enough to consider decreasing sequences $\{E_j\}$. When X and Y are Banach function spaces (with the same underlying measure space), we say that X is *almost compactly embedded in* Y, and write $X \overset{*}{\hookrightarrow} Y$, if, for every sequence $\{E_j\}$ of measurable sets such that $E_j \to \emptyset$ a.e., we have

$$\lim_{j\to\infty} \sup_{\|u\|_X \le 1} \left\| u\chi_{E_j} \right\|_Y = 0.$$

Replacement of $\{E_j\}$ by $\{\cup_{k\ge j} E_k\}$ shows that in this definition the sequence $\{E_j\}$ may be taken to be non-increasing. This notion is useful in establishing the compactness of embeddings of Sobolev spaces. For a connected account of it, based on results given in [159] and [76], we refer to [146]. Among the important properties established in these references are the following, in which X, Y are assumed to be Banach function spaces over a σ-finite measure space (R, μ):

(i) If $X \hookrightarrow\hookrightarrow Y$, then $X \overset{*}{\hookrightarrow} Y$.
(ii) $X \overset{*}{\hookrightarrow} Y$ if and only if $Y' \overset{*}{\hookrightarrow} X'$.
(iii) $X \overset{*}{\hookrightarrow} Y$ if and only if for every sequence $\{f_k\}$ of $\mu-$measurable functions on R satisfying $\|f_k\|_X \le 1$ and $f_k \to 0$ $\mu-$a.e. we have $\|f_k\|_Y \to 0$.
(iv) If (R, μ) is completely atomic, then $X \overset{*}{\hookrightarrow} Y$ if and only if $X \hookrightarrow\hookrightarrow Y$.
(v) If (R, μ) is nonatomic and $\mu(R) = \infty$, then there is no pair X, Y such that $X \overset{*}{\hookrightarrow} Y$.
(vi) Suppose that (R, μ) is nonatomic and $0 < \mu(R) < \infty$. Then $X \overset{*}{\hookrightarrow} Y$ implies that

$$\lim_{t\to 0+} \phi_Y(t)/\phi_X(t) = 0;$$

the converse is false: see [107], p. 286. Moreover, $X \overset{*}{\hookrightarrow} Y$ if and only if

$$\lim_{t\to 0+} \sup_{\|f\|_X \le 1} \sup_{\mu(E)\le t} \|f\chi_E\|_Y = 0;$$

or equivalently,

$$\lim_{t\to 0+} \sup_{\|f\|_X \le 1} \left\| f^* \chi_{[0,t)} \right\|_{\overline{Y}} = 0.$$

From (vi) it follows immediately that if Ω is a bounded open subset of $\mathbb{R}^n$ and $1 \le q < p < \infty$, then $L_p(\Omega) \overset{*}{\hookrightarrow} L_q(\Omega)$. In fact, a similar argument shows that

$L_p(\Omega) \overset{*}{\hookrightarrow} L_{p,r}(\Omega)$ if $1 < p < r$ and $r \in (1, \infty)$, which since $L_{p,r}(\Omega) \hookrightarrow L_q(\Omega)$ if $p > q$ gives a sharpening of the earlier result.

1.4 The Palais–Smale Condition

Let X be a uniformly convex Banach space and suppose that $G \in C^1(X, \mathbb{R})$, so that the Fréchet derivative G' of G belongs to $B(X, X^*)$. A point $x \in X$ is said to be *critical* (for G) if $G'(x) = 0$; otherwise x is called *regular*. A real number λ is a *critical value* of G if there is a critical point $x \in X$ such that $G(x) = \lambda$; otherwise λ is a r*egular value* of G (even if $\lambda \notin G(X)$).

Now let $M := \{u \in X\colon G(u) = 1\}$; assumed to be non-empty, M is a C^1 manifold (see, for example, [51]); and we suppose that 1 is a regular value of G. Given $u \in M$,

$$T_u M := \left\{x \in X\colon \left\langle x, G'(u)\right\rangle = 0\right\}$$

is the *tangent space* T_uM of M at u; the norm on the dual space $(T_uM)^*$ will be denoted by $\left\| \cdot | \, (T_uM)^* \right\|$. Let $\Phi \in C^1(X, \mathbb{R})$ and represent its restriction to M by $\widetilde{\Phi}$. For each $u \in M$ the norm of the derivative of $\widetilde{\Phi}$ at u is

$$\left\| \widetilde{\Phi}'(u) \right\|_* := \left\| \Phi'(u) | \, (T_uM)^* \right\| .$$

The functional Φ is said to satisfy the *Palais–Smale condition at level c* (written $\Phi \in (PS)_{c,M}$) if every sequence $\{u_j\}$ in M such that

$$\lim_{j\to\infty} \Phi\left(u_j\right) = c \text{ and } \lim_{j\to\infty} \left\| \widetilde{\Phi}'(u_j) \right\|_* = 0$$

has a convergent subsequence.

Given any $k \in \mathbb{N}$, the unit sphere of $\mathbb{R}^k$ is denoted by S^k and we write

$$C_o\left(S^k, M\right) = \left\{h \in C\left(S^k, M\right)\colon h \text{ is odd}\right\} .$$

Finally, we state a theorem due to Cuesta [46].

Theorem 1.8 *Let $\Phi \in C^1(X, \mathbb{R})$ be even, suppose that $k \in \mathbb{N}$, set*

$$d = \inf_{h \in C_o\left(S^k, M\right)} \max_{z \in S^k} \Phi\left(h(z)\right)$$

and assume that $d \in \mathbb{R}$. If $\Phi \in (PS)_{d,M}$, then there exists $u \in M$ such that $\Phi(u) = d$ and $\widetilde{\Phi}'(u) = 0$.

The proof uses the Ekeland variational principle [71] and is too long to reproduce here. Application of this result will be made in Chapter 4 in connection with the second eigenvalue of the fractional p-Laplacian.

1.5 Inequalities

Here we collect some inequalities that will be useful later on, following largely the presentation of Appendix B of [29]. For shortness of presentation we introduce the function $J_p\colon \mathbb{R} \to [0, \infty)$ defined for each $p \in (1, \infty)$ by $J_p(t) = |t|^{p-2}\, t$. Note that on $[0, \infty)$ it is convex when $p > 2$ and concave if $p \in (1, 2]$.

Lemma 1.9 *Suppose that $a, b \in \mathbb{R}$ and $ab \le 0$. Then*

$$|a - b|^{p-2}\,(a - b)a \ge \begin{cases} |a|^p - (p-1)\,|a-b|^{p-2}\,ab & \text{if } p \in (1, 2], \\ |a|^p - (p-1)\,|a|^{p-2}\,ab & \text{if } p \in (2, \infty). \end{cases}$$

Proof We may suppose that $a \ge 0$ and $b \le 0$. When $p > 2$, the convexity of J_p implies that

$$J_p(x) + (y - x)J_p^{'}(x) \le J_p(y),\, 0 \le x \le y; \tag{1.5.1}$$

similarly, when $p \in (1, 2]$ we have

$$J_p(x) + (y - x)J_p^{'}(y) \le J_p(y),\, 0 \le x \le y. \tag{1.5.2}$$

Suppose that $p \in (1, 2]$. Then from (1.5.2) with $y = a - b$ and $x = a$, we see that

$$\begin{aligned} |a - b|^{p-2}\,(a - b) = J_p(a - b) &\ge J_p(a) - bJ_p^{'}(a - b) \\ &= |a|^p - (p-1)\,|a - b|^{p-2}\,b, \end{aligned}$$

from which the desired result follows immediately. The argument when $p > 2$ is similar, this time using (1.5.1). □

Lemma 1.10 *Let $p \in (1, \infty)$. Then there is a constant $c = c(p) > 0$ such that for all $a, b \in \mathbb{R}$,*

$$|a - b|^p \le |a|^p + |b|^p + c\left(|a|^2 + |b|^2\right)^{(p-2)/2} |ab|\,.$$

Proof When $ab \ge 0$ it is enough to suppose that $a, b \ge 0$ and $a \ge b$. But then

$$|a - b|^p \le a^p \le |a|^p + |b|^p$$

and the result follows. On the other hand, if $ab \le 0$, then we may assume that $a \ge 0$ and $b \le 0$, so that $b = -d$ for some $d \ge 0$. We have to show that

$$(a + d)^p \le a^p + d^p + c\left(a^2 + d^2\right)^{(p-2)/2} ad.$$

This is obvious when $a = 0$, and so it is enough to prove that

$$(1 + x)^p \le 1 + x^p + c\left(1 + x^2\right)^{(p-2)/2} x.$$

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Since

$$\lim_{x\to 0+}\frac{(1+x)^p-1-x^p}{\left(1+x^2\right)^{(p-2)/2}x}=\lim_{x\to\infty}\frac{(1+x)^p-1-x^p}{\left(1+x^2\right)^{(p-2)/2}x}=p,$$

the claim follows and completes the proof. □

Lemma 1.11 *For all $a, b \in \mathbb{R}$,*

$$\left(|b|^{p-2}\,b-|a|^{p-2}\,a\right)(b-a)\geq\begin{cases}(p-1)\,|b-a|^2\left(|a|^2+|b|^2\right)^{-(2-p)/2} & \text{if } p\in(1,2],\\ 2^{2-p}\,|b-a|^p & \text{if } p\in(2,\infty).\end{cases}$$

Proof Suppose that $p \in (1,2]$. Then

$$\begin{aligned}\left(J_p(b)-J_p(a)\right)(b-a)&=(b-a)\int_0^1\frac{d}{dt}J_p\left((1-t)a+tb\right)dt\\&=(p-1)(b-a)^2\int_0^1|(1-t)a+tb|^{p-2}\,dt.\end{aligned}$$

Since

$$|(1-t)a+tb|^2\leq(1-t)\,|a|^2+t\,|b|^2\leq|a|^2+|b|^2\,,$$

the desired inequality follows.

When $p \in (2,\infty)$ we argue as in [128] and use the identity

$$\begin{aligned}\left(|b|^{p-2}\,b-|a|^{p-2}\,a\right)(b-a)&=\frac{|b|^{p-2}+|a|^{p-2}}{2}\,|b-a|^2\\&\quad+\frac{\left(|b|^{p-2}-|a|^{p-2}\right)\left(|b|^2-\left|a^2\right|\right)}{2},\end{aligned}$$

from which it is immediate that

$$\begin{aligned}\left(|b|^{p-2}\,b-|a|^{p-2}\,a\right)(b-a)&\geq 2^{-1}\left(|b|^{p-2}+|a|^{p-2}\right)|b-a|^2\\&\geq 2^{2-p}\,|b-a|^p\,.\end{aligned}$$

□