Canad. Math. Bull. Vol. **62** (3), 2019 pp. 565–576
http://dx.doi.org/10.4153/S0008439518000073

The Coniveau Filtration on K_1 for Some Severi–Brauer Varieties

Eoin Mackall

Abstract. We produce an isomorphism $E_\infty^{m,-m-1} \cong \mathrm{Nrd}_1(A^{\otimes m})$ between terms of the K-theory coniveau spectral sequence of a Severi–Brauer variety X associated with a central simple algebra A and a reduced norm group, assuming A has equal index and exponent over all finite extensions of its center and that $\mathrm{SK}_1(A^{\otimes i}) = 1$ for all $i > 0$.

Notation and conventions

- We work over a fixed base field k.
- A variety is a separated scheme of finite type over a field.
- For a prime p, we write $v_p(-)$ for the p-adic valuation.

1 Introduction

Some K-cohomology groups were studied, and computed, for Severi–Brauer varieties associated with algebras of square-free degree in [MS82]. As an application of these computations one can compute the Chow groups of these Severi–Brauer varieties and find they are torsion free. Chow groups of arbitrary Severi–Brauer varieties X have been studied in depth and, in certain degrees, are known to be torsion free (*e.g.*, $\mathrm{CH}^0(X)$ is free trivially; $\mathrm{CH}^1(X)$ is torsion free by [Art82]; $\mathrm{CH}_0(X)$ is torsion free by [CM06]; if X is associated with an algebra whose index equals its exponent, then $\mathrm{CH}^2(X)$ is torsion free by [Kar98]).

The Chow groups of Severi–Brauer varieties are not always torsion free. Their torsion subgroups have also been studied in depth. In [Kar98], Karpenko shows that if X is a Severi–Brauer variety associated with an algebra having differing index and exponent, $\mathrm{CH}^2(X)$ sometimes contains a nontrivial torsion subgroup that surjects onto torsion in the graded group associated with the coniveau filtration on the Grothendieck group $\mathrm{G}_0(X)$. In a different direction, Merkurjev [Mer95] has shown there is sometimes nontrivial torsion in the Chow groups of Severi–Brauer varieties that occurs in codimension 3 or higher; this torsion cannot be detected by Karpenko's methods, since it is contained in the kernel of the canonical epimorphism from $\mathrm{CH}(X)$ onto the graded group associated with the coniveau filtration on the Grothendieck group $\mathrm{G}_0(X)$.

Recently, Karpenko computed the Chow ring of a Severi–Brauer variety associated with a central simple algebra with equal index and exponent under the assumption the

Received by the editors June 6, 2018; revised October 9, 2018.
Published online on Cambridge Core April 10, 2019.
AMS subject classification: **19E08,** 14C35.
Keywords: Severi–Brauer variety, coniveau spectral sequence.

Chow ring is generated by Chern classes [Kar17]. In this computation, the Chow ring turns out to be torsion free. Without the assumption that the Chow ring is generated by Chern classes, any nontrivial torsion in the Chow ring of such a Severi–Brauer variety will come from nontrivial differentials in the K-theory coniveau, or Brown–Gersten–Quillen, spectral sequence.

This article stemmed from exploring the possibility of torsion in the Chow group of a Severi–Brauer variety associated with an algebra A with index equal to its exponent. Hopefully, it will be of use in further study of this problem.

Section 2 is mainly for reference and introducing notation. In Section 3 we prove a series of lemmas that will be used for the main results of Sections 4 and 5.

In Section 4, we compute the $E_\infty^{m,-m-1}$ terms of the K-theory coniveau spectral sequence for any Severi–Brauer variety X associated with an algebra A satisfying the following properties: the index of A is a power of a prime p, the exponent of A equals the index of A over all finite extensions of the center of A, and the reduced Whitehead groups $\mathrm{SK}_1(A^{\otimes r}) = 1$ vanish for all $r \geq 1$. This result is a direct generalization of the known computation for the terms $E_\infty^{m,-m}$, and the proof of the main theorem manages to describe both simultaneously. The main result is Theorem 4.2; its proof is elementary, but it requires some involved arguments comparing the reduced norms of certain tensor powers of a given algebra.

In Section 5, we show how to prove the general case stated in the abstract using the primary case of Section 4.

2 On the K-theory of a Severi–Brauer Variety

The material in this section was developed in detail by Quillen [Qui73]. The K-theory coniveau spectral sequence, or the Brown–Gersten–Quillen spectral sequence, is a fourth quadrant cohomological spectral sequence

$$E_1^{p,q} = \coprod_{x\in X^{(p)}} \mathrm{K}_{-p-q}(k(x)) \Longrightarrow \mathrm{G}_{-p-q}(X),$$

where $X^{(p)}$ denotes the set of codimension p points of X. For a variety X, the spectral sequence converges, and for a regular variety X, one can identify the E_2-terms with K-cohomology groups

$$E_2^{p,q} = \mathrm{H}^p(X, \mathcal{K}_{-q}) \Longrightarrow \mathrm{G}_{-p-q}(X).$$

Recall that the K-cohomology groups $\mathrm{H}^p(X, \mathcal{K}_q)$ are defined to be the homology of a complex

$$\mathrm{H}^p(X, \mathcal{K}_q) = \mathrm{H}\Big(\coprod_{x\in X^{(p-1)}} \mathrm{K}_{q-p+1}(k(x)) \longrightarrow \coprod_{x\in X^{(p)}} \mathrm{K}_{q-p}(k(x)) \longrightarrow \coprod_{x\in X^{(p+1)}} \mathrm{K}_{q-p-1}(k(x))\Big).$$

In particular, the groups $\mathrm{H}^p(X, \mathcal{K}_q) = 0$ whenever $p > q$ or $p > \dim(X)$.

The coniveau filtration is the filtration appearing in the abutment of the K-theory coniveau spectral sequence. If X is a regular variety (which is all that is worked with in this note), then there are natural isomorphisms $\mathrm{K}_i(X) \cong \mathrm{G}_i(X)$, and by transporting the filtration on G-theory to K-theory, we get a coniveau filtration on the groups $\mathrm{K}_i(X)$. The j-th term of this filtration on $\mathrm{K}_i(X)$ is denoted $\mathrm{K}_i(X)^j$ below. We write $\mathrm{K}_i(X)^{j/j+1}$ for the quotient $\mathrm{K}_i(X)^j/\mathrm{K}_i(X)^{j+1}$.

The K-theory of a Severi–Brauer variety X associated with a central simple algbera A was computed by Quillen in terms of the tautological bundle ζ_X on X.

Theorem 2.1 ([Qui73, §8, Theorem 4.1]) *Let X be the Severi–Brauer variety of a central simple algebra A. Then for every $i \geq 0$ the group homomorphism*

$$\bigoplus_{j=0}^{\deg(A)-1} \mathrm{K}_i(A^{\otimes j}) \longrightarrow \mathrm{K}_i(X)$$

induced by the exact functor that takes a left $A^{\otimes i}$-module M to $\zeta_X^{\otimes i} \otimes_{A^{\otimes i}} M$ is an isomorphism.

Crucial in our computation will be the reduced norm subgroups of a central simple k-algebra. For this, let L be a Galois splitting field for A. The *first reduced norm* of A is defined to be the unique map making the following diagram commutative:

$$\begin{array}{ccc} \mathrm{K}_1(A_L) & \xrightarrow{\det} & \mathrm{K}_1(L) \\ \uparrow & & \uparrow \\ \mathrm{K}_1(A) & \xrightarrow{\mathrm{Nrd}_1} & \mathrm{K}_1(k). \end{array}$$

The vertical arrows in this diagram are induced by extension of scalars. Similarly we define the *zeroth reduced norm* of A to be the map $\mathrm{Nrd}_0 : \mathrm{K}_0(A) \to \mathrm{K}_0(k)$ taking the class of an A-module M to the k-vector space of dimension $\mathrm{rdim}_A(M)$, the reduced dimension of M. For $i = 0, 1$ we will often use the abbreviation $\mathrm{Nrd}_i(\mathrm{K}_i(A)) := \mathrm{Nrd}_i(A)$.

The kernel of the map Nrd_i is called the *i-th reduced Whitehead group* and is denoted $\mathrm{SK}_i(-)$. Note that the group $\mathrm{SK}_0(A)$ necessarily vanishes, since Nrd_0 is injective with image the subgroup generated by the index of A, $\mathrm{ind}(A)\mathbb{Z} \subset \mathrm{K}_0(k) = \mathbb{Z}$. The group $\mathrm{SK}_1(A)$ does not vanish in general.

For any finite field extension E of k, the extension of scalars map

$$\rho^*_{E/k} : \mathrm{K}_i(X) \longrightarrow \mathrm{K}_i(X_E)$$

is the sum of the maps $\mathrm{K}_i(A^{\otimes j}) \to \mathrm{K}_i(A_E^{\otimes j})$ in the decomposition of Theorem 2.1. In the other direction, the pushforward $\rho_{E/k*} : \mathrm{K}_i(X_E) \to \mathrm{K}_i(X)$ is given by the sum of the norm maps $\mathrm{K}_i(A_E^{\otimes j}) \to \mathrm{K}_i(A^{\otimes j})$ in the same decomposition. If $i = 0$, then the norm map is characterized componentwise by having image the number

$$\rho_{E/k*}(\mathrm{K}_0(A_E)) = [E:k]\frac{\mathrm{rdim}_{A_E}(M)}{\mathrm{rdim}_A(N)} \subset \mathrm{K}_0(A) = \mathbb{Z},$$

where M, N are simple modules under A_E, A, respectively. The image of the norm maps when $i = 1$ are more complicated to describe. In the simple situation we work in, these images can be described fairly explicitly. We do this in detail in the next section.

https://doi.org/10.4153/S0008439518000073 Published online by Cambridge University Press

3 Relations Between Reduced Norms

In this section we fix a central simple algebra A over k, and we set X to be the Severi–Brauer variety associated with A.

Our first objective is to describe the image of the reduced norm using splitting fields of A.

Lemma 3.1 *Let A be a central simple algebra. Then for every finite field extension L of k and for $i = 0, 1$, the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{K}_i(A_L) & \xrightarrow{\mathrm{Nrd}_i} & \mathrm{K}_i(L) \\ \downarrow{\scriptstyle N_{A_L/A}} & & \downarrow{\scriptstyle N_{L/k}} \\ \mathrm{K}_i(A) & \xrightarrow{\mathrm{Nrd}_i} & \mathrm{K}_i(k), \end{array}$$

where both $N_{A_L/A}$ and $N_{L/k}$ are the norm maps induced by restriction of scalars.

Moreover, the subgroup $\mathrm{Nrd}_i(A)$ is generated by the images $N_{L/k}(\mathrm{K}_i(L))$, as L varies over all finite extensions of k that split A. This can be reduced further. The subgroup $\mathrm{Nrd}_i(A)$ is generated by the images $N_{L/k}(\mathrm{K}_i(L))$, as L varies over all finite extensions of k that are maximal subfields of the underlying division algebra of A.

Proof The commutativity of the diagram is clear when $i = 0$, and is well known (see [GS06, Proposition 2.8.11]) when $i = 1$.

The only claim that needs to be proved is the last one: the subgroup $\mathrm{Nrd}_i(A)$ is generated by norms of maximal subfields of the underlying division algebra of A. In the case $i = 0$, the claim follows from the fact that such a field has degree $\mathrm{ind}(A)$ over k, so we are left proving the case $i = 1$.

For the proof when $i = 1$, we'll use Morita invariance to reduce to the case A is a division algebra, and we'll use [GS06, Proposition 2.6.3], which says $\mathrm{Nrd}_1(x) = N_{K/k}(x)$ for any element x of a maximal subfield K contained in A. Any element x of A is contained in some maximal subfield (indeed, if F is a maximal element in the collection of subfields of A containing $k(x)$, then the centralizer of F in A is F itself; this is known to be equivalent to being a maximal subfield), so taking the composition

$$A^\times \twoheadrightarrow \mathrm{K}_1(A) \xrightarrow{\mathrm{Nrd}_1} \mathrm{K}_1(k)$$

of the natural surjection and the reduced norm gives the result by the commutativity of the given diagram. ■

The K-theory of the Severi–Brauer variety X relies heavily on the tensor powers of the algebra A due to the decomposition of Theorem 2.1. Because of this, we'll need to investigate certain relations between the reduced norms $\mathrm{Nrd}_i(A)$ and $\mathrm{Nrd}_i(A^{\otimes r})$ for varying $r \geq 0$. It will be necessary in our formulation of these relations to introduce some condition on the index of A over finite extensions. From now on we'll say that an algebra A satisfies condition (C) if

(C) $\qquad \mathrm{ind}(A_E) = \exp(A_E)$ for any finite extension E/k.

Example 3.2 Any central simple algebra of square-free index satisfies condition (C) trivially. Any central simple algebra over a finite extension of $\mathbb{Q}_p$ satisfies condition (C). Central simple algebras over function fields of surfaces, with base a separably closed field, having index coprime to the characteristic of the base also satisfy condition (C); see [dJ04].

Moreover, if a central simple algebra A satisfies condition (C), then so do the tensor powers of A. This is because, given a central simple algebra A with equal index and exponent, the indices of all tensor powers of A can be explicitly determined. If the index of A was a power of a prime p, say p^n, then $A^{\otimes p}$ has index p^{n-1}; *cf.* [Kar98, Example 3.9]. The general case follows easily from this one.

Remark 3.3 There exists a cyclic algebra A of index and exponent 4, over a field F of characteristic 2, along with a finite purely inseparable field extension E/F with $[E{:}F] = 2$ and such that $\operatorname{ind}(A_E) = 4$ and $\exp(A_E) = 2$ (*cf.* [Per41, Theorem 4]).

Lemma 3.4 *Let A be a central simple k-algebra with* $\operatorname{ind}(A) = p^n$ *for some* $n \geq 0$ *and let* $i = 0$ *or* $i = 1$. *Then*

$$\mathrm{Nrd}_i(A^{\otimes j}) = \mathrm{Nrd}_i(A^{\otimes p^{v_p(j)}})$$

for any $j > 0$.

Proof By Lemma 3.1 the subgroup $\mathrm{Nrd}_i(A^{\otimes j}) \subset \mathrm{K}_i(k)$ is generated by the norm subgroups $N_{L/k}(\mathrm{K}_i(L))$, as L varies over all finite extension of k splitting $A^{\otimes j}$. The set of such fields is the same for $A^{\otimes j}$ and $A^{\otimes p^{v_p(j)}}$, which proves the claim. ∎

Lemma 3.5 *Let A be a central simple k-algebra with* $\operatorname{ind}(A) = p^n = \exp(A)$ *for some prime p and some* $n \geq 0$. *Assume that A satisfies condition* (C). *Then for* $i = 0, 1$ *the containments*

$$\mathrm{Nrd}_i(A^{\otimes p^a}) \supset \mathrm{Nrd}_i(A^{\otimes p^b}) \supset \mathrm{Nrd}_i(A^{\otimes p^a})^{p^{a-b}}$$

hold for all $a \geq b \geq 0$.

Proof By Lemma 3.1, the subgroup $\mathrm{Nrd}_i(A^{\otimes j}) \subset \mathrm{K}_i(k)$ is generated by the norm subgroups $N_{L/k}(\mathrm{K}_i(L))$, as L varies over all finite extension of k splitting $A^{\otimes j}$. If such an L would split $A^{\otimes p^b}$, then L would also split $A^{\otimes p^a}$. Hence we have the inclusion $\mathrm{Nrd}_i(A^{\otimes p^b}) \subset \mathrm{Nrd}_i(A^{\otimes p^a})$.

To show the inclusion $\mathrm{Nrd}_i(A^{\otimes p^a})^{p^{a-b}} \subset \mathrm{Nrd}_i(A^{\otimes p^b})$, we work in two cases. If $a \geq n$, then $A^{\otimes p^a}$ is split; if L is a maximal subfield of the underlying division algebra of $A^{\otimes p^b}$, then $[L{:}k] = p^{n-b}$ (see Example 3.2) and

$$\mathrm{Nrd}_i(A^{\otimes p^a})^{p^{a-b}} \subset p^{n-b}\mathrm{K}_i(k) = N_{L/k}(\mathrm{K}_i(k)) \subset \mathrm{Nrd}_i(A^{\otimes p^b}).$$

Otherwise, when $a < n$, let L be a maximal subfield of the underlying division algebra of $A^{\otimes p^a}$. Then L has degree $[L{:}k] = p^{n-a}$, the algebra A_L has exponent dividing p^a and, since we are assuming condition (C), index dividing p^a. If E is a maximal subfield of the underlying division algebra of $A_L^{\otimes p^b}$, then $[E{:}L]$ divides p^{a-b}.

Again by Lemma 3.1, we have the inclusion

$$N_{E/k}(\mathrm{K}_i(E)) \subset \mathrm{Nrd}_i(A^{\otimes p^b}),$$

since E splits $A^{\otimes p^b}$. It follows that for any element x of $\mathrm{K}_i(L) \subset \mathrm{K}_i(E)$, we have that

$$N_{E/k}(x) = N_{L/k}(N_{E/L}(x)) = N_{L/k}(x^{[E:L]}) = N_{L/k}(x)^{[E:L]}$$

is contained in $\mathrm{Nrd}_i(A^{\otimes p^b})$. The proof is then complete, since we've shown the collection of elements $N_{L/k}(x)^{p^{a-b}}$, as L varies over all maximal subfields of the underlying division algebra of $A^{\otimes p^a}$ and x varies over $\mathrm{K}_i(L)$, are contained in $\mathrm{Nrd}_i(A^{\otimes p^b})$, and these form a generating set by Lemma 3.1. ∎

Lemma 3.6 *Let A be a central simple k-algebra with* $\mathrm{ind}(A) = p^n = \exp(A)$ *for some prime p and some $n \geq 0$. Assume that A satisfies condition* (C). *Then for $i = 0, 1$, there is containment*

$$\mathrm{Nrd}_i(A^{\otimes a})^{\binom{a}{b}} \subset \mathrm{Nrd}_i(A^{\otimes b})$$

for all $a \geq b > 0$.

Proof The proof continues by working in cases: assuming either $v_p(a) \leq v_p(b)$ or $v_p(a) > v_p(b)$. In the first case, $v_p(a) \leq v_p(b)$, we appeal to Lemmas 3.4 and 3.5 to find

$$\mathrm{Nrd}_i(A^{\otimes a}) = \mathrm{Nrd}_i(A^{\otimes p^{v_p(a)}}) \subset \mathrm{Nrd}_i(A^{\otimes p^{v_p(b)}}) = \mathrm{Nrd}_i(A^{\otimes b}).$$

In the second case, $v_p(a) > v_p(b)$, we appeal to the second containment of Lemma 3.5. That is to say, by Lemma 3.7 below, we find that $v_p(\binom{a}{b}) \geq v_p(a) - v_p(b)$ so that

$$\mathrm{Nrd}_i(A^{\otimes a})^{\binom{a}{b}} \subset \mathrm{Nrd}_i(A^{\otimes p^{v_p(a)}})^{p^{v_p(a)-v_p(b)}} \subset \mathrm{Nrd}_i(A^{\otimes p^{v_p(b)}}) = \mathrm{Nrd}_i(A^{\otimes b})$$

by applying Lemma 3.4 for the first inclusion, Lemma 3.5 for the second inclusion, and Lemma 3.4 for the last equality. ∎

The lemma needed for the above is the following.

Lemma 3.7 *If $a > b$ and $v_p(a) > v_p(b)$, then $v_p(\binom{a}{b}) \geq v_p(a) - v_p(b)$.*

Proof More generally, for any pair of integers $a > b$, one can show that $\frac{a}{(a,b)}$ divides the binomial coefficient $\binom{a}{b}$. The claim follows from noting that

$$v_p\left(\frac{a}{(a,b)}\right) = v_p(a) - v_p((a,b)) = v_p(a) - v_p(b).$$

First, write $(a, b) = na + mb$ with n, m both integers. Then

$$\frac{(a,b)}{a}\binom{a}{b} = \frac{(na+mb)}{a}\binom{a}{b} = n\binom{a}{b} + \frac{mb}{a}\binom{a}{b} = n\binom{a}{b} + m\binom{a-1}{b-1}$$

with the latter sum an integer. ∎

To go from an algebra of p-primary index to an arbitrary central simple algebra A, see Proposition 5.1, we will need a characterization of $\mathrm{Nrd}_i(A)$ in terms of the primary components of A when A is division. For this, we fix a primary decomposition

$$A \cong A_{p_1} \otimes \cdots \otimes A_{p_s}$$

https://doi.org/10.4153/S0008439518000073 Published online by Cambridge University Press

with $p_1, \dots, p_s$ the primes dividing $\mathrm{ind}(A)$ (such decompositions exist with the factors unique up to isomorphism, see [GS06, Proposition 4.5.16]). For each algebra A_{p_j} we fix a maximal subfield F_{p_j}, necessarily of degree a power of p_j over k. We set F^{p_j} to be a composite of the fields $F_{p_1}, \dots, F_{p_{j-1}}, F_{p_{j+1}}, \dots, F_{p_s}$ with the j-th field being omitted, contained in some fixed algebraic closure L.

Lemma 3.8 *In the notation above, and for* $i = 0, 1$,

$$\mathrm{Nrd}_i(A) = \bigcap_{j=1}^{s} \mathrm{Nrd}_i(A_{F^{p_j}})$$

inside of $\mathrm{K}_i(L)$.

Proof If $s = 1$, the lemma is trivial, so we can assume that $s > 1$.

The inclusion $\subset$ is immediate from Lemma 3.1, since a field E splitting A also necessarily splits each of the $A_{F^{p_j}}$.

For the other inclusion, $\supset$, we let x be an element of the intersection. By Lemma 3.1 this means we have equalities

$$\begin{gathered} x = N_{E_{1,1}/F^{p_1}}(y_{1,1}) \cdots N_{E_{1,r_1}/F^{p_1}}(y_{1,r_1}) \\ \vdots \\ x = N_{E_{s,1}/F^{p_s}}(y_{s,1}) \cdots N_{E_{s,r_s}/F^{p_s}}(y_{s,r_s}) \end{gathered}$$

for some elements $y_{j,k}$ of fields $E_{j,k}$ splitting $A_{F^{p_j}}$, respectively. It follows from these equalities that x is an element of $B = \mathrm{K}_i(F^{p_1}) \cap \cdots \cap \mathrm{K}_i(F^{p_s})$. If $i = 0$, then B is just $\mathrm{ind}(A)\mathbb{Z}$. If $i = 1$, then, since by construction the degrees $[F^{p_j}{:}k]$ are divisible by all primes dividing $\mathrm{ind}(A)$ except for p_j, we have $\gcd([F^{p_1}{:}k], \dots, [F^{p_s}{:}k]) = 1$ and $B = k^{\times}$.

Applying the norm, from F^{p_j} to k, to the corresponding expression above for x, we find the elements

$$N_{F^{p_j}/k}(x) = N_{E_{j,1}/k}(y_{j,1}) \cdots N_{E_{j,r_j}/k}(y_{j,r_j})$$

are contained in $\mathrm{Nrd}_i(A)$, for every $1 \le j \le s$, since each $E_{j,k}$ splits $A_{F^{p_j}}$ and so necessarily also splits A. Since x is already contained in $\mathrm{K}_i(k)$, taking the norm also yields equalities

$$N_{F^{p_j}/k}(x) = x^{[F^{p_j}:k]}.$$

Finally, as x is in the subgroup spanned by these powers, x is contained in $\mathrm{Nrd}_i(A)$, completing the proof. ∎

4 The Coniveau Filtration on K_i for a p-primary Algebra

We fix a prime p throughout. We fix a central simple algebra A with index $\mathrm{ind}(A) = p^n$ and exponent $\exp(A) = p^n$ for some $n > 0$. We write X for the Severi–Brauer variety of A.

This section describes the groups $\mathrm{K}_i(X)^j$ and $\mathrm{K}_i(X)^{j/j+1}$ for $j \ge 0$ assuming A satisfies condition (C) and either $i = 0$ or, $i = 1$ and $\mathrm{SK}_1(A^{\otimes r}) = 1$ for all $r \ge 1$. In the case $i = 0$, this result was shown in [Kar98, Proposition 3.3] (condition (C) is not

https://doi.org/10.4153/S0008439518000073 Published online by Cambridge University Press

needed in this result). Although the only new result is when $i = 1$, the proof does not depend on this assumption.

We note that the assumption $\mathrm{SK}_1(A^{\otimes r})$ is trivial for all powers r is another way of stating that $\mathrm{K}_1(X) \to \mathrm{K}_1(X_L)$ is injective for a splitting field L of A. The reason the latter, more natural, assumption is not given is because it is often easier to check that the groups $\mathrm{SK}_1(A^{\otimes r})$ are trivial. Note that the analogous statement is also true replacing $i = 1$ with $i = 0$ in the above, so that the map $\mathrm{K}_0(X) \to \mathrm{K}_0(X_L)$ is always injective.

Lemma 4.1 *Suppose B is an arbitary central simple algebra and let Y be the Severi–Brauer variety of B. Let L be a splitting field for B. Then, for $i = 0, 1$, the pullback $\mathrm{K}_i(Y) \to \mathrm{K}_i(Y_L)$ is injective if and only if the groups $\mathrm{SK}_i(B^{\otimes j})$ are trivial for all $j \geq 0$.*

Proof The diagram

$$\begin{tikzcd} \mathrm{K}_i(B_L^{\otimes r}) \arrow[r, equal, "\mathrm{Nrd}_i"] & \mathrm{K}_i(L) \\ \mathrm{K}_i(B^{\otimes r}) \arrow[u, "\pi_r^*"] \arrow[r, "\mathrm{Nrd}_i"] & \mathrm{K}_i(k) \arrow[u, hook] \end{tikzcd}$$

commutes, where the vertical arrows are the extension of scalars maps. Since the right-vertical arrow is always an injection, we find $\mathrm{SK}_i(B^{\otimes r}) = \ker(\pi_r^*)$. The claim then follows from Theorem 2.1 by summing over all $r \geq 0$. ∎

As in the above lemma, let B be an arbitrary central simple algebra and let Y be the associated Severi–Brauer variety. If L is a splitting field for B, then $\mathrm{K}_0(Y_L)$ is generated as a group by the powers γ^i, from $i = 0$ to $\deg(B) - 1$, of the element γ representing the class of $\mathcal{O}_{Y_L}(-1)$. By Lemma 4.1, the pullback $\mathrm{K}_0(Y) \to \mathrm{K}_0(Y_L)$ is injective, and we identify $\mathrm{K}_0(Y)$ with its image in $\mathrm{K}_0(Y_L)$. Similarly, the group $\mathrm{K}_1(Y_L)$ is a sum of groups $L^\times \gamma^i$ as i ranges from $i = 0$ to $i = \deg(B) - 1$. If $\mathrm{SK}_1(B^{\otimes r}) = 1$ for all $r \geq 1$, then the pullback $\mathrm{K}_1(Y) \to \mathrm{K}_1(Y_L)$ is injective and we identify $\mathrm{K}_1(Y)$ with its image in $\mathrm{K}_1(Y_L)$.

Theorem 4.2 *Assume A satisfies condition* (C). *Let L be a splitting field for A. If $i = 0$, or if $i = 1$ and $\mathrm{SK}_1(A^{\otimes r}) = 1$ for all $r \geq 1$, then there is an equality (with notation as above)*

$$\mathrm{K}_i(X) \cap \mathrm{K}_i(X_L)^j = \mathrm{Nrd}_i(A^{\otimes j})(\gamma - 1)^j + \cdots + \mathrm{Nrd}_i(A^{\otimes \deg(A)-1})(\gamma - 1)^{\deg(A)-1}$$

for all $0 \leq j \leq \deg(A) - 1$. For $j < 0$, or for $j > \deg(A) - 1$, the groups $\mathrm{K}_i(X)^j = 0$ vanish.

Proof The claim when $j < 0$ or $j > \deg(A) - 1$ is immediate: the first of these is by definition; the second follows from the fact $(\gamma - 1)^{\deg(A)} = 0$ in $\mathrm{K}_0(X)$. Recall (*cf.* [Pey95, Proposition 3.6]), the coniveau filtration on $\mathrm{K}_i(X_L)$ is given by

$$\mathrm{K}_i(X_L)^j = \mathrm{K}_i(A_L^{\otimes j})(\gamma - 1)^j + \cdots + \mathrm{K}_i(A_L^{\otimes \deg(A)-1})(\gamma - 1)^{\deg(A)-1},$$

where $\gamma = [\mathcal{O}(-1)]$ is the class of the tautological line bundle in $\mathrm{K}_0(X_L)$. Under the pullback $\mathrm{K}_i(X) \to \mathrm{K}_i(X_L)$, the groups $\mathrm{K}_i(A^{\otimes j})$ are identified with the subgroups

https://doi.org/10.4153/S0008439518000073 Published online by Cambridge University Press

$\mathrm{Nrd}_i(A^{\otimes j}) \subset \mathrm{K}_i(L)$. Hence, we identify

$$\mathrm{K}_i(X) = \mathrm{Nrd}_i(k)\cdot 1 + \mathrm{Nrd}_i(A)\gamma + \cdots + \mathrm{Nrd}_i(A^{\otimes \deg(A)-1})\gamma^{\deg(A)-1}.$$

We claim

$$(\ast)\quad \mathrm{K}_i(X) \cap \mathrm{K}_i(X_L)^j = \mathrm{Nrd}_i(A^{\otimes j})(\gamma-1)^j + \cdots + \mathrm{Nrd}_i(A^{\otimes \deg(A)-1})(\gamma-1)^{\deg(A)-1}.$$

The proof utilizes the following lemmas.

Lemma 4.3 *Let A and L be as in Theorem 4.2. Fix an element b in $\mathrm{Nrd}_i(A^{\otimes k})$ with $k \geq 0$ and $i = 0$ or $i = 1$. Then for any sequence of integers $(n_j)_{j\geq 0}$ an equality*

$$bx^k = \sum_{j\geq 0} a_j(x+n_j)^j$$

inside of the free $\mathrm{K}_i(L)$-module $\mathrm{K}_i(L)[x]$ implies a_j is contained in $\mathrm{Nrd}_i(A^{\otimes j})$ for all $j \geq 0$.

Proof By assumption $a_k = b$ is contained in $\mathrm{Nrd}_i(A^{\otimes k})$. By descending induction on j, we assume that each a_j is contained in $\mathrm{Nrd}_i(A^{\otimes j})$ for all j larger than some fixed $l \geq 0$. Then by expanding the right side of the given equality and comparing coefficients yields

$$a_l = -\sum_{j=l+1}^{k} n_j^{j-l}\binom{j}{l}a_j,$$

which is contained in $\mathrm{Nrd}_i(A^{\otimes l})$ due to Lemma 3.6 applied to each $\binom{j}{l}a_j$. ∎

Lemma 4.4 *Keeping notation as above, we have*

$$\sum_{j\geq 0}\mathrm{Nrd}_i(A^{\otimes j})\gamma^j = \sum_{j\geq 0}\mathrm{Nrd}_i(A^{\otimes j})(\gamma-1)^j$$

inside of $\mathrm{K}_i(X_L)$.

Proof Setting $n_j = -1$ for all $j \geq 0$ in Lemma 4.3, and setting $x = \gamma$, shows the forward containment. Setting $n_j = 1$ for all $j \geq 0$, and setting $x = \gamma - 1$, shows the reverse containment. ∎

Continuing with the proof of Theorem 4.2, we have

$$\begin{aligned}\mathrm{K}_i(X) \cap \mathrm{K}_i(X_L)^j &= \sum_{n\geq 0}\mathrm{Nrd}_i(A^{\otimes n})\gamma^n \cap \sum_{n\geq j}\mathrm{K}_i(L)(\gamma-1)^n \\ &= \sum_{n\geq 0}\mathrm{Nrd}_i(A^{\otimes n})(\gamma-1)^n \cap \sum_{n\geq j}\mathrm{K}_i(L)(\gamma-1)^n \\ &= \sum_{n\geq j}\mathrm{Nrd}_i(A^{\otimes n})(\gamma-1)^n,\end{aligned}$$

as claimed. Here we used Lemma 4.4 to go from the first line to the second. ∎

https://doi.org/10.4153/S0008439518000073 Published online by Cambridge University Press

Corollary 4.5 *Let L be an algebraic closure of k. Assume that A satisfies condition* (C). *Let $i = 0$ or $i = 1$ and assume $\mathrm{SK}_i(A^{\otimes r}) = 1$ for all $r \geq 1$. Then we have an equality*

$$\mathrm{K}_i(X)^j = \mathrm{K}_i(X) \cap \mathrm{K}_i(X_L)^j$$

for all $j \geq 0$.

Proof It's clear we have the inclusion $\mathrm{K}_i(X)^j \subset \mathrm{K}_i(X) \cap \mathrm{K}_i(X_L)^j$. By Theorem 4.2, there is an equality

$$\mathrm{K}_i(X) \cap \mathrm{K}_i(X_L)^j = \mathrm{Nrd}_i(A^{\otimes j})(\gamma-1)^j + \cdots + \mathrm{Nrd}_i(A^{\otimes \deg(A)-1})(\gamma-1)^{\deg(A)-1}.$$

To show the reverse containment $\mathrm{K}_i(X) \cap \mathrm{K}_i(X_L)^j \subset \mathrm{K}_i(X)^j$, we go by induction on the index. That is to say: if E is a finite extension of k splitting A, then we have containment $\mathrm{K}_i(X_E) \cap \mathrm{K}_i(X_L)^j \subset \mathrm{K}_i(X_E)^j$ and for our induction hypothesis, we assume this containment holds for all fields E with $\mathrm{ind}(A_E) < \mathrm{ind}(A)$.

If E is a finite extension of k with $\mathrm{ind}(A_E) < \mathrm{ind}(A)$, then, using our induction hypothesis and the assumption A satisfies condition (C), we have

$$\begin{aligned}
\mathrm{K}_i(X)^j &= \rho^*_{L/k}(\mathrm{K}_i(X)^j) \\
&\supset \rho^*_{L/k}(\rho_{E/k*}(\mathrm{K}_i(X_E)^j)) \\
&= \rho_{E/k_*}\big(\mathrm{Nrd}_i(A_E^{\otimes j})(\gamma-1)^j + \cdots + \mathrm{Nrd}_i(A_E^{\otimes \deg(A)-1})(\gamma-1)^{\deg(A)-1}\big).
\end{aligned}$$

Expanding a product $(\gamma-1)^r$ and taking $\rho_{E/k*}$ shows that

$$\rho_{E/k*}(a(\gamma-1)^r) = N_{E/k}(a)(\gamma-1)^r.$$

Since all elements of $\mathrm{Nrd}_i(A^{\otimes r})$ are norms from finite extensions E of k splitting $A^{\otimes r}$ by Lemma 3.1, it follows that $\mathrm{K}_i(X) \cap \mathrm{K}_i(X_L)^j$ is generated by the groups on the right of the containment above. ∎

Corollary 4.6 *Let $i = 0$, or $i = 1$ and $\mathrm{SK}_i(A^{\otimes r}) = 1$ for all $r \geq 0$. Assume that A satisfies condition* (C). *Then there is an isomorphism*

$$\mathrm{K}_i(X)^{j/j+1} \cong \mathrm{Nrd}_i(A^{\otimes j})$$

for all $0 \leq j \leq \deg(A) - 1$. For other j these groups vanish.

Proof This follows immediately from Theorem 4.2 and Corollary 4.5. ∎

5 The Coniveau Filtration on K_i for a Central Simple Algebra

In this section we assume B is a central simple algebra with $\mathrm{ind}(B_E) = \exp(B_E)$ for all finite field extensions E/k. We let Y be the Severi–Brauer variety of B.

Proposition 5.1 *If $i = 0$, or if $i = 1$ and $\mathrm{SK}_1(B^{\otimes r}) = 1$ for all $r \geq 0$, then there is an isomorphism*

$$\mathrm{K}_i(Y)^{j/j+1} \cong \mathrm{Nrd}_i(B^{\otimes j})$$

for all $0 \leq j \leq \deg(B) - 1$. For other j these groups vanish.

https://doi.org/10.4153/S0008439518000073 Published online by Cambridge University Press

Proof Using a result of Karpenko ([Kar00, Example 10.20]), we can assume that B is a division algebra throughout the proof.

Fix a primary decomposition

$$B \cong B_{p_1} \otimes \cdots \otimes B_{p_s}$$

with $p_1, \dots, p_s$ the primes dividing $\mathrm{ind}(B)$. We can assume that $s > 1$, as the result has been proved above otherwise. For each algebra B_{p_j}, we fix a maximal subfield F_{p_j}, necessarily of degree a power of p_j over k. We set F^{p_j} to be a composite of the fields $F_{p_1}, \dots, F_{p_{j-1}}, F_{p_{j+1}}, \dots, F_{p_s}$, the j-th field being omitted, contained in some fixed algebraic closure L of k

We first observe an equality

$$\mathrm{K}_i(Y) \cap \mathrm{K}_i(Y_L)^j = \mathrm{Nrd}_i(B^{\otimes j})(\gamma-1)^j + \cdots + \mathrm{Nrd}_i(B^{\otimes \deg(B)-1})(\gamma-1)^{\deg(B)-1}.$$

Indeed, by Lemma 3.8 and the explicit description of $\mathrm{K}_i(Y)$ given by Lemma 4.1, we have

$$\mathrm{K}_i(Y) = \mathrm{K}_i(Y_{F^{p_1}}) \cap \cdots \cap \mathrm{K}_i(Y_{F^{p_s}})$$

inside of $\mathrm{K}_i(Y_L)$. Hence, we get equalities

$$\begin{aligned}
&\mathrm{K}_i(Y) \cap \mathrm{K}_i(Y_L)^j \\
&\quad = \mathrm{K}_i(Y_{F^{p_1}}) \cap \cdots \cap \mathrm{K}_i(Y_{F^{p_s}}) \cap \mathrm{K}_i(Y_L)^j \\
&\quad = \bigcap_{r=1}^{s} \big(\mathrm{K}_i(Y_{F^{p_r}}) \cap \mathrm{K}_i(Y_L)^j \big) \\
&\quad = \bigcap_{r=1}^{s} \big(\mathrm{Nrd}_i(B_{F^{p_r}})(\gamma-1)^j + \cdots + \mathrm{Nrd}_i(B_{F^{p_r}}^{\otimes \deg(B)-1})(\gamma-1)^{\deg(B)-1} \big) \\
&\quad = \mathrm{Nrd}_i(B^{\otimes j})(\gamma-1)^j + \cdots + \mathrm{Nrd}_i(B^{\otimes \deg(B)-1})(\gamma-1)^{\deg(B)-1}.
\end{aligned}$$

A careful reading of the proof of Corollary 4.5 shows that the assumption A has p-primary index was unnecessary. Hence, the corollary can be applied to B as well to show $\mathrm{K}_i(Y) = \mathrm{K}_i(Y) \cap \mathrm{K}_i(Y_L)^j$ and the result follows. ∎

Acknowledgment I'd like to thank an anonymous referee for helpful advice and simplifications to the proof of Theorem 4.2.

References

[Art82] M. Artin, *Brauer-Severi varieties, Brauer groups in ring theory and algebraic geometry*. (Wilrijk, 1981), Lecture Notes in Mathematics, 917, Springer, Berlin–New York, 1982, pp. 194–210.

[CM06] V. Chernousov and A. Merkurjev, *Connectedness of classes of fields and zero-cycles on projective homogeneous varieties*. Compos. Math. 142(2006), no. 6, 1522–1548. https://doi.org/10.1112/S0010437X06002363.

[dJ04] A. J. de Jong, *The period-index problem for the Brauer group of an algebraic surface*. Duke Math. J. 123(2004), no. 1, 71–94. https://doi.org/10.1215/S0012-7094-04-12313-9.

[GS06] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*. Cambridge Studies in Advanced Mathematics, 101, Cambridge University Press, Cambridge, 2006. https://doi.org/10.1017/CBO9780511607219.

[Kar98] N. A. Karpenko, *Codimension 2 cycles on Severi–Brauer varieties*. K-Theory 13(1998), no. 4, 305–330. https://doi.org/10.1023/A:1007705720373.

[Kar00] N. A. Karpenko, *Cohomology of relative cellular spaces and of isotropic flag varieties*. Algebra i Analiz 12(2000), no. 1, 3–69.

[Kar17] N. A. Karpenko, *Chow groups of some generically twisted flag varieties*. Ann. K-Theory 2(2017), no. 2, 341–356. https://doi.org/10.2140/akt.2017.2.341.

[Mer95] A. S. Merkurjev, *Certain K-cohomology groups of Severi–Brauer varieties*. In: *K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992)*, Proc. Sympos. Pure Math., 58, American Mathematical Society, Providence, RI, 1995, pp. 319–331.

[MS82] A. S. Merkurjev and A. A. Suslin, *K-cohomology of Severi–Brauer varieties and the norm residue homomorphism*. Izv. Akad. Nauk SSSR Ser. Mat. 46(1982), no. 5, 1011–1046, 1135–1136.

[Per41] S. Perlis, *Scalar extensions of algebras with exponent equal to index*. Bull. Amer. Math. Soc. 47(1941), 670–676. https://doi.org/10.1090/S0002-9904-1941-07536-1.

[Pey95] E. Peyre, *Products of Severi–Brauer varieties and Galois cohomology*. In: *K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992)*, Proc. Sympos. Pure Math., 58, American Mathematical Society, Providence, RI, 1995, pp. 369–401.

[Qui73] D. Quillen, *Higher algebraic K-theory. I*. In: *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, Lecture Notes in Mathematics, 341, Springer, Berlin, 1973, pp. 85–147.

Mathematical & Statistical Sciences, University of Alberta, Edmonton, AB
e-mail: mackall@ualberta.ca