IV

Local solvability of vector fields

In this chapter we study in detail an important class of locally integrable vector fields: those which are locally solvable. The most basic question one can ask concerning the solvability of a vector field L is whether, given a smooth right-hand side f, there exists a solution, at least locally and not subjected to any additional condition, of the equation $Lu = f$. For real vector fields very satisfactory theorems stating local existence of solutions under very mild hypotheses of regularity have been known since long ago, and it came as a surprise when Hans Lewy published in 1956 his now famous example of a nonlocally solvable vector field. Indeed, if $f \in C^\infty(\mathbb{R}^3)$ is conveniently chosen, the equation

$$(\partial_x + i\partial_y - (x+iy)\partial_z)u = f, \qquad (x, y, z) \in \mathbb{R}^3$$

does not have distribution solutions in any open subset of $\mathbb{R}^3$ (**[L2]**). In the first part of this chapter we focus on vector fields in two variables; in this case, a priori estimates are known to hold under weaker assumptions on the regularity of the coefficients than in the general case. In Section IV.1 we motivate condition $(\mathcal{P})$ with simple examples and prove a priori estimates in L^p and in a mixed norm that involves the Hardy space $h^1(\mathbb{R})$. While the first kind of estimate gives, by duality, local solvability in L^p, $1 < p < \infty$, the latter kind gives local solvability in $L^\infty[\mathbb{R}; \mathrm{bmo}(\mathbb{R})]$ which serves as a substitute for local solvability in L^∞, a property that is not implied by $(\mathcal{P})$, as is shown by the example described at the end of Section IV.1.1. On the other hand, in some applications—this is indeed the case for the similarity principle described in the Epilogue—solvability in the larger space of mixed norm $L^\infty[\mathbb{R}; \mathrm{bmo}(\mathbb{R})]$ suffices. Some technical properties of the space $h^1(\mathbb{R})$ that are useful for the proof of a priori estimates will only be presented later in Appendix A. In Section IV.2 we still consider vector fields in two variables and study the existence of smooth solutions when the right-hand side is smooth. The sufficiency of condition $(\mathcal{P})$ for local solvability in any number of variables is discussed in Section IV.3, while Section IV.4 is devoted to its necessity.

IV.1 Planar vector fields

We shall consider vector fields defined in an open subset $\Omega \subset \mathbb{R}^2$

$$Lu = A(x,t)\frac{\partial u}{\partial t} + B(x,t)\frac{\partial u}{\partial x} \tag{IV.1}$$

with complex coefficients $A, B \in C^\infty(\Omega)$ such that

$$|A(x,t)| + |B(x,t)| > 0, \quad (x,t) \in \Omega. \tag{IV.2}$$

Since our point of view is local, most of the time the behavior of L outside a neighborhood of the point under study is irrelevant. This means that we can modify the coefficients of L off that neighborhood in order to assume that they are defined throughout $\mathbb{R}^2$ and we shall often do so. The sort of properties of L we shall be interested in will not change by multiplication of L by a nonvanishing factor. Since (IV.2) implies that either A or B does not vanish in a neighborhood of a given point (assume as well that it is A), we may multiply L by A^{-1} and obtain the new vector field $\tilde{L} = A^{-1}L$ which has the form

$$\tilde{L}u = \frac{\partial u}{\partial t} + \tilde{B}(x,t)\frac{\partial u}{\partial x}. \tag{IV.3}$$

Write $\tilde{B}(x,t) = \tilde{a}(x,t) + i\tilde{b}(x,t)$ with $\tilde{a}$ and $\tilde{b}$ real, and assume that they are defined for $|x| < \rho$, $|t| < \rho$.

Lemma IV.1.1. *In appropriate new local coordinates $\xi = x$, $s = s(x,t)$ defined in a neighborhood of the origin, the vector field $\tilde{L}$ assumes the form*

$$\tilde{L}u = \frac{\partial u}{\partial s} + ib(\xi,s)\frac{\partial u}{\partial \xi}, \tag{IV.4}$$

with $b(\xi,s)$ real-valued.

Proof. Consider the ODE

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}s} = \tilde{a}(x,t), & x(0) = \xi, \\[2ex] \dfrac{\mathrm{d}t}{\mathrm{d}s} = 1, & t(0) = 0, \end{cases}$$

with solution $(x(\xi, s), t(\xi, s))$ given by

$$\begin{cases} x(\xi, s) = \xi + \int_0^s \tilde{a}(x(\xi, \sigma), \sigma)\, \mathrm{d}\sigma, \\ t(\xi, s) = s. \end{cases}$$

Observe that $x(\xi, 0) = \xi$ so $(\partial x/\partial \xi)(0,0) = 1$; also $(\partial t/\partial \xi)(0,0) = 0$ and $(\partial t/\partial s)(0,0) = 1$ so the Jacobian determinant $\det[\partial(x, t)/\partial(\xi, s)]$ assumes the value 1 at $x = s = 0$, granting that $(\xi, s) \longleftrightarrow (x, t)$ is, at least locally, a smooth change of variables. The chain rule gives

$$\frac{\partial}{\partial s} = \frac{\partial}{\partial t} + \tilde{a}(x, t)\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x},$$

so in the new coordinates we have $\tilde{L} = \partial_s + i(\tilde{b}/(\partial x/\partial \xi))\partial_\xi = \partial_s + ib\partial_\xi$. $\square$

The reductions just described show that in the study of local problems for a planar vector field L with smooth coefficients we may always assume that L is of the form

$$L = \frac{\partial}{\partial t} + ib(x, t)\frac{\partial}{\partial x} \tag{IV.5}$$

with $b(x, t)$ real and defined for all $(x, t) \in \mathbb{R}^2$.

Definition IV.1.2. *Let L be a vector field defined in an open set $\Omega \subset \mathbb{R}^2$, $p \in \Omega$. We say that L is locally solvable at p if there exists a neighborhood $U = U(p)$ such that for all $f \in C^\infty(\Omega)$ there exists $u \in \mathcal{D}'(\Omega)$ such that $Lu - f$ vanishes identically on U. If L is locally solvable at every point $p \in \Omega$ we say that L is locally solvable in Ω.*

Remark IV.1.3. Observe that Definition IV.1.2 means that given p there exists a fixed open subset $U \ni p$ such that for every $f \in C^\infty(\Omega)$ there exists $u \in \mathcal{D}'(\Omega)$ such that the equation $Lu = f$ holds on U. A moment's reflection shows that we would get an equivalent definition by requiring instead that for every $f \in C_c^\infty(U)$ there exists $u \in \mathcal{D}'(U)$ such that $Lu = f$ in U. It is less evident that we also get an equivalent definition if we require that for every $f \in C^\infty(\Omega)$ there exists $u \in \mathcal{D}'(\Omega)$ such that $Lu - f$ vanishes on a neighborhood $U(p, f)$ of p that may depend on both f and p. However, a category argument shows that if this happens we may always take U independent of f for fixed p and the apparently weaker requirement is in fact equivalent to that given in Definition IV.1.2 (*cf.* Theorem VII.6.1).

In order to acquire some insight on local solvability let us consider the simpler case in which the coefficient $b(x, t)$ of the vector field (IV.5) is actually independent of x, i.e.,

$$L = \frac{\partial}{\partial t} + ib(t)\frac{\partial}{\partial x},$$

and we wish to study the local solvability of L in a neighborhood of the origin. In other words, we wish to find a distribution u such that $Lu = f$ where $f \in C_c^\infty(\mathbb{R}^2)$ is given. We shall perform a partial Fourier transform in the variable x and denote by $\widehat{u}$ and $\widehat{f}$ the transforms of u and f respectively, so the transformed equation becomes

$$\frac{\mathrm{d}\widehat{u}}{\mathrm{d}t} - b(t)\xi\widehat{u} = \widehat{f}, \quad \text{where } \widehat{f}(\xi, t) = \int_{\mathbb{R}} \mathrm{e}^{-ix\xi} f(x, t)\,\mathrm{d}x.$$

Using a standard formula for the linear ODE with parameter ξ, we find a solution $\widehat{u}$

$$\widehat{u}(\xi, t) = \int_{T(\xi)}^{t} \mathrm{e}^{(B(t)-B(s))\xi}\widehat{f}(\xi, s)\,\mathrm{d}s, \quad \text{where } B(t) = \int_0^t b(\tau)\,\mathrm{d}\tau.$$

Changing the endpoint of integration $T(\xi)$ amounts to adding a solution of the homogeneous equation for each value of the parameter ξ. Thus, we see that it is very easy to find (many) solutions of the transformed equation, but in order to get a solution of the original equation we need that $\widehat{u}(\xi, t)$ be tempered in ξ, at least for t in a certain range $|t| < T$, so that we can define u as the inverse partial Fourier transform of $\widehat{u}$. The difficulty comes from the risk of growth at infinity arising from the factor $\mathrm{e}^{(B(t)-B(s))\xi}$; notice that since $\xi \mapsto \widehat{f}(\xi, s)$ is in $\mathcal{S}(\mathbb{R})$ uniformly in s its rapid decay can overpower a factor of polynomial growth but to control factors with exponential growth by the decay of $\widehat{f}$ is not possible. A sensible attitude to avoid exponential growth is then to search for conditions that allow—after a convenient choice of $T(\xi)$—that $(B(t) - B(s))\xi \le 0$ whenever $|t| < T$ and s is in the interval with endpoints $T(\xi)$ and t. Of course, the sign of $(B(t) - B(s))\xi$ does not change if ξ is multiplied by a positive number so we need only define two values for $T(\xi)$: $T(\xi) = T^+$ for $\xi > 0$ and $T(\xi) = T^-$ for $\xi < 0$. Let us concentrate first on the case $\xi > 0$. *We need to find T^+ such that for all $|t| < T$ and s in the interval with endpoints $\{T^+, t\}$ the following inequality holds:*

$$B(t) - B(s) = \int_s^t b(\tau)\,\mathrm{d}\tau \le 0.$$

We immediately see that if $b(\tau) \le 0$ it will be enough to set $T^+ \doteq -T$ to obtain what we wish! Similarly, if $b(\tau) \ge 0$ the choice $T^+ \doteq T$ does the job, because to require that s be in the interval with endpoints $\{T, t\}$ simply means that $t < s < T$. So, if $b(0) \neq 0$ we may take T small enough so that $b(\tau)$ does not vanish in $(-T, T)$ and then define $T^+ = \pm T$ according to the sign of $b(0)$. Let us assume now that $b(0) = 0$. If $b(\tau)$ does not change sign in $(-T, T)$ for some $T > 0$ we already know how to proceed. What if $b(\tau)$ changes sign in $(-T, T)$? Well, suppose there is a point $t_0 \in (-T, T)$ such that $b(\tau) \geq 0$ for $\tau \in (-T, t_0]$ and $b(\tau) \leq 0$ for $\tau \in [t_0, T]$. In this case, we take $T^+ = t_0$ and notice that $\int_s^t b(\tau)\,\mathrm{d}\tau \leq 0$ both for $t_0 < s < t$ and for $t < s < t_0$. It is easy to convince oneself that those are all the cases for which a good choice of T^+ is possible. Indeed, if $b(\tau_0) < 0$ and $b(\tau_1) > 0$ for some $-T < \tau_0 < \tau_1 < T$ no choice of T^+ will work. We would be forced to take $T^+ > \tau_1$ to guarantee that $\int_s^t b(\tau)\,\mathrm{d}\tau \leq 0$ for $t < s$, s, t close to τ_1, but this would imply that $\int_s^t b(\tau)\,\mathrm{d}\tau > 0$ for $t < s$, s, t close to τ_0. In other words, *we must prevent that $b(t)$ changes sign from minus to plus as t increases*. The analysis of the case $\xi < 0$ and the choice of T^- will tell us that we must as well prevent that $b(t)$ changes sign from plus to minus as t increases and both conclusions imply together that $b(t)$ cannot change sign at all.

Remark IV.1.4. If we were studying the local solvability of the differential/pseudo-differential operator

$$L = \frac{\partial}{\partial t} - b(t)|D_x|,$$

where $|D_x|$ is the operator defined by $\widehat{|D_x|u}(\xi, t) = |\xi|\widehat{u}(\xi, t)$, this would lead us to consider the ODE

$$\frac{\mathrm{d}\widehat{u}}{\mathrm{d}t} - b(t)|\xi|\widehat{u} = \widehat{f}$$

and to require that $(B(t) - B(s))|\xi| \leq 0$. This time the sign of ξ does not matter and we are only forced to prevent sign changes of $b(\tau)$ from minus to plus.

Let us return to the problem of finding a solution to the equation $Lu = f$ when the coefficient $b(t)$ does not depend on x and we further assume that $t \mapsto b(t)$ does not change sign for $|t| < T$. Assuming that $b(t) \geq 0$, a solution is given by $u(x, t) = u^+(x, t) + u^-(x, t)$ where

$$u^+(x, t) = \frac{1}{2\pi}\int_0^\infty \int_T^t \mathrm{e}^{ix\xi + (B(t)-B(s))\xi}\widehat{f}(\xi, s)\,\mathrm{d}s\,\mathrm{d}\xi, \tag{IV.6}$$

$$u^-(x, t) = \frac{1}{2\pi}\int_{-\infty}^0 \int_{-T}^t \mathrm{e}^{ix\xi + (B(t)-B(s))\xi}\widehat{f}(\xi, s)\,\mathrm{d}s\,\mathrm{d}\xi, \quad |t| < T. \tag{IV.7}$$

The exponential in the integrals that define u^+ and u^- is bounded by 1 because the exponent always has nonpositive real part. The integrand is bounded by $|\widehat{f}(\xi, s)|$, which is rapidly decreasing in ξ as $|\xi| \to \infty$, in particular, $u(x, t)$

is continuous and bounded. Differentiating under the integral sign we always obtain integrable integrands, showing that our solution $u \in C^\infty(\mathbb{R} \times (-T, T))$.

Definition IV.1.5. *We say that the operator L given by* (IV.5) *satisfies condition $(\mathcal{P})$ at $p = (x_0, t_0)$ if there is a neighborhood $(x_0 - \delta, x_0 + \delta) \times (t_0 - \delta, t_0 + \delta)$ of p such that for every $x \in (x_0 - \delta, x_0 + \delta)$ the function $(t_0 - \delta, t_0 + \delta) \ni t \mapsto b(x, t)$ does not change sign. If L satisfies condition $(\mathcal{P})$ at every point of an open set Ω we say that L satisfies condition $(\mathcal{P})$ in Ω.*

The importance of this definition comes from the following:

Theorem IV.1.6. *The operator L given by* (IV.5) *is locally solvable at p if and only if it satisfies condition $(\mathcal{P})$ at p.*

We will not prove Theorem IV.1.6 here. The ‘if’ part of the theorem will follow from Corollary IV.1.10 presented later in this section while the ‘only if’ part will be discussed in Section IV.4 under the assumption that L is locally integrable.

Remark IV.1.7. In the case of a coefficient independent of x, if condition $(\mathcal{P})$ is satisfied in a rectangle Ω, it follows that either $b(t) \geq 0$ in Ω or $b(t) \leq 0$ in Ω, but this is not the general situation. For instance, if $b(x, t) = x$ we see that L satisfies condition $(\mathcal{P})$ in $\mathbb{R}^2$ but b is positive for $x > 0$ and negative for $x < 0$.

Remark IV.1.8. If L satisfies condition $(\mathcal{P})$ in a rectangle Ω centered at p and $\chi(x, t) \in C_c^\infty(\Omega)$ is identically 1 in a neighborhood of p, replacing $b(x, t)$ by $\chi(x, t)b(x, t)$ gives an operator $\tilde{L}$ that satisfies condition $(\mathcal{P})$ everywhere and coincides with L in a neighborhood of p. Furthermore, it is apparent that L is locally solvable at p if and only if $\tilde{L}$ is locally solvable at p. Thus, when studying the local solvability of an operator that satisfies condition $(\mathcal{P})$ in a neighborhood of p we may assume without loss of generality that $b(x, t)$ is compactly supported and condition $(\mathcal{P})$ is satisfied in $\mathbb{R}^2$.

Returning to the case in which the coefficient $b(t)$ is independent of x, observe that the solution u of $Lu = f$ furnished by (IV.6) and (IV.7) when $b \geq 0$ may be written in operator form as $u = Kf$, $K = K^+ + K^-$. Take a test function $\varphi \in C_c^\infty(\mathbb{R} \times (-T, T))$ and set $f = L\varphi$ and $u = Kf$. We see that $Lu = f = L\varphi$. Moreover, since $\widehat{f}(\xi, t) \equiv 0$ for $|t| \geq T$ we see that u is supported in $\mathbb{R} \times [-T, T]$. Thus $w = u - \phi$ satisfies $Lw = 0$ and vanishes for $t \leq -T$. By uniqueness in the Cauchy problem we conclude that $\varphi = KL\varphi$.

Using Parseval's identity it is easy to derive that for fixed $t < s < T$ the $L^2(\mathbb{R})$ norm of

$$x \mapsto (2\pi)^{-1} \int_0^\infty e^{ix\xi + (B(t)-B(s))\xi} \widehat{f}(\xi, s)\, d\xi$$

is bounded by $\|f(\cdot, s)\|_{L^2(\mathbb{R})}$. This implies

$$\|K^+ f(\cdot, t)\|_{L^2(\mathbb{R})} \leq \int_{-T}^{T} \|f(\cdot, s)\|_{L^2(\mathbb{R})}\, ds, \quad x \in \mathbb{R},\ |t| < T.$$

Integrating this inequality in t between $-T$ and T we obtain

$$\int_{-T}^{T} \|K^+ f(\cdot, s)\|_{L^2(\mathbb{R})}\, ds \leq 2T \int_{-T}^{T} \|f(\cdot, s)\|_{L^2(\mathbb{R})}\, ds.$$

The same inequality holds for K^-, so we obtain the following mixed norm estimate:

$$\|Kf\|_{L^1[(-T,T);L^2(\mathbb{R})]} \leq CT\, \|f\|_{L^1[(-T,T);L^2(\mathbb{R})]}. \tag{IV.8}$$

Now apply (IV.8) to $f = L\varphi$ and $Kf = \varphi \in C_c^\infty(\mathbb{R} \times (-T, T))$ to get the a priori inequality

$$\|\varphi\|_{L^1[(-T,T);L^2(\mathbb{R})]} \leq CT\, \|L\varphi\|_{L^1[(-T,T);L^2(\mathbb{R})]}, \quad \varphi \in C_c^\infty(\mathbb{R} \times (-T, T)). \tag{IV.9}$$

Observe that the transpose ${}^tL$ defined by $\langle L\varphi, \psi\rangle = < \varphi, {}^tL\psi >$ for all test functions $\varphi, \psi \in C_c^\infty(\mathbb{R}^2)$ is given by

$${}^tL = -L,$$

so (IV.9) may also be written as

$$\|\varphi\|_{L^1[(-T,T);L^2(\mathbb{R})]} \leq CT\, \|{}^tL\varphi\|_{L^1[(-T,T);L^2(\mathbb{R})]}, \tag{IV.10}$$

for every $\varphi \in C_c^\infty(\mathbb{R} \times (-T, T))$. It is a remarkable fact that essentially the same formulas that yield an a priori estimate for the simple case in which b is independent of t also give, in spite of technical complications, the same a priori estimate for the case of a general $b(x, t)$. We will prove a priori estimates like (IV.10) for a general vector field (IV.5) that satisfies condition $(\mathcal{P})$. More precisely,

THEOREM IV.1.9. *Let L given by* (IV.5) *satisfy condition* $(\mathcal{P})$ *in a neighborhood U of the origin and fix numbers p and q satisfying $1 < p < \infty$, $1 \leq q \leq \infty$. Then, there exist $T_0 > 0$, $a > 0$, and $C > 0$ such that for any $0 < T \leq T_0$ the following a priori estimate holds for every $\varphi \in C_c^\infty((-a, a) \times (-T, T))$:*

$$\|\varphi\|_{L^q[(-T,T);L^p(\mathbb{R})]} \leq CT\, \|{}^tL\varphi\|_{L^q[(-T,T);L^p(\mathbb{R})]}. \tag{IV.11}$$

Moreover, the constants T_0 and C depend only on p, q, and $\|b_x\|_{L^\infty(U)}$.

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Before embarking on the rather long proof of Theorem IV.1.9, let us state a standard consequence that implies the local solvability of L.

Corollary IV.1.10. *Let L given by* (IV.5) *satisfy condition* ($\mathcal{P}$) *in a neighborhood of the origin, let $1 < p' < \infty$ and $1 \le q' \le \infty$ be given. Then there exist $T_0 > 0$, $C > 0$ such that for any $0 < T \le T_0$ and $f(x,t) \in L^{q'}[\mathbb{R}; L^{p'}(\mathbb{R})]$ there exists $u \in L^{q'}[\mathbb{R}; L^{p'}(\mathbb{R})]$, with norm*

$$\|u\|_{L^{q'}[\mathbb{R};L^{p'}(\mathbb{R})]} \le CT\|f\|_{L^{q'}[\mathbb{R};L^{p'}(\mathbb{R})]]}$$

that satisfies the equation

$$Lu = f \quad in \quad \mathbb{R}\times(-T,T). \tag{IV.12}$$

Since $L^{q'}[\mathbb{R}; L^{p'}(\mathbb{R})] \simeq L^{p'}(\mathbb{R}^2)$ when $p' = q'$, L is locally solvable in $L^{p'}$ for any $1 < p' < \infty$.

Proof. We shall use the notation $\Omega_T = \mathbb{R}\times(-T,T)$. Let p, q be the conjugate exponents, $p = p'/(p'-1)$, $q = q'/(q'-1)$. Take C and T_0 as granted by Theorem IV.1.9 and for some $0 < T < T_0$ consider the linear functional

$${}^tLC_c^\infty(\Omega_T) \ni {}^tL\varphi(x,t) \mapsto \lambda({}^tL\varphi) = \int_{\mathbb{R}^2} f(x,t)\,\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

The inequalities

$$|\lambda({}^tL\varphi)| \le \|f\|_{L^{q'}[\mathbb{R};L^{p'}(\mathbb{R})]}\|\varphi\|_{L^q[\mathbb{R};L^p(\mathbb{R})]} \le CT\|{}^tL\varphi\|_{L^q[\mathbb{R};L^p(\mathbb{R})]}$$

show that λ is well-defined and continuous in the $L^q[\mathbb{R}; L^p(\mathbb{R})]$ norm. By the Hahn–Banach theorem this functional can be extended to the whole space $L^q[\mathbb{R}; L^p(\mathbb{R})]$ without increasing its norm that is bounded by CT. By the Riesz representation theorem this extension is represented by integration against a function $u \in L^{q'}[\mathbb{R}; L^{p'}(\mathbb{R})]$ with norm $\|u\|_{L^\infty[\mathbb{R};L^q(\mathbb{R})]} \le CT$. For $\varphi \in C_c^\infty(\Omega_T)$ we have

$$\int_{\mathbb{R}^2} u(x,t)\,{}^tL\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t = \lambda({}^tL\varphi) = \int_{\mathbb{R}^2} f(x,t)\,\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t,$$

which means that $Lu = f$ in Ω_T in the sense of distributions. □

IV.1.1 A priori estimates in L^p

To prove Theorem IV.1.9 let us start by observing that we may assume without loss of generality that $b(x,t)$ has compact support, L satisfies condition ($\mathcal{P}$) everywhere, and $\|b_x\|_{L^\infty(\mathbb{R})} \le C\|b_x\|_{L^\infty(U)}$. The transpose ${}^tL$ of L is given by ${}^tL = -L - ib_x$ so if an a priori estimate like (IV.11) is proved for L instead

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of ${}^tL$ it will easily imply the estimate for ${}^tL$ since the contribution of the bounded zero-order term ib_x can be absorbed by taking T_0 small enough. In other words, it is enough to prove (IV.11) with L in the place of ${}^tL$.

When dealing with the case $b(t)$ we already saw the advantage of considering separately the cases $\xi > 0$ and $\xi < 0$ (microlocalization) and this corresponds to writing $1 = H(\xi) + H(-\xi)$, where $H(\xi)$ is the Heaviside function, defined as $H(\xi) = 1$ for $\xi > 0$ and $H(\xi) = 0$ for $\xi < 0$. It will be convenient—although not strictly necessary—to substitute this rough partition of unity by a smooth one, so we consider a test function $\chi \in C_c^\infty(-2, 2)$ such that $\chi(\xi) = 1$ for $|\xi| \leq 1$ and set

$$\psi^+(\xi) = \begin{cases} 1 - \chi(\xi), & \text{if} \quad \xi \geq 0, \\ 0 & \text{if} \quad \xi \leq 0, \end{cases}$$

and

$$\psi^-(\xi) = \begin{cases} 0 & \text{if} \quad \xi \geq 0, \\ 1 - \chi(\xi), & \text{if} \quad \xi \leq 0, \end{cases}$$

so we have $1 = \chi(\xi) + \psi^+(\xi) + \psi^-(\xi)$. Given $\varphi \in \mathcal{S}(\mathbb{R}_x \times \mathbb{R}_t)$, for each fixed t we have a decomposition

$$\begin{aligned} \varphi(\cdot, t) &= P_0\varphi(\cdot, t) + P^+\varphi(\cdot, t) + P^-\varphi(\cdot, t) \\ &= \varphi_0(\cdot, t) + \varphi^+(\cdot, t) + \varphi^-(\cdot, t), \end{aligned} \tag{IV.13}$$

where

$$\begin{aligned} P_0\varphi(x, t) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi}\, \chi(\xi)\, \widehat{\varphi}(\xi, t)\, d\xi, \\ P^+\varphi(x, t) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi}\, \psi^+(\xi)\, \widehat{\varphi}(\xi, t)\, d\xi, \\ P^-\varphi(x, t) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi}\, \psi^-(\xi)\, \widehat{\varphi}(\xi, t)\, d\xi. \end{aligned}$$

Set $B(x, t) = \int_0^t b(x, \tau)\, d\tau$ and define

$$\begin{aligned} K^+f(x, t) &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{T(x)}^t e^{ix\xi + (B(x,t) - B(x,s))\xi}\, \tilde{\psi}^+(\xi) \widehat{f}(\xi, s)\, ds\, d\xi \\ &= \int_{T(x)}^t \int_{\mathbb{R}} e^{ix\xi + (B(x,t) - B(x,s))|\xi|}\, \widehat{\tilde{P}^+ f}(\xi, s)\, \frac{d\xi}{2\pi}\, ds, \end{aligned} \tag{IV.14}$$

where $T(x) = T$ if $\sup_t b(x, t) > 0$ and $T(x) = -T$ if $\inf_t b(x, t) < 0$ (notice that these conditions exclude each other because $t \mapsto b(x, t)$ does not change sign). The function $0 \leq \tilde{\psi}^+(\xi) \leq 1$ is supported in $[0, \infty)$ and chosen so that $\tilde{\psi}^+(\xi) = 1$ for ξ in the support of $\psi^+(\xi)$. This implies that $\tilde{P}^+P^+ = P^+$. If

$\sup_t b(x,t) = \inf_t b(x,t) = 0$ we set $T(x) = T$. In particular, $T(x)$ is constant on the open set $\inf_t b(x,t) < 0$ and also constant in its complement. It follows that if $T(x)$ is not continuous at the point x then $t \mapsto b(x,t)$ vanishes identically. Since the integrand in the definition of K^+ vanishes for $\xi < 1$ we had the right to replace ξ by $|\xi|$ in (IV.14). We now recall that the Fourier transform of the Poisson kernel of the half upper plane $\{(x,y) \in \mathbb{R}^2 : \ y > 0\}$

$$P_y(x) = \frac{1}{\pi}\frac{y}{y^2 + x^2}$$

is

$$\int_{\mathbb{R}} e^{-ix\xi} P_y(x)\,dx = e^{-y|\xi|}.$$

This is still true for $y = 0$ if we interpret $P_0(x)$ as a limit in the distribution sense: $P_0(x) = \lim_{y\searrow 0} P_y(x) = \delta(x) =$ Dirac's delta. In view of this fact, a common pseudo-differential notation for the convolution $P_y * g$ is $e^{-y|D_x|}g$. Thus, for x, t, s fixed, the inner integral in (IV.14) may be written as the convolution $P_y * \tilde{f}^+$ with $\tilde{f}^+(x,s) = \tilde{P}^+ f(x,s)$ and $y = B(x,s) - B(x,t)$, i.e., as $e^{-(B(x,s)-B(x,t))|D_x|}\tilde{f}^+$. Notice that $B(x,s) - B(x,t) \geq 0$ when s belongs to the interval with endpoints $\{t, T(x)\}$ because of the way $T(x)$ was defined. For any function $g(x)$ in $L^p(\mathbb{R})$, let us write $g^\perp(x) = \sup_{y>0} |P_y * g(x)|$. We thus have

$$|K^+ f(x,t)| \leq \int_{-T}^{T} (\tilde{f}^+)^\perp(x,s)\,ds. \tag{IV.15}$$

It is well known that $g^\perp(x) \leq Mg(x)$, where M denotes the Hardy–Littlewood maximal function

$$Mg(x) = \sup_{r>o} \frac{1}{2r}\int_{x-r}^{x+r} |g(t)|\,dt$$

and that $\|Mg\|_{L^p} \leq C_p\|g\|_{L^p}$, $1 < p < \infty$. This shows that

$$\|K^+ f(\cdot,t)\|_{L^p(\mathbb{R})} \leq C\int_{-T}^{T} \|\tilde{f}^+(\cdot,s)\|_{L^p(\mathbb{R})}\,ds \leq C\int_{-T}^{T} \|f(\cdot,s)\|_{L^p(\mathbb{R})}\,ds$$

where we have used that $\tilde{P}^+$ is bounded in $L^p(\mathbb{R})$ for $1 < p < \infty$ because it is a pseudo-differential operator of order zero. Raising the inequality to the power q and using Hölder's inequality we get

$$\|K^+ f(\cdot,t)\|^q_{L^p(\mathbb{R})} \leq CT^{q/q'}\|f\|^q_{L^q[(-T,T);L^p(\mathbb{R})]}$$

so integrating between $-T$ and T with respect to t and taking the $1/q$th power we obtain

$$\|K^+ f\|_{L^q[(-T,T);L^p(\mathbb{R})]} \leq CT\|f\|_{L^q[(-T,T);L^p(\mathbb{R})]}. \tag{IV.16}$$

Next we have to see the effect of K^+ on $L\varphi$, $\varphi \in C_c^\infty(\Omega_T) = C_c^\infty(\mathbb{R} \times (-T, T))$. Observe that since $\varphi(x, \pm T) \equiv 0$ it follows that $\varphi^+(\xi, \pm T) = P^+\varphi(\xi, \pm T) \equiv 0$, in particular $\varphi^+(\xi, T(x)) = 0$ for any $\xi \in \mathbb{R}$. Let us compute

$$K^+\varphi_t^+(x, t) = \int_{T(x)}^t \int_{\mathbb{R}} \mathrm{e}^{ix\xi + (B(x,t) - B(x,s))|\xi|} \, \frac{\mathrm{d}\widehat{\varphi^+}(\xi, s)}{\mathrm{d}s} \, \frac{\mathrm{d}\xi}{2\pi} \, \mathrm{d}s.$$

Note that we have used that $\tilde{P}^+\varphi^+ = \varphi^+$. We integrate by parts in s. The boundary term is

$$\frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{ix\xi} \varphi^+(\xi, t) \, \mathrm{d}\xi = \varphi^+(x, t)$$

and the integral term is

$$I = \int_{T(x)}^t \int_{\mathbb{R}} \mathrm{e}^{ix\xi + (B(x,t) - B(x,s))|\xi|} \, b(x, s) \, |\xi| \widehat{\varphi^+}(\xi, s) \, \frac{\mathrm{d}\xi}{2\pi} \, \mathrm{d}s.$$

Since $|\xi| = \xi$ on the support of φ^+ and $i\xi\widehat{\varphi^+} = \widehat{\varphi_x^+}$ we have

$$K^+\varphi_t^+(x, t) = \varphi^+(x, t) - i \int_{T(x)}^t b(x, s) \mathrm{e}^{-(B(x,s) - B(x,t))|D_x|} \varphi_x^+(x, s) \, \mathrm{d}s. \tag{IV.17}$$

We may write $b(x, s)\mathrm{e}^{-(B(x,s)-B(x,t))|D_x|}\varphi_x^+ = \left[b, \mathrm{e}^{-(B(x,s)-B(x,t))|D_x|}\right]\varphi_x^+ + \mathrm{e}^{-(B(x,s)-B(x,t))|D_x|} b\varphi_x^+$. Thus, (IV.17) may be rewritten as

$$K^+L\varphi^+(x, t) = \varphi^+(x, t) + R^+\varphi^+(x, t) \tag{IV.18}$$

where

$$R^+\varphi^+(x, t) = \int_{T(x)}^t \left[b, \mathrm{e}^{-(B(x,s)-B(x,t))|D_x|}\right] \varphi_x^+(x, s) \, \mathrm{d}s. \tag{IV.19}$$

It follows from (IV.16) and (IV.18) that

$$\begin{aligned} \|\varphi^+\|_{L^q[\mathbb{R}; L^p(\mathbb{R})]} &\le \|K^+L\varphi^+\|_{L^q[\mathbb{R}; L^p(\mathbb{R})]} + \|R^+\varphi^+\|_{L^q[\mathbb{R}; L^p(\mathbb{R})]} \\ &\le CT\|L\varphi^+\|_{L^q[\mathbb{R}; L^p(\mathbb{R})]} + \|R^+\varphi^+\|_{L^q[\mathbb{R}; L^p(\mathbb{R})]}, \end{aligned} \tag{IV.20}$$

so (IV.20) will imply (IV.11) for φ^+ with L in the place of ${}^tL$ if the error term $\|R^+\varphi^+\|_{L^q[\mathbb{R}; L^p(\mathbb{R})]}$ can be absorbed. At this point we need

LEMMA IV.1.11. *Let $b(x)$, $x \in \mathbb{R}$, be a Lipschitz function with Lipschitz constant K, $\varphi \in \mathcal{S}(\mathbb{R})$. There is a constant $C > 0$ such that*

$$\left\| \sup_{\varepsilon > 0} \left| \left[b, \mathrm{e}^{-\varepsilon|D_x|}\right] \varphi_x \right| \right\|_{L^p(\mathbb{R})} \le C_p K \|\varphi\|_{L^p(\mathbb{R})}.$$

PROOF. We have

$$\left[b, \mathrm{e}^{-\varepsilon|D_x|}\right]\varphi_x(x) = \int (b(x)-b(y))P_\varepsilon(x-y)\varphi'(y)\,\mathrm{d}y.$$

After an integration by parts we may write

$$\begin{aligned}&\left[b,\mathrm{e}^{-\varepsilon|D_x|}\right]\varphi_x(x)\\ &\quad= P_\varepsilon * (b'\varphi)(x) + \int (b(x)-b(y))[P_\varepsilon]'(x-y)\varphi(y)\,\mathrm{d}y.\end{aligned} \tag{IV.21}$$

As we already saw, $\sup_{\varepsilon>0}|P_\varepsilon * (b'\varphi)| \leq M(b'\varphi) \leq \|b'\|_{L^\infty} M\varphi$, where M is the Hardy–Littlewood maximal operator. The second term may be majorized by $K|Q_\varepsilon| * |\varphi|(x)$ where

$$Q_\varepsilon(x) = \frac{1}{\varepsilon}Q(x/\varepsilon), \qquad Q(x) = x\frac{\mathrm{d}P}{\mathrm{d}x}(x) = \frac{-2x^2}{(1+x^2)^2}.$$

Since the function $|Q(x)|$ has an integrable even majorant it follows **[S1]** that $\sup_{\varepsilon>0}|Q_\varepsilon| * |\varphi| \leq CM(\varphi)$. Therefore, $\left|\left[b, \mathrm{e}^{-\varepsilon|D_x|}\right]\varphi'(x)\right| \leq CM\varphi(x)$ and the L^p boundedness of the Hardy–Littlewood operator M grants the desired estimate. □

We may now estimate the error term in (IV.20). Since

$$\left|\left[b, \mathrm{e}^{-(B(x,s)-B(x,t))|D_x|}\right]\varphi_x^+(x,s)\right| \leq \sup_{y>0}\left|\left[b, \mathrm{e}^{-y|D_x|}\right]\varphi_x^+(x,s)\right|$$

it follows from (IV.19) and Lemma IV.1.11 that

$$\|R^+\varphi^+(\cdot,t)\|_{L^p(\mathbb{R})} \leq C\int_{-T}^{T}\|\varphi^+(\cdot,s)\|_{L^p(\mathbb{R})}\,\mathrm{d}s.$$

We already showed that from this inequality follows the estimate

$$\|R^+\varphi^+\|_{L^q[(-T,T);L^p(\mathbb{R})]} \leq CT\|\varphi^+\|_{L^q[(-T,T);L^p(\mathbb{R})]}.$$

Taking account of (IV.20) we obtain

$$\|\varphi^+\|_{L^q[(-T,T);L^p(\mathbb{R})]} \leq CT\|L\varphi^+\|_{L^q[(-T,T);L^p(\mathbb{R})]} + CT\|\varphi^+\|_{L^q[(-T,T);L^p(\mathbb{R})]}.$$

Write $L\varphi^+ = LP^+\varphi = P^+L\varphi + [L, P^+]\varphi = P^+L\varphi + [-bD_x, P^+]\varphi$. Since P^+ is a pseudo-differential of order zero, P^+ is bounded in $L^p(\mathbb{R})$ and so is the commutator $[-bD_x, P^+]$ with norm proportional to $\|b_x\|_{L^\infty}$ (see **[S2**, page 309], for the continuity in L^2 which implies the L^p continuity, $1<p<\infty$, by the Calderón–Zygmund theory). Thus,

$$\|\varphi^+\|_{L^q[(-T,T);L^p(\mathbb{R})]} \leq CT\|L\varphi\|_{L^q[(-T,T);L^p(\mathbb{R})]} + CT\|\varphi\|_{L^q[(-T,T);L^p(\mathbb{R})]}.$$

In a similar way, we may prove

$$\|\varphi^-\|_{L^q[(-T,T);L^p(\mathbb{R})]} \le CT\|L\varphi\|_{L^q[(-T,T);L^p(\mathbb{R})]} + CT\|\varphi\|_{L^q[(-T,T);L^p(\mathbb{R})]}.$$

It remains to estimate φ_0, which is easier. We define

$$K_0 f(x,t) = \int_{-T}^{t}\int_{\mathbb{R}} \mathrm{e}^{ix\xi+(B(x,t)-B(x,s))\xi}\,\widehat{P_0 f}(\xi,s)\,\frac{\mathrm{d}\xi}{2\pi}\,\mathrm{d}s,$$

and notice that $\widehat{P_0 f}(\xi,s) = \chi(\xi)\widehat{f}(\xi,s)$ is supported in $|\xi| \le 2$ so the exponential remains bounded independently of the sign of the exponent. Reasoning with K_0 as we did with K^+ we derive

$$\|\varphi_0\|_{L^q[(-T,T);L^p(\mathbb{R})]} \le CT\|L\varphi\|_{L^q[(-T,T);L^p(\mathbb{R})]} + CT\|\varphi\|_{L^q[(-T,T);L^p(\mathbb{R})]}.$$

Since $\varphi = \varphi_0 + \varphi^+ + \varphi^-$ we obtain

$$\|\varphi\|_{L^q[(-T,T);L^p(\mathbb{R})]} \le CT\|L\varphi\|_{L^q[(-T,T);L^p(\mathbb{R})]} + CT\|\varphi\|_{L^q[(-T,T);L^p(\mathbb{R})]},$$

which implies, assuming that $CT_0 < 1/2$ and $0 < T < T_0$, that

$$\|\varphi\|_{L^q[(-T,T);L^p(\mathbb{R})]} \le 2CT\|L\varphi\|_{L^q[(-T,T);L^p(\mathbb{R})]}.$$

This proves Theorem IV.1.9.

REMARK IV.1.12. Although the coefficient $b(x,t)$ was assumed to be smooth in the proof of estimate (IV.11), all steps can be carried out assuming only that $b(x,t)$ is continuous and b_x is bounded, so Theorem IV.1.9 and its Corollary IV.1.10 remain valid under these hypotheses.

Consider a finite rectangle $U = (-T,T)\times(-T,T)$. In view of Corollary IV.1.10, for every $f \in L^p(U)$ we may find $u \in L^p(U)$ such that $Lu = f$ in U. Since $L^p(U)$ decreases as p increases from 1 to ∞ the value of p may be considered as a degree of regularity of the functions that belong to $L^p(U)$. If we fix a function $f \in L^\infty(\mathbb{R}^2)$, Corollary IV.1.10 tells us that for any $p < \infty$ we may find a function $u_p \in L^p(U_p)$, $U_p = (-T(p),T(p))\times(-T(p),T(p))$ solving the equation $Lu_p = f$ in U_p. Unfortunately, $T(p) \to 0$ as $p \to \infty$, so we cannot hope to find a convergent subsequence of the sequence of solutions u_p, $p = 1,2,3,\ldots$ The question arises whether we can find a local solution of $Lu = f$ with $u \in L^\infty$. The answer, in general, is no—as the following example shows. Consider the smooth function of one variable

$$B(t) = \begin{cases} \exp(-1/t), & \text{if} \quad t \ge 0, \\ -\exp(1/t), & \text{if} \quad t \le 0, \end{cases}$$

with derivative

$$b(t) = B'(t) = \frac{1}{t^2}\exp(-1/|t|),$$

and define the differential operator on $\mathbb{R}^2$

$$L = \frac{\partial}{\partial t} - ib(t)\frac{\partial}{\partial x}.$$

It is easily verified that L satisfies condition $(\mathcal{P})$ and $L^t = -L$. The function $B(t)$ is strictly increasing for $-\infty < t < \infty$ and has an inverse $\beta(s)$: $(-1, 1) \to (-\infty, \infty)$ given by $\beta(\pm|s|) = \pm 1/|\log|s||$. There is a homeomorphism $\Psi(x, s) = (x, \beta(s)) : \mathbb{R} \times (-1, 1) \to \mathbb{R} \times (-\infty, \infty)$ which is a diffeomorphism for $0 < |s| < 1$. Let $u \in L^\infty(\mathbb{R}^2)$ and $f \in L^\infty(\mathbb{R}^2)$ be such that

$$Lu = f \tag{IV.22}$$

in the sense of distributions and set

$$v(x, s) = u(x, \beta(s)), \qquad g(x, s) = \frac{f(x, \beta(s))}{s \log^2 s}.$$

Lemma IV.1.13. *Let L, $u(x, t)$, $f(x, t)$, $v(x, s)$, $g(x, s)$ be as above. Then, $v(x, s) \in L^\infty$, $g(x, s) \in L^1_{\mathrm{loc}}$ and*

$$\frac{\partial}{\partial \bar{z}} v = \frac{1}{2} g \qquad \text{for} \quad -1 < s < 1, \tag{IV.23}$$

in the sense of distributions. In particular, if w is any solution of

$$\frac{\partial}{\partial \bar{z}} w = g,$$

in a neighborhood of the origin, w must be essentially bounded in a neighborhood of the origin.

Proof. If U is an open subset of $\mathbb{R}^2$ and $V = \Psi^{-1} U$, then V is an open subset of $\mathbb{R} \times (-1, 1)$ and its Lebesgue measure is given by

$$m(U) = \int_V \frac{1}{|s| \log^2 |s|}\, \mathrm{d}x\, \mathrm{d}s.$$

It follows that the Borel measure $\mu(X) = m(\Psi(X))$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R} \times (-1, 1)$, since $|s|^{-1} \log^{-2} |s|$ is locally integrable in $\mathbb{R} \times (-1, 1)$. Thus, $v(x, s)$ and $g(x, s)$ are measurable, v is bounded, g is locally integrable, and for every $\phi \in C_c^\infty(\mathbb{R} \times (-1, 1))$ we have the identity

$$2 \int v(x, s) \frac{\partial \phi(x, s)}{\partial \bar{z}}\, \mathrm{d}x \mathrm{d}s = \int g(x, s) \phi(x, s)\, \mathrm{d}x \mathrm{d}s, \tag{IV.24}$$

as follows from the change of variables $(x, s) = (x, B(t))$ in both integrals. Indeed, $\psi(x, t) = \phi(x, B(t))$ is a test function and (IV.24) becomes $\langle u, L^t \phi \rangle = \langle f, \phi \rangle$, which is precisely (IV.22). Furthermore, if $w/2$ is a local solution of

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(IV.24) it follows that $w-2v$ is holomorphic in a neighborhood of the origin and w must be locally bounded. $\square$

By the lemma, we will have our example if we show that for an appropriate choice of $f \in L^\infty$, equation (IV.23) has a solution which is not locally bounded in any neighborhood of the origin. We choose f so that $F = f \circ \Psi$ is the characteristic function χ of the sector K described in polar coordinates by $0 \le r \le 1/2$, $0 \le \theta \le \pi/4$. Hence, $g = \beta' \chi \in L^1_c(\mathbb{R}^2)$ and a solution $w(x, s)$ of (IV.23) is obtained by convolution of $g/2$ with the standard fundamental solution of the Cauchy–Riemann operator. Thus,

$$\Re w(x, s) = \frac{1}{2\pi} \int_K \frac{x - x'}{|x-x'|^2 + |s-s'|^2} \, \frac{1}{s' \ln^2 s'} \, \mathrm{d}x' \mathrm{d}s'.$$

We see that for $(x, s) = (0, 0)$, the integral above is given by

$$\frac{1}{2\pi} \int_0^{\pi/4} \int_0^{1/2} \frac{-\cos\theta}{r \sin\theta \log^2(r\sin\theta)} \, \mathrm{d}r \mathrm{d}\theta = \frac{1}{2\pi} \int_0^{\pi/4} \frac{1}{\sin\theta \log[(\sin\theta)/2]} \, \mathrm{d}\theta$$
$$= -\infty$$

and it is easy to conclude that $\Re w$ cannot be essentially bounded in any neighborhood of the origin. Indeed, if (x_n, s_n) is any sequence such that $x_n < 0$ and $(x_n, s_n) \to (0, 0)$, the integrand in $\Re w(x_n, s_n)$ remains negative and by Fatou's lemma $\liminf_{(x_n, s_n) \to (0,0)} \Re w(x_n, s_n) \le \Re w(0, 0) = -\infty$. Hence, $w(x, s)$ cannot remain essentially bounded in $\{(x, s) : x < 0,\ x^2 + s^2 < \varepsilon^2\}$ for any $\varepsilon > 0$.

Take $q' = \infty$ and write p instead of p' in Corollary IV.1.10. If $f \in L^\infty$ we can obtain local solutions of $Lu = f$ in $L^\infty[(-T, T); L^p(\mathbb{R})]$ for any $1 < p < \infty$ but, as we just saw, we cannot find in general a solution $u \in L^\infty[(-T, T); L^\infty(\mathbb{R})] \simeq L^\infty(\Omega_T)$. Many results in analysis that hold for $1 < p < \infty$ and fail for $p = \infty$ become true if L^∞ is replaced by a space of functions of bounded mean oscillation. In our situation the remedy is to replace the space L^∞ by the space $\mathrm{bmo}(\mathbb{R})$, dual of the semilocal (or localizable) Hardy space $h^1(\mathbb{R})$.

IV.1.2 A priori estimates in h^1

We recall some facts about the real Hardy spaces $H^1(\mathbb{R})$, a particular instance of the spaces introduced by Stein and Weiss in [**SW**], and its semilocal version $h^1(\mathbb{R})$ introduced by Goldberg [**G**]. In many situations $H^1(\mathbb{R})$ is an advantageous substitute for $L^1(\mathbb{R})$ ([**S2**]), as the latter does not behave well in many respects, for instance, concerning the continuity of singular integral operators. Let us choose a function $\Phi \geq 0 \in C_c^\infty([-1/2, 1/2])$, with $\int \Phi \mathrm{d}z = 1$. Write $\Phi_\varepsilon(z) = \varepsilon^{-1}\Phi(z/\varepsilon)$, $z \in \mathbb{R}$, and set

$$M_\Phi f(z) = \sup_{0<\varepsilon<\infty} |(\Phi_\varepsilon * f)(z)|.$$

Then [**S2**]

$$H^1(\mathbb{R}) = \{f \in L^1(\mathbb{R}) : \quad M_\Phi f \in L^1(\mathbb{R})\}.$$

A space of distributions is called semilocal if it is invariant under multiplication by test functions. The space $H^1(\mathbb{R})$, is not: ψu may not belong to $H^1(\mathbb{R})$ for $\psi \in C_c^\infty(\mathbb{R})$ and $u \in H^1(\mathbb{R})$. A way around this is the definition of the semilocal (or localizable) Hardy space—better suited for the study of PDEs—$h^1(\mathbb{R})$ ([**G**], [**S2**]) by means of the truncated maximal function

$$m_\Phi f(z) = \sup_{0<\varepsilon\leq 1} |(\Phi_\varepsilon * f)(z)|;$$

$$h^1(\mathbb{R}) = \{f \in L^1(\mathbb{R}) : \quad m_\Phi f \in L^1(\mathbb{R})\},$$

which is stable under multiplication by test functions (we will systematically denote by $\mathcal{S}$ the Schwartz space of rapidly decreasing functions and by $\mathcal{S}'$ its dual, i.e., the space of tempered distributions). It turns out that if Φ is substituted in the definition of $h^1(\mathbb{R})$ by any other function $\Phi \in \mathcal{S}(\mathbb{R})$ only subjected to $\int \Phi \neq 0$, this will not change the space $h^1(\mathbb{R})$. Moreover, $h^1(\mathbb{R})$ is a Banach space with the norm

$$\|f\|_{h^1} = \|m_\Phi f\|_{L^1},$$

and $H^1 \subset h^1 \subset L^1$. Of course, this norm depends on the choice of Φ but different Φ's will give equivalent norms, moreover, if $\mathcal{A} \subset \mathcal{S}$ is a bounded subset, there is a constant $C = C(\mathcal{A}) > 0$ such that $\|m_\phi f\|_{L^1} \leq C\|m_\Phi f\|_{L^1}$ for all $f \in \mathcal{S}$ and $\phi \in \mathcal{A}$. In fact, more is true: denoting by $\mathcal{M}f(x) = \sup_{\phi\in\mathcal{A}} m_\phi f(x)$ the grand maximal function associated with $\mathcal{A}$ it follows that $\|\mathcal{M}f\|_{L^1} \leq C\|m_\Phi f\|_{L^1}$.

We now describe the atomic decomposition of $h^1(\mathbb{R})$ ([**G**], [**S2**]). An $h^1(\mathbb{R})$ atom is a bounded, compactly supported function $a(z)$ satisfying the following properties: there exists an interval I containing the support of a such that

(1) $|a(z)| \leq |I|^{-1}$, a.e., with $|I|$ denoting the Lebesgue measure of I;
(2) if $|I| < 1$, we further require that $\int a(z)\,\mathrm{d}z = 0$.

Any $f \in h^1$ can be written as an infinite linear combination of h^1 atoms, more precisely, there exist scalars λ_j and h^1 atoms a_j such that $\sum_j |\lambda_j| < \infty$ and the series $\sum_j \lambda_j a_j$ converges to f both in h^1 and in $\mathcal{S}'$. Furthermore,

$\|f\|_{h^1} \sim \inf \sum_j |\lambda_j|$, where the infimum is taken over all atomic representations. Another useful fact is that the atoms may be assumed to be smooth functions. A simple consequence of the atomic decomposition is that $h^1(\mathbb{R})$ is stable under multiplication by Lipschitz functions $b(x)$: if a satisfies (1) with $|I| \geq 1$ it follows that $a(x)b(x)/\|b\|_{L^\infty}$ also does. If $|I| < 1$ and the center of I is x_0 we may write $a(x)b(x) = b(x_0)a(x) + (b(x) - b(x_0))a(x) = \beta_1(x) + \beta_2(x)$. Then $\beta_1(x)/\|b\|_{L^\infty}$ satisfies (1) and (2) (with the same I) while $\beta_2(x)/K$ satisfies (1) for the interval I' of center x_0 and length 1, where K is the Lipschitz constant of $a(x)$. It follows that $f \mapsto bf$ is bounded with constant $\leq \|b\|_{L^\infty} + K$ in $h^1(\mathbb{R})$. A refinement of this argument shows that $h^1(\mathbb{R})$ is stable under multiplication by more general continuous functions including Hölder functions, as we now describe. Let ω be a modulus of continuity, meaning that $\omega : [0, \infty) \longrightarrow \mathbb{R}^+$ is continuous, increasing, $\omega(0) = 0$ and $\omega(2t) \leq C\omega(t)$, $0 < t < 1$. Consider the Banach space $C_\omega(\mathbb{R})$ of bounded continuous functions $f : \mathbb{R} \longrightarrow \mathbb{C}$ such that

$$|f|_{C_\omega} \doteq \sup_{x \neq y} \frac{|f(y) - f(x)|}{\omega(|x - y|)} < \infty,$$

equipped with the norm $\|f\|_{C_\omega} = \|f\|_{L^\infty} + |f|_{C_\omega}$. Note that C_ω is only determined by the behavior of $\omega(t)$ for values of t close to 0. We will show in Lemma A.1.1 in the Appendix that if the modulus of continuity $\omega(t)$ satisfies

$$\frac{1}{h}\int_0^h \omega(t)\,\mathrm{d}t \leq C\left(1 + \log\frac{1}{h}\right)^{-1}, \quad 0 < h < 1, \tag{IV.25}$$

then $h^1(\mathbb{R})$ is stable under multiplication by functions $\in C_\omega(\mathbb{R})$. Note that the modulus of continuity $\omega(t) = t^r$, $0 < r < 1$, that defines the Hölder space C^r, satisfies (IV.25).

Consider now a first-order linear differential operator in two variables

$$L = \frac{\partial}{\partial t} + ib(x,t)\frac{\partial}{\partial x} + c(x,t), \qquad x, t \in \mathbb{R}. \tag{IV.26}$$

We assume that for some $0 < r < 1$

(i) *$c(x, t) \in C^r(\mathbb{R}^2)$;*

(ii) *$b(x, t)$ is real and of class C^{1+r}, i.e., for all multi-indexes $|\alpha| \leq 1$, $D^\alpha b$ is bounded and $D^\alpha b \in C^r(\mathbb{R}^2)$;*

(iii) *for any $x \in \mathbb{R}$ the function $t \mapsto b(x, t)$ does not change sign.*

Of course, (iii) means that the operator L given by (IV.26) satisfies condition $(\mathcal{P})$. We now introduce the space $L^1[\mathbb{R}_t; h^1(\mathbb{R}_x)]$ of measurable functions $u(x,t)$ such that, for almost every $t \in \mathbb{R}$, $x \mapsto u(x,t) \in h^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} \|u(\cdot, t)\|_{h^1} \, \mathrm{d}t \le C < \infty.$$

The dual of the space $L^1[\mathbb{R}; h^1(\mathbb{R})]$ is (canonically isomorphic to) the space $L^\infty[\mathbb{R}; \mathrm{bmo}(\mathbb{R})]$ (see page 174).

When proving a priori estimates for norms involving Hardy spaces, the role of the coefficient $c(x,t)$ will be small and its contribution may be absorbed. For that reason, it is convenient to assume initially that $c(x,t) \equiv 0$ and we shall do so for a long time in the computations that follow. We will withdraw the temporary hypothesis only after we have proved our estimates with the additional assumption that $c(x,t) \equiv 0$.

Proposition IV.1.14. *Let the operator L given by* (IV.26) *with* $c(x,t) \equiv 0$ *satisfy (ii) and (iii), and let* $\alpha > 0$ *be given. Then there exist operators* $K, R : C_c^\infty((-\alpha, \alpha) \times (-T, T)) \longrightarrow L^1[(-T,T); h^1(\mathbb{R}_x)]$ *and constants* $C > 0$ *and* $T_0 > 0$ *such that*

$$KLu = u + Ru, \tag{IV.27}$$

$$\|Ku\|_{L^1[(-\alpha,\alpha)\times(-T,T)]} \le CT \|u\|_{L^1[\mathbb{R}; h^1(\mathbb{R})]}, \tag{IV.28}$$

$$\|Ru\|_{L^1[(-\alpha,\alpha)\times(-T,T)]} \le CT \|u\|_{L^1[\mathbb{R}; h^1(\mathbb{R})]}, \tag{IV.29}$$

for all $u \in C_c^\infty([-\alpha, \alpha] \times [-T, T])$, $0 < T \le T_0$.

This is a technical proposition that does not have an immediate duality consequence due to the fact that the norm on the left-hand side of the estimates is a weaker norm than that on the right-hand side and it should be regarded as an intermediate step towards a better estimate to be obtained later. The proof of Proposition IV.1.14 is similar to that of Theorem IV.1.9; in particular, the operators K and R referred to in (IV.27) were implicitly used in its proof, for instance, $K = K^+ + K^- + K_0$, $R = R^+ + R^- + R_0$ with K^+ given by (IV.14), R^+ given by (IV.19) and so on. So the first step will be to prove the analogue of (IV.27) for K^+. This will follow from a slight modification of (IV.15). Let us consider a *restricted* maximal function

$$g^\perp(x) = \sup_{0<y<1} |P_y * g(x)|.$$

Notice that the sup is now taken for values of y between 0 and 1 instead of $0 < y < \infty$ as we did in (IV.15), but we keep the same notation $g \mapsto g^\perp$. Assuming without loss of generality that $b(x,t)$ has compact support and taking T small we may assume that $|B(x,t)-B(x,s)|<1$ in formula (IV.14), so we get

$$|K^+u(x,t)| \le \int_{-T}^{T} (\tilde{u}^+)^{\perp}(x,s)\,\mathrm{d}s. \tag{IV.30}$$

Before we continue with the proof of the estimates, we state and prove some lemmas. The first one deals with the nonlocal space H^1.

Lemma IV.1.15. *Let $Q \in C^1(\mathbb{R})$ be an integrable function such that*

$$|Q'(x)| \le \frac{C}{1+|x|^2}, \quad x \in \mathbb{R}$$

for some $C>0$. Then, for some $C>0$,

$$\int_{\mathbb{R}} M_Q f(x)\,\mathrm{d}x = \int_{\mathbb{R}} \sup_{y>0} |Q_y * f(x)|\,\mathrm{d}x \le C\,\|f\|_{H^1(\mathbb{R})}, \quad f \in H^1(\mathbb{R}).$$

Proof. By the atomic decomposition we may assume that $f(x)=a(x)$ is an H^1-atom supported in an interval (x_0-r, x_0+r). Assume initially that $x_0=0$. We have

$$|(Q_y * a)(x)| \le \|Q_y\|_{L^1}\,\|a\|_{L^\infty} \le \frac{C}{r}, \quad x \in \mathbb{R},$$

and we easily derive that

$$\int_{-2r}^{2r} M_Q a(x)\,\mathrm{d}x \le C.$$

Recalling that $\int a(x)\,\mathrm{d}x = 0$ we may write

$$(Q_y * a)(x) = \int_{-r}^{r} a(z)(Q_y(x-z) - Q_y(x))\,\mathrm{d}z.$$

By the mean value theorem, we get for $|z|<r$

$$|Q_y(x-z) - Q_y(x)| \le \frac{1}{y^2}|Q'(\xi/y)\,z| \le \frac{Cr}{y^2+\xi^2} \le \frac{Cr}{\xi^2}$$

for some $\xi \in (x-r, x+r)$. If $|x|>2r$, it follows that $|\xi|>|x|/2$. Thus,

$$\sup_{y>0} |(Q_y * a)(x)| \le \frac{Cr}{x^2}, \quad |x|>2r,$$

and

$$\int_{|x|\ge 2r} M_Q a(x)\,\mathrm{d}x \le Cr \int_{2r}^{\infty} \frac{\mathrm{d}t}{t^2} \le C'.$$

This shows that $\int M_Q a\,\mathrm{d}x \le C$. In the general case we consider a translated atom $\widetilde{a}(x) = a(x+x_0)$ which is centered at the origin and observe that $\|M_Q a\|_{L^1} = \|M_Q \widetilde{a}\|_{L^1}$ because $M_Q\widetilde{a}(x) = M_Q a(x+x_0)$. □

We return to the semilocal Hardy space h^1 in the next lemma.

LEMMA IV.1.16. *Let* $0 < \alpha < \infty$, *let* P *be the Poisson kernel in* $\mathbb{R}^2_+$ *and let* Q *be an integrable function satisfying* $|Q'(x)| \le C/(1+|x|^2)$ *as in the previous lemma. There exists* $C > 0$ *such that*

$$\int_{-\alpha}^{\alpha} \sup_{0<y<1} |P_y * f(x)| \,\mathrm{d}x \le C \, \|f\|_{h^1(\mathbb{R})}, \quad f \in h^1(\mathbb{R}),$$

$$\int_{-\alpha}^{\alpha} \sup_{0<y<1} |Q_y * f(x)| \,\mathrm{d}x \le C \, \|f\|_{h^1(\mathbb{R})}, \quad f \in h^1(\mathbb{R}).$$

PROOF. The first inequality follows from the second one, as P satisfies the hypothesis required for Q. To prove the second inequality we need only show that there exists $C > 0$ such that

$$\int_{-\alpha}^{\alpha} \sup_{0<y<1} |Q_y * a| \le C$$

for all h^1-atoms a. Let a be an h^1-atom supported in the interval $I = (x_0 - r, x_0 + r)$. If $r > 1/2$ we observe that

$$\sup_{0<y<1} |(Q_y * a)(x)| \le \sup_{0<y<1} \|a\|_{L^\infty} \|Q_y\|_{L^1} \le |I|^{-1} \|Q\|_{L^1} \le \|Q\|_{L^1} \le C,$$

so the integral we must estimate is majorized by $2C\alpha$. If $r \le 1/2$ the atom a must satisfy the moment condition and it is also an H^1-atom so the required inequality holds even for $\alpha = \infty$ by the proof of Lemma IV.1.15. $\square$

In view of (IV.30) and the first inequality of Lemma IV.1.16 we obtain

$$\|K^+ u\|_{L^1[(-\alpha,\alpha)\times(-T,T)]} \le CT \|u^+\|_{L^1[\mathbb{R};h^1(\mathbb{R})]}. \tag{IV.31}$$

To obtain a similar inequality for R^+ we use (IV.19) to derive

$$|R^+ u(x,t)| \le \int_{-T}^{T} \sup_{0<y<1} \left| \left[b, \mathrm{e}^{-y|D_x|}\right] u_x^+ \right| (x,s) \,\mathrm{d}s. \tag{IV.32}$$

We already saw that

$$\begin{aligned}
&\left[b, \mathrm{e}^{-y|D_x|}\right] u^+(x,t) \\
&\quad = P_y * (b_x u^+)(x,t) + \int \frac{b(x,t) - b(y,t)}{x-y} Q_y(x-y) u^+(y,t) \,\mathrm{d}y \\
&\quad = P_y * (b_x u^+)(x,t) + \int Q_y(x-y) \beta^x(y,t) u^+(y,t) \,\mathrm{d}y,
\end{aligned}$$

with

$$Q_y(x) = \frac{1}{y} Q(x/y), \qquad Q(x) = x \frac{\mathrm{d}P}{\mathrm{d}x}(x) = \frac{-2x^2}{(1+x^2)^2}$$

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and

$$\beta^x(y,t) = \begin{cases} \dfrac{b(x,t)-b(y,t)}{x-y}, & \text{if} \quad y \neq x, \\ b_x(x,s'), & \text{if} \quad y = x. \end{cases}$$

Using once more Lemma IV.1.16 we see that the norm in $L^1[(-\alpha,\alpha)\times(-T,T)]$ of the term $P_y * (b_x u^+)(x,t)$ is dominated by

$$\|b_x u^+\|_{L^1[\mathbb{R};h^1(\mathbb{R})]} \leq \|u^+\|_{L^1[\mathbb{R};h^1(\mathbb{R})]}$$

where we have used that multiplication by $b_x \in C^r$ is a bounded operation in $h^1(\mathbb{R})$. Concerning the second term, observe that it may be written as a convolution $Q_y * (\beta^x u^+)(x)$ (note however that the factor β^x depends on the point at which the convolution is evaluated). The main tool to estimate the second term is

LEMMA IV.1.17. *Let* $0 < \alpha < \infty$. *Let* $Q \in C^1(\mathbb{R})$ *satisfy*

$$|Q(x)| + |Q'(x)| \leq \frac{C}{1+x^2}, \quad x \in \mathbb{R}$$

for some $C > 0$ *and assume that* $\beta \in L^\infty(\mathbb{R}^2)$ *is such that for some* $K > 0$

$$|\beta(x,y) - \beta(x,x_0)| \leq K\frac{|y-x_0|}{|x-x_0|}, \quad \textit{if} \quad |x-x_0| \geq 2|y-x_0|.$$

Then there exists $C = C(\beta, Q) > 0$ *such that, for every* $f \in h^1(\mathbb{R})$, *the inequality*

$$\int_{-\alpha}^{\alpha} \sup_{0<y<1} |Q_y * (\beta^x f)(x)|\,\mathrm{d}x \leq C\|f\|_{h^1(\mathbb{R})} \quad \textit{holds,}$$

with $\beta^x(y) = \beta(x,y)$.

PROOF. Let a be an h^1-atom, with $s(a) \subset I = (x_0 - r, x_0 + r)$. If $r > 1/2$ we have

$$\begin{aligned} \int_{-\alpha}^{\alpha} \sup_{0<y<1} |Q_y * (\beta^x a)(x)|\,\mathrm{d}x &= \int_{-\alpha}^{\alpha} \sup_{0<y<1} \left| \int Q_y(x-z)\beta^x(z)a(z)\,\mathrm{d}z \right| \mathrm{d}x \\ &\leq \int_{-\alpha}^{\alpha} \|\beta\|_{L^\infty} \|a\|_{L^\infty} \|Q_y\|_{L^1}\,\mathrm{d}x \\ &\leq 2\alpha \|\beta\|_{L^\infty} \|Q\|_{L^1}. \end{aligned}$$

Let us tackle the case $r \leq 1/2$ assuming initially that $x_0 = 0$. The estimate

$$\int_{-2r}^{2r} \sup_{0<\varepsilon<1} |Q_\varepsilon * (\beta^x a)(x)|\,\mathrm{d}x \leq Cr\|a\|_{L^\infty} \leq C'$$

is, as usual, easily obtained. Keeping in mind that $\int a(y)\mathrm{d}y = 0$ and writing

$$\begin{aligned} Q_\varepsilon(x-y)\beta^x(y) - Q_\varepsilon(x)\beta^x(0) = (Q_\varepsilon(x-y) - Q_\varepsilon(x))\beta^x(0) \\ + Q_\varepsilon(x-y)(\beta^x(y) - \beta^x(0)) \end{aligned}$$

we get the estimate

$$\begin{aligned} |Q_\varepsilon * (\beta^x a)(x)| &\le \int_{-r}^{r} |(Q_\varepsilon(x-y)\beta^x(y) - Q_\varepsilon(x)\beta^x(0)|\,|a(y)|\,\mathrm{d}y \\ &\le \int \frac{1}{\varepsilon^2} \sup_{|y|\le r} |Q'((x-y)/\varepsilon)|\, \|\beta\|_{L^\infty}\, |y|\,|a(y)|\,\mathrm{d}y \\ &\quad + \int \frac{K}{\varepsilon} \sup_{|y|\le r} |Q((x-y)/\varepsilon)|\, \frac{|y|}{|x|}\, |a(y)|\,\mathrm{d}y. \end{aligned}$$

Since $|x| > 2r$ and $|y| < r$ imply that $|x-y| \ge |x|/2$, using the decay of Q and Q' we see that

$$\begin{aligned} \frac{1}{\varepsilon^2} \sup_{|y|\le r} |Q'((x-y)/\varepsilon)| \le \frac{C}{\varepsilon^2 + x^2} \le \frac{C}{x^2}, \\ \frac{1}{\varepsilon} \sup_{|y|\le r} |Q((x-y)/\varepsilon)| \le \frac{C\varepsilon}{\varepsilon^2 + x^2} \le \frac{C}{x^2} \end{aligned}$$

for $|x| > 2r$ so

$$\begin{aligned} |Q_\varepsilon * (\beta^x a)(x)| &\le C(\beta, Q)\frac{r}{x^2} \int_{-r}^{r} |a(y)|\,\mathrm{d}y \\ &\le C(\beta, Q)\frac{r}{x^2}. \end{aligned}$$

Thus,

$$\int_{|x|>2r} \sup_{0<\varepsilon<1} |Q_\varepsilon * (\beta^x a)(x)|\,\mathrm{d}x \le C(\beta, Q)\, r \int_{2r}^{\infty} \frac{1}{t^2}\,\mathrm{d}t \le C(\beta, Q).$$

In the general case, we reason as before with $\widetilde{a}(x) = a(x + x_0)$, which is an atom centered at the origin, and $\widetilde{\beta}(x, y) = \beta(x + x_0, y + x_0)$, which satisfies the same inequalities as $\beta(x, y)$, then observe that

$$Q_\varepsilon * (\beta^x a)(x) = Q_\varepsilon * (\widetilde{\beta}^{x-x_0}\widetilde{a})(x - x_0)$$

so

$$\int \sup_{0<\varepsilon<1} |Q_\varepsilon * (\beta^x a)(x)|\,\mathrm{d}x = \int \sup_{0<\varepsilon<1} |Q_\varepsilon * (\widetilde{\beta}^x \widetilde{a})(x)|\,\mathrm{d}x \le C(\beta, Q).$$

□

REMARK IV.1.18. A function $\beta(x,y)$ satisfying the hypothesis of Lemma IV.1.17 can be obtained by setting

$$\beta^x(y) = \begin{cases} \dfrac{b(x)-b(y)}{x-y}, & \text{if} \quad y \neq x, \\ b'(x), & \text{if} \quad y = x, \end{cases}$$

if $b(x)$ and $b'(x)$ are bounded, as is easily seen.

Returning to the estimate of the second term $Q_y * (\beta^x u^+)(x)$ in the expression of $[b, \mathrm{e}^{-y|D_x|}]u^+$ we point out that Lemma IV.1.17 can indeed be applied for any fixed t to $\beta(x,y) = \beta^x(y,t)$, so using Lemma IV.1.17 and (IV.32) we get

$$\|R^+ u\|_{L^1[(-\alpha,\alpha)\times(-T,T)]} \leq CT \|u^+\|_{L^1[\mathbb{R};h^1(\mathbb{R})]}. \tag{IV.33}$$

Using (IV.31), (IV.33), their analogues for K^-, K_0, R^-, R_0 and the fact that $P^\pm$ and P_0 are pseudo-differential operators of order zero acting on the variable x, so the norm of u^+, u^- and u_0 in $h^1(\mathbb{R})$ are bounded by that of u, we may prove estimates (IV.28) and (IV.29) concluding the proof of Proposition IV.1.14.

Consider now a test function $\varphi \in C_c^\infty([-\alpha,\alpha]\times[-T,T])$. It follows easily from (IV.27) that

$$\|\varphi\|_{L^1[(-\alpha,\alpha)\times(-T,T)]} \leq \|KL\varphi\|_{L^1[(-\alpha,\alpha)\times(-T,T)]} + \|R\varphi\|_{L^1[(-\alpha,\alpha)\times(-T,T)]}$$

which, in view of (IV.28) and (IV.29), implies

$$\|\varphi\|_{L^1[(-\alpha,\alpha)\times(-T,T)]} \leq CT \left(\|L\varphi\|_{L^1[\mathbb{R};h^1(\mathbb{R})]} + \|\varphi\|_{L^1[\mathbb{R};h^1(\mathbb{R})]} \right). \tag{IV.34}$$

Notice that we cannot absorb the term $\|\varphi\|_{L^1[\mathbb{R};h^1(\mathbb{R})]}$ by taking T small because it involves a stronger norm than that of the left-hand side. Thus, we wish to obtain a similar but sharper estimate in which the norm $\|\varphi\|_{L^1[\mathbb{R};h^1(\mathbb{R})]}$ also appears as well on the left-hand side. To achieve this we make use of the mollified Hilbert transform $\widetilde{H}$ defined by $\widehat{\widetilde{H}f} = (1-\chi)\widehat{Hf}$, where H denotes the usual Hilbert transform, $\chi \in C_c^\infty(-2,2)$, $\phi = 1$, for $|\xi| \leq 1$. Here the usefulness of $\widetilde{H}$, which is a pseudo-differential operator of order zero, derives mainly from the fact that it can be used to define an equivalent norm on $h^p(\mathbb{R})$ without appealing to maximal functions, as granted by the following estimates (*cf.* **[G]**):

$$C_1 \|\widetilde{H}f\|_{h^1} \leq \|f\|_{h^1} \leq C_2(\|f\|_{L^1} + \|\widetilde{H}f\|_{L^1}), \quad f \in h^1(\mathbb{R}).$$

Another ingredient is the following lemma.

LEMMA IV.1.19. *Let $r(D)$ be a pseudo-differential of order zero with symbol $r(x,\xi) = r(\xi)$ independent of x. Assume that for some $C > 0$ the following inequality holds:*

$$\|f\|_{h^1} \le C(\|f\|_{L^1} + \|r(D)f\|_{L^1}), \qquad f \in h^1.$$

Let K be the kernel of $r(D)$ and for each $\varepsilon > 0$ write

$$\begin{aligned} r(D)f(x) &= < \chi(\varepsilon(x-\cdot))K, f > + < (1-\chi(\varepsilon(x-\cdot)))K, f > \\ &= r_1^\varepsilon(D)f(x) + r_2^\varepsilon(D)f(x), \end{aligned}$$

where $\chi \in C_c^\infty(-2,2)$ with $\chi(y) = 1$ for $|y| \le 1$. Then there exists ε_0 such that for all $0 < \varepsilon \le \varepsilon_0$ there exist constants $C_1 = C_1(\varepsilon)$, $C_2 = C_2(\varepsilon) > 0$ such that

$$\|f\|_{h^1} \le C_1(\|f\|_{L^1} + \|r_1^\varepsilon(D)f\|_{L^1}) \le C_2\|f\|_{h^1}. \tag{IV.35}$$

PROOF. For each $\varepsilon > 0$, $r_1^\varepsilon(D)$ is a pseudo-differential operator of order zero, thus bounded in h^1, so

$$\|f\|_{L^1} + \|r_1^\varepsilon(D)f\|_{L^1} \le \|f\|_{h^1} + \|r_1^\varepsilon(D)f\|_{h^1} \le C_2(\varepsilon)\|f\|_{h^1}.$$

On the other hand, $\|r_2^\varepsilon(D)f\|_{L^1} \le \|K_2^\varepsilon\|_{L^1}\|f\|_{L^1}$ and $\|K_2^\varepsilon\|_{L^1} \to 0$ as $\varepsilon \to 0$. Therefore, there exists $\varepsilon_0 > 0$ such that $\|K_2^\varepsilon\|_{L^1} \le 1/2C$ for $0 < \varepsilon \le \varepsilon_0$. Thus

$$\begin{aligned} \|f\|_{h^1} &\le C(\|f\|_{L^1} + \|r(D)f\|_{L^1}) \\ &\le C\left(\|f\|_{L^1(\mathbb{R})} + \|r_1^\varepsilon(D)f\|_{L^1(\mathbb{R})} + \frac{1}{2C}\|f\|_{L^1}\right) \\ &\le C(\|f\|_{L^1} + \|r_1^\varepsilon(D)f\|_{L^1}) + \frac{1}{2}\|f\|_{h^1}, \end{aligned}$$

which implies

$$\|f\|_{h^1} \le 2C(\|f\|_{L^1} + \|r_1^\varepsilon(D)f\|_{L^1}).$$

□

REMARK IV.1.20. Notice that $r_1^\varepsilon(D)$ is given by convolution with a distribution supported in the interval $(-2/\varepsilon, 2/\varepsilon)$, in particular if $u \in \mathcal{E}'([-r,r])$—i.e., if u is distribution supported in the interval $[-r,r]$—$r_1^\varepsilon(D)u$ is supported in the interval $[-r-2\varepsilon^{-1}, r+2\varepsilon^{-1}]$.

We are now able to prove a stronger estimate. We will show that there exist constants C and $T_0 > 0$ such that for any $0 < T \le T_0$ and $\varphi \in C_c^\infty((-a,a)\times(-T,T))$,

$$\|\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))} \le CT\|L\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))}. \tag{IV.36}$$

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Given $\phi \in C_c^\infty((-a,a)\times(-T,T))$ set

$$\widehat{\widetilde{H}\varphi(\cdot,t)}(\xi) = (1-\chi)(\xi)\widehat{H\varphi(\cdot,t)}(\xi)$$

where H is the Hilbert transform and $\chi \in C_c^\infty(-2,2)$, $\chi(\xi)=1$ for $|\xi|\le 1$. The symbol of $\widetilde{H}$ is equal to $h(\xi)=\psi^+(\xi)-\psi^-(\xi)$, where ψ^+ and ψ^- are the symbols of the operators P^+ and P^- already used. We see that $\widetilde{H}$ is a pseudo-differential operator satisfying the hypotheses of Lemma IV.1.19 and we may write it as a sum $\widetilde{H}=\widetilde{H}_1^\varepsilon+\widetilde{H}_2^\varepsilon$ where $\widetilde{H}_1^\varepsilon : \mathcal{E}'((-a,a)) \to \mathcal{E}'((-a',a'))$ satisfies (IV.35), i.e.,

$$\|\varphi(\cdot,t)\|_{h^1(\mathbb{R}_x)} \le C(\|\varphi(\cdot,t)\|_{L^1(-a,a)} + \|\widetilde{H}_1^\varepsilon\varphi(\cdot,t)\|_{L^1(-a',a')}), \tag{IV.37}$$

for some $C>0$. Since $\widetilde{H}_1^\varepsilon\varphi(x,t)\in C_c^\infty((-a',a')\times(-T,T))$, applying (IV.34) (with a' in the place of a) to $\widetilde{H}_1^\varepsilon\varphi$ we get

$$\begin{aligned}\|\widetilde{H}_1^\varepsilon\varphi\|_{L^1((-T,T)\times(-a',a'))}&\\ \le C\,T\left(\|L\widetilde{H}_1^\varepsilon\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))} + \|\widetilde{H}_1^\varepsilon\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))}\right).\end{aligned} \tag{IV.38}$$

Since $L\widetilde{H}_1^\varepsilon = \widetilde{H}_1^\varepsilon L + [L,\widetilde{H}_1^\varepsilon]$ and, invoking Proposition A.2.2 in the Appendix A, we may claim that $\widetilde{H}_1^\varepsilon$ as well as $[L,\widetilde{H}_1^\varepsilon]$ are bounded operators in $h^1(\mathbb{R}_x)$. It follows from (IV.38) that

$$\begin{aligned}\|\widetilde{H}_1^\varepsilon\varphi\|_{L^1((-T,T)\times(-a',a'))}&\\ \le C\,T(\|L\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))} + \|\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))}).\end{aligned} \tag{IV.39}$$

Integrating (IV.37) with respect to t and using (IV.39) we see that

$$\begin{aligned}\|\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))} &\le C(\|\varphi\|_{L^1((-T,T)\times(-a,a))} + \|\widetilde{H}_1^\varepsilon\varphi\|_{L^1((-T,T)\times(-a',a'))})\\ &\le CT(\|\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))} + \|L\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))}).\end{aligned}$$

It is now enough to choose T_0 such that $CT\le 1/2$ if $T\le T_0$ to get

$$\|\varphi\|_{L^1((-T,T);h^1(\mathbb{R}_x))} \le 2C\,T\|L\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))}$$

as desired. We may now state

THEOREM IV.1.21. *Let the operator L given by* (IV.26) *satisfy (i), (ii) and (iii) and let $a>0$. Then there exist constants $C>0$ and $T_0>0$ such that*

$$\|u\|_{L^1((-T,T);h^1(\mathbb{R}_x))} \le CT\|Lu\|_{L^1((-T,T);h^1(\mathbb{R}_x))}, \tag{IV.40}$$

for all $u\in C_c^\infty([-a,a]\times[-T,T])$, $0<T\le T_0$.

Proof. We have already proved (IV.36) assuming that $c(x,t) \equiv 0$ which is the same as (IV.40). In the general case we write $L = L_0 + c$ and since (IV.36) holds for L_0 we obtain

$$\begin{aligned}\|u\|_{L^1((-T,T);h^1(\mathbb{R}_x))} &\leq CT\left(\|Lu\|_{L^1((-T,T);h^1(\mathbb{R}_x))} + \|cu\|_{L^1((-T,T);h^1(\mathbb{R}_x))}\right)\\ &\leq CT\left(\|Lu\|_{L^1((-T,T);h^1(\mathbb{R}_x))} + C_1\|u\|_{L^1((-T,T);h^1(\mathbb{R}_x))}\right)\end{aligned}$$

as multiplication by a C^r function is a bounded operator in the space $L^1((-T,T);h^1(\mathbb{R}))$. Taking T small so that $CC_1T < 1/2$, we obtain (IV.40). $\square$

The a priori inequality (IV.40) has a standard duality consequence which we now describe. The dual of $h^1(\mathbb{R})$, denoted by $\mathrm{bmo}(\mathbb{R})$, may be identified ([**G**]) with the space of locally integrable functions $f(x)$ such that $\sup_{|I|<1}|I|^{-1}\int_I|f - f_I| < \infty$ and $\sup_{|I|\geq 1}|I|^{-1}\int_I|f| < \infty$, where we have denoted by I an arbitrary interval and by f_I the mean of f on I. In particular, $\mathrm{bmo}(\mathbb{R})$ is contained in $\mathrm{BMO}(\mathbb{R})$, the space of bounded mean oscillation functions. Then, (IV.40) implies local solvability in $L^\infty([-T,T],\mathrm{bmo}(\mathbb{R}_x))$ for the formal transpose L^t. Now, L and $-L^t$ have the same principal part, so L and $-L^t$ satisfy simultaneously the hypotheses of Theorem IV.1.21. Summing up,

Theorem IV.1.22. *Let the operator*

$$L = \frac{\partial}{\partial t} + ib(x,t)\frac{\partial}{\partial x} + c(x,t)$$

satisfy (i), (ii) and (iii). There is a neighborhood $U = (-a,a)\times(-T,T)$ of the origin such that for every function $f \in X = L^\infty(\mathbb{R}_t,\mathrm{bmo}(\mathbb{R}_x))$ there exists a function $u \in X$ which solves $Lu = f$ in U, with norm

$$\|u\|_{L^\infty(\mathbb{R}_t,\mathrm{bmo}(\mathbb{R}_x))} \leq CT\|f\|_{L^\infty(\mathbb{R}_t,\mathrm{bmo}(\mathbb{R}_x))}.$$

In particular, the size of u can be taken arbitrary small by letting $T \to 0$.

We conclude this section by proving consequences of Theorems IV.1.21 and IV.1.22 that can be stated in a more invariant form that does not depend on a special coordinate system. In Theorems IV.1.21 and IV.1.22, the operator L has a special form which is instrumental in obtaining a priori estimates with minimal assumptions on the regularity of the coefficients but, at least heuristically, after a suitable change of variables any first-order operator of principal type has this form as we saw in Lemma IV.1.1. On the other hand, for operators with rough coefficients this change of variables imposes a loss of regularity on the coefficients of the transformed operator. One should also observe the loss of derivatives caused in the process of deriving estimates in terms of the original variables from estimates obtained in the new variables by the behavior of local Hardy norms under composition with diffeomorphisms. For this reason we now deal with operators having C^{2+r} coefficients in the principal part. Since we are dealing with mixed norms, the roles of t and x cannot be interchanged and we must consider changes of variables that preserve the privileged role of t. Consider a general first-order operator defined in an open subset $\Omega \subset \mathbb{R}^2$ that contains the origin

$$Lu = A(x,t)\frac{\partial u}{\partial t} + B(x,t)\frac{\partial u}{\partial x} + C(x,t)\,u$$

with complex coefficients $A, B \in C^{2+r}(\Omega)$, $0 < r < 1$, $C \in C_\omega(\Omega)$. Assume that the lines $t = \text{const.}$ are noncharacteristic, which amounts to saying that $|A(x,t)| > 0$, $(x,t) \in \Omega$. Since the properties we are studying do not change if L is multiplied by a nonvanishing function of class C^{2+r}, we may assume without loss of generality that $A \equiv 1$, i.e.,

$$Lu = \frac{\partial u}{\partial t} + B(x,t)\frac{\partial u}{\partial x} + C(x,t)\,u.$$

Write $B(x,t) = \tilde{a}(x,t) + i\tilde{b}(x,t)$ with $\tilde{a}$ and $\tilde{b}$ real. In convenient new local coordinates $\xi = \xi(x,t)$, $s = t$, the expression of L is

$$\tilde{L} = \partial_s + i(\tilde{b}/(\partial x/\partial \xi))\partial_\xi + C(x(\xi,s),s) = \partial_s + ib\partial_\xi + c,$$

where b is real of class C^{1+r} and $c \in C_\omega$. If L satisfies the Nirenberg–Treves condition $(\mathcal{P})$ so does $\tilde{L}$, due to the invariance of this property that will be discussed in the next section (the coefficients are supposed to be smooth for simplicity in that section but the arguments adapt to the present situation). Multiplying the coefficients b and c by a cut-off function $\chi \geq 0 \in C_c^\infty(\mathbb{R}^2)$ that is identically equal to 1 in the neighborhood of the origin we now have an operator L' with smooth coefficients and globally defined in $\mathbb{R}^2$ that satisfies the hypotheses of Theorem IV.1.21 and agrees with $\tilde{L}$ in a neighborhood of the origin. Thus, the a priori estimate (IV.40) holds for L' in the variables (ξ, s). Let $u'(\xi,s) \in C_c^\infty(\mathbb{R}^2)$ be supported in a sufficiently small neighborhood of the origin and set $u(x,t) = u'(\xi(x,t),t)$, where $(x,t) \mapsto (\xi,s)$ is the inverse of $(\xi,s) \mapsto (x,t)$, thus of class C^{2+r}. Invoking the invariance of $h^1(\mathbb{R})$ under diffeomorphisms of class C^2 discussed in Proposition IV.3.1 we conclude that if u' is supported in a convenient neighborhood of the origin we have

$$C_1 \int_{\mathbb{R}} \|u(\cdot,t)\|_{h^1(\mathbb{R}_x)}\,\mathrm{d}t \leq \int_{\mathbb{R}} \|u'(\cdot,s)\|_{h^1(\mathbb{R}_\xi)}\,\mathrm{d}s \leq C_2 \int_{\mathbb{R}} \|u(\cdot,t)\|_{h^1(\mathbb{R}_x)}\,\mathrm{d}t$$

and this shows that the a priori estimate (IV.40) for L' implies an analogous estimate for L, using the fact that $Lu(x,t) = L'u'(\xi(x,t),t)$. Summing up, THEOREM IV.1.23. *Let L given by*

$$Lu = A(x,t)\frac{\partial u}{\partial t} + B(x,t)\frac{\partial u}{\partial x} + C(x,t)\,u$$

be defined in a neighborhood of the origin, with complex coefficients $A, B \in C^{2+r}(\Omega)$, $0 < r < 1$, $C \in C_\omega(\Omega)$. Assume that the level curves $t =$ constant are noncharacteristic for L and that L satisfies the Nirenberg–Treves condition $(\mathcal{P})$. Then there exist constants $a > 0$, $C > 0$ and $T_0 > 0$ such that

$$\|u\|_{L^1(\mathbb{R}_t; h^1(\mathbb{R}_x))} \le CT\|Lu\|_{L^1(\mathbb{R}_t; h^1(\mathbb{R}_x))},$$

for all $u \in C_c^\infty([-a,a] \times [-T,T])$, $0 < T \le T_0$. Hence, for every function $f \in X = L^\infty(\mathbb{R}_t, \mathrm{bmo}(\mathbb{R}_x))$ there exists a function $u \in X$ which solves $Lu = f$ in a neighborhood U of the origin, with norm

$$\|u\|_X \le CT\|f\|_X.$$

IV.2 Solvability in C^∞

In the last section we introduced the local solvability condition $(\mathcal{P})$ in Definition IV.1.5 assuming that the vector field L was in the special form

$$L = \frac{\partial}{\partial t} + ib(x,t)\frac{\partial}{\partial x} \tag{IV.41}$$

with $b(x,t)$ real, smooth, and defined for all $(x,t) \in \mathbb{R}^2$. However, to require that $t \mapsto b(x,t)$ does not change sign is not *per se* a coordinate-free definition because we are demanding that a particular coefficient (namely, $b(x,t)$) does not take opposite signs on sets of a special kind (namely, $\{x\} \times \mathbb{R}$). It order to find more invariant ways to formulate condition $(\mathcal{P})$ it is convenient to find larger sets on which $b(x,t)$ keeps its sign unchanged. Assume that L given by (IV.41) satisfies $(\mathcal{P})$. Then the sets

$$A^+ = \{x \in \mathbb{R} : \quad \sup_t b(x,t) > 0\} \text{ and } A^- = \{x \in \mathbb{R} : \quad \inf_t b(x,t) < 0\}$$

are open and disjoint, and the complement of its union $F = \mathbb{R} \backslash A^+ \cup A^-$ is a closed set with the property that $b(x,t) = 0$ on $F \times \mathbb{R}$. Write A^+ and A^- in terms of their connected components

$$A^+ = \bigcup_j (a_j^+, b_j^+), \qquad A^- = \bigcup_j (a_j^-, b_j^-).$$

If $x \in (a_j^+, b_j^+)$ there exists $t \in \mathbb{R}$ such that $b(x,t) > 0$ so we see that $b(x,t) \ge 0$ on $(a_j^+, b_j^+) \times \mathbb{R}$ and similarly $b(x,t) \le 0$ on $(a_j^-, b_j^-) \times \mathbb{R}$. There is an easy

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way to describe invariantly the open sets $\Omega_j^+ = (a_j^+, b_j^+) \times \mathbb{R}$ and $\Omega_j^- = (a_j^-, b_j^-) \times \mathbb{R}$: they are the orbits of dimension two of the pair of vector fields $\{X \doteq \Re L, Y \doteq \Im L\}$. Indeed, $\Omega_j^\pm$ is a union of vertical lines, so invariant under the flow of X, and it is also invariant under the flow of Y because Y vanishes on its boundary, so if $p \in \Omega_j^\pm$ the $\mathcal{O}$ orbit $\mathcal{O}(p)$ of $\{X, Y\}$ through p is contained in $\Omega_j^\pm$. Now, $\mathcal{O}(p)$ is an orbit of maximal dimension, thus open and connected, and being invariant under the flow of X it is of the form $(a, b) \times \mathbb{R}$ with $a_j^\pm \le a < b \le b_j^\pm$. Since $\{a\} \times \mathbb{R}$ is contained in the boundary of $\mathcal{O}(p)$, $b(x, t)$ must vanish identically on $\{a\} \times \mathbb{R}$ so $a \notin (a_j^\pm, b_j^\pm)$ and similarly $b \notin (a_j^\pm, b_j^\pm)$, which proves that $\Omega_j^\pm = \mathcal{O}(p)$. On the other hand, the sets $\{x\} \times \mathbb{R}$, $x \in F$, are precisely the orbits of dimension one of $\{X, Y\}$. Since $a_j^+, b_j^+, a_j^-, b_j^- \in F$ we see that a two-dimensional orbit is bounded by two one-dimensional orbits in case its orthogonal projection onto the x-axis is a finite interval, by one one-dimensional orbit if its projection has exactly one finite endpoint and, of course, the boundary is empty if the projection is the whole real line. To give a coordinate-free formulation of the fact that $b(x, t)$ does not change sign on two-dimensional orbits we look at $X \wedge Y \in C^\infty(\mathbb{R}^2; \bigwedge^2(T(\mathbb{R}^2)))$. Since $\bigwedge^2(T(\mathbb{R}^2)))$ has a global nonvanishing section $e_1 \wedge e_2$, $X \wedge Y$ is a real multiple of $e_1 \wedge e_2$ and this gives a meaning to the requirement that $\Re L \wedge \Im L$ does not change sign on any two-dimensional orbits of $\{\Re L, \Im L\}$. Note that when L has the form (IV.41) we have seen that this happens if and only if L satisfies $(\mathcal{P})$.

Consider now a vector field defined in an open subset $\Omega \subset \mathbb{R}^2$

$$Lu = A(x, t)\frac{\partial u}{\partial t} + B(x, t)\frac{\partial u}{\partial x} \tag{IV.42}$$

with complex coefficients $A, B \in C^\infty(\Omega)$ such that

$$|A(x, y)| + |B(x, y)| > 0, \quad (x, t) \in \Omega.$$

Definition IV.2.1. *We say that the operator L given by* (IV.42) *satisfies condition $(\mathcal{P})$ in Ω if $\Re L \wedge \Im L$ does not change sign on any two-dimensional orbit of L, i.e., on any two-dimensional orbit of the pair of real vector fields $\{\Re L, \Im L\}$.*

The previous discussion shows that the coordinate-free Definition IV.2.1 reduces to Definition IV.1.5 when L is in the form (IV.41).

Let $\varphi(x, t) \in C^\infty(\mathbb{R}^2)$, set

$$Z(x, t) = x + i\varphi(x, t), \tag{IV.43}$$

and consider the vector field

$$L = \frac{\partial}{\partial t} - \frac{i\varphi_t(x,t)}{1+i\varphi_x(x,t)}\frac{\partial}{\partial x} = \frac{\partial}{\partial t} - \frac{Z_t}{Z_x}\frac{\partial}{\partial x}. \tag{IV.44}$$

Thus, $Z(x,t)$ is a global first integral of L, i.e., $LZ = 0$ and $\mathrm{d}Z \neq 0$ everywhere.

LEMMA IV.2.2. *Let $Z(x,t)$ and L be given by* (IV.43) *and* (IV.44) *respectively. Then, L satisfies $(\mathcal{P})$ in $\mathbb{R}^2$ if and only if $\mathbb{R} \ni t \mapsto \varphi(x,t)$ is monotone for every $x \in \mathbb{R}$.*

PROOF. We have

$$X = \frac{\partial}{\partial t} + \frac{\varphi_t\varphi_x}{1+\varphi_x^2}\frac{\partial}{\partial x}, \qquad Y = -\frac{\varphi_t}{1+\varphi_x^2}\frac{\partial}{\partial x},$$

so

$$X \wedge Y = \frac{\varphi_t(x,y)}{1+\varphi_x^2}\,\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial t}.$$

Note that X and Y are linearly dependent at a point if and only if φ_t vanishes at that point. Thus, the one-dimensional orbits of L are vertical lines $x = \text{constant}$ on which φ_t vanishes identically. Since the two-dimensional orbits of L are bounded by 0, 1 or 2 one-dimensional orbits we see that each two-dimensional orbit Ω_j, $j = 1, 2, \ldots$, is of the form $(a_j, b_j) \times \mathbb{R}$. If L satisfies $(\mathcal{P})$ then φ_t does not assume opposite signs on Ω_j, say, $\varphi_t \geq 0$ on Ω_j so $t \mapsto \varphi(x,t)$ is monotone increasing for all $a_j < x < b_j$. If $x \notin (a_j, b_j)$ for any j it follows that the point of coordinates $(x, 0)$ belongs to a one-dimensional orbit, so $\varphi_t(x,t) = 0$, $-\infty < t < \infty$, and $t \mapsto \varphi(x,t)$ is constant. This shows that $t \mapsto \varphi(x,t)$ is monotone for every $x \in \mathbb{R}$. Conversely, assume that $t \mapsto \varphi(x,t)$ is monotone for every $x \in \mathbb{R}$ and let $(a_j, b_j) \times \mathbb{R}$ be a two-dimensional orbit. Given $x_0 \in (a_j, b_j)$ we have that $t \mapsto \varphi_t(x_0, t)$ has a consistent sign, say $\varphi_t(x_0, t) \geq 0$. We must show that $\varphi_t(x,t) \geq 0$ for all $a_j < x < b_j$. Indeed, if $\varphi_t(x_1, t) < 0$ for some $x_1 \in (a_j, b_j)$ and $t \in \mathbb{R}$, it is easy to see that there exist an intermediate point x_2 between x_0 and x_1 such that $\varphi_t(x_2, t) = 0$ for all $t \in \mathbb{R}$. Then $\{x_2\} \times \mathbb{R}$ is a one-dimensional orbit and must be disjoint of the two-dimensional orbit $(a_j, b_j) \times \mathbb{R}$, a contradiction to the fact that $x_2 \in (a_j, b_j)$. □

From now on, we assume that L given by (IV.44) satisfies condition $(\mathcal{P})$ and we wish to find a local solution $Lu = f$ with $u \in C^\infty$ when $f \in C^\infty$. We start from estimate (IV.11) in Theorem IV.1.9, with L in the place of ${}^tL$, $q = p = 2$. There exists $a, T, C > 0$ such that, for every $u \in C_c^\infty((-a,a) \times (-T,T))$,

$$\|u(x,t)\|_{L^2(\mathbb{R}^2)} \leq C\|Lu(x,t)\|_{L^2(\mathbb{R}^2)}. \tag{IV.45}$$

Modifying $\varphi(x,t)$ outside a neighborhood of the origin as in the proof of Theorem IV.1.9, we may assume that φ_t and φ_x are compactly supported and that $a=\infty$. The a priori estimate (IV.45) may be extended using Friedrichs' lemma to any $u\in L^2_c((-T,T)\times\mathbb{R})$ such that $Lu\in L^2_c((-T,T)\times\mathbb{R})$.

We wish to extend (IV.45) in two ways: first, we want to know that the inequality is still valid when $u(x,t)$ is not regular enough to be in $L^2(\mathbb{R}^2)$ although $Lu(x,t)$ is known be in $L^2(\mathbb{R}^2)$; second, we wish to consider estimates for Sobolev norms. We write

$$M=Z_x^{-1}\partial_x,\qquad D=-L^2-\lambda M^2,$$

where $\lambda>0$ is a large parameter. Then L and M commute, which implies that L and D also do so. A consequence of this fact that can be expressed in terms of their respective symbols $\ell(x,t,\xi,\tau)=i(\tau+\Lambda(x,t)\xi)$, $\Lambda=-Z_t/Z_x$, $d(x,t,\xi,\tau)=-(\ell^2+\lambda m^2)(x,t,\xi,\tau)$, $m(x,t,\xi,\tau)=iZ_x^{-1}(x,t)\xi$, is expressed by the identity

$$\{\ell,d\}(x,t,\xi,\tau)=0,\qquad (x,t,\xi,\tau)\in\mathbb{R}^4$$

where $\{\ell,d\}$ denotes the Poisson bracket performed in all variables. Note that

$$d(x,t,\xi,\tau)=\tau^2-2\frac{Z_t}{Z_x}\xi\tau+\frac{Z_t^2+\lambda}{Z_x^2}\xi^2$$

so for λ large $|\Im d|\le C\,\Re d$ and also $d(x,t,\xi,\tau)=0$ implies $\tau=\xi=0$, i.e., D is a uniformly elliptic second-order operator with smooth bounded coefficients.

Consider a pseudo-differential operator $P(x,t,D_x,D_t)$ of order s and type $(\rho,\delta)=(1,0)$ with symbol $p(x,t,\xi,\tau)$, that is,

$$Pu(x,t)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\mathrm{e}^{i(x\xi+t\tau)}\,p(x,t,\xi,\tau)\,\widehat{u}(\xi,\tau)\,\mathrm{d}\xi\mathrm{d}\tau.$$

The first term in the expansion of the symbol of the commutator $[L,P]$ is given by $-i\{\ell,p\}(x,t,\xi)$ by a well-known formula from the calculus of pseudo-differential operators. Thus, $[L,P]$ is a pseudo-differential operator with the same order s. However, if $p(x,t,\xi)=F(d(x,t,\xi))$ with F holomorphic on the range of d, it follows that

$$\{\ell,p\}(x,t,\xi,\tau)=\{\ell,F\circ d\}(x,t,\xi,\tau)=(F'\circ d)\,\{\ell,d\}(x,t,\xi,\tau)=0.$$

We see that in this case $[L,P]$ has order $s-1$, i.e., it commutes with L to a higher degree than in the general situation, a fact we will explore. We already saw that the range of $d(x,t,\xi,\tau)$ is contained in a closed cone of the complex plane of the form $|\Im z|\le C\,\Re z$ and it follows that for any real $\epsilon>0$ the range of $1+\epsilon d(x,t,\xi,\tau)$ has positive real part. Consider the pseudo-differential operator $P^\epsilon(x,t,D_x,D_t)$ with symbol

$$p^\epsilon(x,t,\xi,\tau)=\frac{\chi(t)}{(1+\epsilon d(x,t,\xi,\tau))^{1/2}},$$

where $\chi(t)\in C_c^\infty(-T,T)$ and $\chi(t)=1$ for $|t|\leq(3/4)T$. We point out that $P^\epsilon(x,t,D_x,D_t)$ has order -1 for $\epsilon>0$ although $\{p^\epsilon\}$ is not a bounded subset of $S^{-1}_{1,0}$. On the other hand, $\{p^\epsilon\}$, $0<\epsilon<1$, remains in a bounded subset of $S^0_{1,0}$ which implies that the norm of P^ϵ in $\mathcal{L}(L^2(\mathbb{R}^2))$ is bounded by a constant independent of $0<\epsilon<1$, $t\in\mathbb{R}$. By the observations made before, the commutator $[L,P^\epsilon]$ has order -2 for fixed $\epsilon>0$ on the open set $\mathbb{R}\times(-3T/4,3T/4)$ and order -1 *uniformly in* $\epsilon>0$, which implies that $\{[L,P^\epsilon]\}$ is a bounded subset of $\mathcal{L}(L^2(\mathbb{R}^2),H^{-1}(\mathbb{R}^2))$, where H^{-1} denotes the Sobolev space of order -1. Furthermore, $P^\epsilon\to I$ weakly as $\epsilon\to 0$.

Consider now a distribution $u(x,t)\in H_c^{-1}(\mathbb{R}^2)$ supported in $\mathbb{R}\times(-T/2,T/2)$ and assume that

- $Lu\in L^2(\mathbb{R}^2)$.

We will show that $u\in L^2(\mathbb{R}^2)$. Indeed, set $u_\epsilon=P^\epsilon u$. Then, $u_\epsilon\in L^2(\mathbb{R}^2)$ and $Lu_\epsilon=P^\epsilon Lu+[L,P^\epsilon]u\in L^2(\mathbb{R}^2)$. Note that the last inclusion is uniform in ϵ and that $[L,P^\epsilon]u\to 0$ in L^2. Applying (IV.45) to u_ϵ we obtain

$$\|u_\epsilon\|_{L^2(\mathbb{R}^2)}\leq C\|Lu_\epsilon\|_{L^2(\mathbb{R}^2)}\leq C_1.$$

Since $u_\epsilon\to u$ weakly as $\epsilon\to 0$ we conclude that $\|u\|_{L^2(\mathbb{R}^2)}\leq C_1$ and

$$\|u\|_{L^2(\mathbb{R}^2)}\leq C\|Lu\|_{L^2(\mathbb{R}^2)}$$

for all $u\in H_c^{-1}(\mathbb{R}\times(-T/2,T/2))$ such that $Lu\in L^2(\mathbb{R}^2)$. Similarly, if $u\in H_c^{s-1}(\mathbb{R}\times(-T/2,T/2))$, $s\in\mathbb{R}$, is such that $Lu\in H_c^s(\mathbb{R}\times(-T/2,T/2))$ we conclude that $u\in H^s(\mathbb{R}^2)$ and

$$\|u\|_{H^s(\mathbb{R}^2)}\leq C_s(\|Lu\|_{H^s(\mathbb{R}^2)}+\|u\|_{H^{s-1}(\mathbb{R}^2)}). \tag{IV.46}$$

To prove (IV.46) we apply (IV.45) to $u_\epsilon=B^\epsilon u$ where B^ϵ is the pseudo-differential operator with symbol

$$b^\epsilon(x,t,\xi,\tau)=\frac{\chi(t)(1+d(x,t,\xi,\tau))^{s/2}}{(1+\epsilon d(x,t,\xi,\tau))^{1/2}}$$

and reason as before. Note that $b^\epsilon\to b=\chi(1+d)^{s/2}$ in the symbol space $S^s_{1,0}$ and that $\|u\|_s\sim\|Bu\|_{L^2}$ if B is the pseudo-differential with symbol

b and $u \in H^s_c(\mathbb{R} \times (-T/2, T/2))$. Furthermore, $[L, B]$ has order $s-1$ on $\mathbb{R} \times (-T/2, t/2)$. Letting $\epsilon \to 0$ we obtain

$$\|Bu\|_{L^2(\mathbb{R}^2)} \le C(\|BLu\|_{L^2(\mathbb{R}^2)} + \|[L, B]u\|_{L^2(\mathbb{R}^2)}),$$

which gives (46). A consequence of (IV.46) is that

$$u \in \mathcal{E}'(\mathbb{R} \times (-T/2, T/2)) \quad \text{and} \quad Lu \in H^s(\mathbb{R}^2)$$

$$\text{imply that} \quad u \in H^s(\mathbb{R}^2).$$

Indeed, if $u \in \mathcal{E}'(\mathbb{R} \times (-T/2, T/2))$ there exists some $\sigma < s$ such that $s - \sigma = k$ is an integer and $u \in H^\sigma_c(\mathbb{R} \times (-T/2, T/2))$. Then $Lu \in H^s(\mathbb{R}^2) \subset H^{s-k}(\mathbb{R}^2)$ and (IV.46) implies that $u \in H^{s-k+1}(\mathbb{R}^2)$. Repeating this process k times we conclude that $u \in H^s(\mathbb{R}^2)$ as wanted. Observe that this implies that $u \in \mathcal{E}'(\mathbb{R} \times (-T/2, T/2))$ must be smooth if $Lu \in C^\infty$.

Another consequence is that if $u \in \mathcal{E}'(\mathbb{R} \times (-T/2, T/2))$ satisfies $Lu = 0$ it must vanish identically (a fact that also follows from uniqueness in the Cauchy problem). Indeed, $Lu = 0$ implies that $u \in C^\infty_c(\mathbb{R} \times (-T/2, T/2))$ and (IV.45) shows that $u = 0$.

Let K denote a closed ball of radius $r < T/2$ centered at the origin of $\mathbb{R}^2$ and let us prove that for any $s \in \mathbb{R}$

$$\|u\|_{H^s(\mathbb{R}^2)} \le C(s)\|Lu\|_{H^s(\mathbb{R}^2)}, \quad u \in C^\infty_c(K). \tag{IV.47}$$

Fix $s \in \mathbb{R}$ and assume by contradiction that for every $j = 1, 2, \ldots,$ there exists $u_j \in C^\infty_c(K)$ such that $\|u_j\|_{H^s(\mathbb{R}^2)} = 1$ and $\|Lu_j\|_{H^s(\mathbb{R}^2)} \le 1/j$. Passing through a subsequence we may assume that $u_j \to u$ in $H^{s-1}(\mathbb{R}^2)$ with $Lu = 0$ and this implies that $u = 0$. On the other hand, (IV.46) gives

$$1 \le \frac{C_s}{j} + C_s \|u_j\|_{H^{s-1}(\mathbb{R}^2)}$$

which, letting $j \to \infty$, contradicts that $u = 0$. Using Friedrichs' lemma we may extend (IV.47) to

$$\|u\|_{H^s(\mathbb{R}^2)} \le C(s)\|Lu\|_{H^s(\mathbb{R}^2)}, \quad \text{if } u \text{ and } Lu \in H^s_c(K). \tag{IV.48}$$

Let us now prove that for every $f \in C^\infty(\mathbb{R}^2)$ there is $u \in C^\infty(\mathbb{R}^2)$ such that $Lu = f$ in K. Denote by $C^\infty(K)$ the quotient of $C^\infty(\mathbb{R}^2)$ by the subspace of those functions which vanish on K to infinite order. This is a Fréchet space and its dual may be identified with $\mathcal{E}'(K)$, the distributions in $\mathcal{E}'(\mathbb{R}^2)$ supported in K.

In order to identify the dual of $C^\infty(K)$ with $\mathcal{E}'(K)$ it is convenient to introduce the pairing

$$\begin{aligned}\langle u(x,t), v(x,t)\rangle &= \int u(x,t)\, v(x,t)\, \mathrm{d}Z(x,t)\wedge \mathrm{d}t\\ &= \int u(x,t)\, v(x,t)\, Z_x(x,t)\, \mathrm{d}x\, \mathrm{d}t,\end{aligned}$$

for which L and $-L$ are formal transposes of each other, i.e., $\langle Lu, v\rangle = -\langle u, Lv\rangle$, $u, v \in C_c^\infty(\mathbb{R}^2)$. This pairing can be extended to $u \in C^\infty(\mathbb{R}^2)$ and $v \in \mathcal{E}'(\mathbb{R}^2)$ and if $v \in \mathcal{E}'(K)$ the value of $\langle Lu, v\rangle$ only depends on the residue class $[u]$ of $u \in C^\infty(\mathbb{R}^2)$ in $C^\infty(K)$ and $[u] \mapsto \langle u, v\rangle$ is clearly continuous. Conversely, given a continuous linear functional λ on $C^\infty(K)$, the continuous linear functional $C^\infty(\mathbb{R}^2) \ni u \mapsto \lambda([u])$ is represented by a compactly supported distribution $v \in \mathcal{E}'(\mathbb{R}^2)$ such that $\lambda([u]) = \langle u, v\rangle$, $u \in C^\infty(\mathbb{R}^2)$. Since $\langle u, v\rangle$ must vanish when u vanishes to infinite order on K we see that v is supported in K. Furthermore, it is clear that $v = 0$ if $\lambda = 0$.

Consider the continuous linear map $T : C^\infty(K) \longrightarrow C^\infty(K)$ defined by $T[u] = [Lu]$, where $[u]$ denotes the residue class of $u \in C^\infty(\mathbb{R}^2)$ in $C^\infty(K)$. Then the range of T is dense; in fact, if μ is a continuous linear functional on $C^\infty(K)$ such that $\langle \mu, T[u]\rangle = 0$, $[u] \in C^\infty(K)$, regarded as an element of $\mathcal{E}'(K)$, μ satisfies the equation $L\mu = 0$ which implies that $\mu = 0$. Thus, to show that T is onto we need only show that the range of T is closed and by the Banach closed range theorem for Fréchet spaces this will follow if we prove that the range of the dual operator T' is closed for the weak$*$ topology. However, $C^\infty(K)$ is reflexive, a consequence of the reflexivity of $C^\infty(\mathbb{R}^2)$, and in this case it is enough to prove that the range of T' is closed for the strong topology (see, e.g., [**T1**], chapter 37). Let the sequence $\mu_j = T'\nu_j = -L\nu_j$, $\nu_j \in \mathcal{E}'(K)$, converge to $\mu \in \mathcal{E}'(K)$. There exist s such that $\{\mu_j\} \subset H^s(\mathbb{R}^2)$ and $\|\mu_j\|_{H^s} \le C$, $j = 1, 2, \ldots$ This implies that $\nu_j \in H^s(\mathbb{R}^2)$ and by (IV.48)

$$\|\nu_j\|_{H^s} \le C(s)\|\mu_j\|_{H^s} \le C'.$$

Passing through a subsequence we may assume that ν_j is convergent in $H^{s-1}(\mathbb{R}^2)$ to some $\nu \in H_c^{s-1}(K)$, showing that $T'\nu = -L\nu = \mu$ so μ is in the range of T'. Thus, the range of T' is closed and so is the range of T, which must be equal to $C^\infty(K)$. In other words, for every $f \in C^\infty(\mathbb{R})$ there is $u \in C^\infty(\mathbb{R})$ such that $Lu - f = 0$ on K. Finally, if $c(x,t)$ is a smooth function we see that we may smoothly solve $Lu + cu = f$ in K. If v, w are smooth, $L(\mathrm{e}^{-v}w) = \mathrm{e}^{-v}(Lw - wLv)$. If we choose $v \in C^\infty$ such that $Lv = c$ on K and then take $w \in C^\infty$ such that $Lw = \mathrm{e}^{v} f$ on K, we see that $u = \mathrm{e}^{-v}w$ satisfies $Lu + cu = f$ on K.

Most of the results we have proved so far in this section are summed up in the following:

Theorem IV.2.3. *Assume that L is a smooth vector field defined in an open subset Ω of the plane and let $c(x,t) \in C^\infty(\Omega)$. If L satisfies $(\mathcal{P})$ in Ω and it is locally integrable then every point $p \in \Omega$ has a neighborhood U such that the equation*

$$Lu + cu = f, \quad f \in C_c^\infty(U)$$

may be solved with $u \in C^\infty(U)$. Conversely, if L is locally solvable in C^∞ then L is locally integrable.

Proof. Only the converse part has not been proved already, and we prove it now. Assume that

$$Lu = A(x,t)\frac{\partial u}{\partial t} + B(x,t)\frac{\partial u}{\partial x}$$

with complex coefficients $A, B \in C^\infty(\Omega)$ such that

$$|A(x,t)| + |B(x,t)| > 0, \quad (x,t) \in \Omega$$

is locally solvable in C^∞. Given a point $p \in \Omega$, that we may as well assume to be the origin, we wish to prove the existence of a smooth function Z, defined in a neighborhood of the origin, such that $LZ = 0$ and $\mathrm{d}Z \neq 0$. Set

$$d(x,t) = \frac{\partial A(x,t)}{\partial t} + \frac{\partial B(x,t)}{\partial x}$$

and find $u \in C^\infty(\Omega)$ such that $Lu = d$ in a rectangle U centered at the origin. Then the 1-form

$$\omega = B(x,t)\mathrm{e}^{-u(x,t)}\,\mathrm{d}t - A(x,t)\mathrm{e}^{-u(x,t)}\,\mathrm{d}x$$

is closed, since

$$\frac{\partial (B\mathrm{e}^{-u})}{\partial x} + \frac{\partial (A\mathrm{e}^{-u})}{\partial t} = \mathrm{e}^{-u}d - \mathrm{e}^{-u}Lu = 0 \quad \text{in } U.$$

Furthermore, ω does not vanish. Since U is simply connected, there exists $Z \in C^\infty(U)$ such that $\mathrm{d}Z = \omega$. So $\mathrm{d}Z \neq 0$ in U and also $LZ = \langle L, \omega\rangle = \mathrm{e}^{-u}\langle A\partial_t + B\partial_x, B\mathrm{d}t - A\mathrm{d}x\rangle = 0$. □

Remark IV.2.4. The assumption in Theorem IV.2.3 that L is locally integrable simplified the construction of smooth solutions but a much more general result is known. In fact, condition $(\mathcal{P})$ alone, formulated in the appropriate way, implies smooth local solvability for operators of principal type of arbitrary order ([**H5**]).

IV.3 Vector fields in several variables

We consider vector fields defined in an open subset $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, that contains the origin,

$$Lu = A(x,t)\frac{\partial u}{\partial t} + \sum_{j=1}^{n} B_j(x,t)\frac{\partial u}{\partial x_j} \tag{IV.49}$$

with complex coefficients $A, B_1, \ldots, B_n \in C^\infty(\Omega)$ such that

$$|A(x,t)| + \sum_{j=1}^{n} |B_j(x,t)| > 0, \quad (x,t) \in \Omega. \tag{IV.50}$$

As in the case $n = 1$ discussed in Section IV.1, we may assume locally that $A = 1$ and then apply a several-variables analogue of Lemma IV.1.1, namely

LEMMA IV.3.1. *In appropriate new local coordinates $x = (x_1, \ldots, x_n)$, t, defined in a neighborhood of the origin, the vector field L assumes the form*

$$Lu = \frac{\partial u}{\partial t} + i\sum_{j=1}^{n} b_j(x,t)\frac{\partial u}{\partial x_j}, \tag{IV.51}$$

with $b_j(x,s)$ real-valued.

As before, it is useful to write $L = X + iY$ with $X = \Re L$ and $Y = \Im L$ and to refer to the orbits of the pair of real vector fields $\{X, Y\}$ as the orbits of L. Note that since X and Y do not vanish simultaneously then L cannot have any orbits of dimension zero. Let Σ be an orbit of L of dimension two and assume that Σ is orientable. There exists a global nonvanishing section $\rho \in C^\infty(\Sigma; \bigwedge^2(T(\Sigma)))$. Both X and Y are tangent to Σ so they may be considered as sections of the tangent bundle $T(\Sigma) \longrightarrow \Sigma$ that produce a section $X \wedge Y$ of the bundle $\bigwedge^2 T(\Sigma) \longrightarrow \Sigma$. Then $X \wedge Y = b\rho$, where b is a smooth real function defined on Σ. If the real function b does not assume opposite signs on Σ we say that $X \wedge Y$ does not change sign on Σ. Note that if ρ_1 is another nonvanishing section of $\bigwedge^2 T(\Sigma) \longrightarrow \Sigma$ then $\rho_1 = \lambda\rho$ with a smooth real $\lambda \neq 0$ and since Σ is connected either $\lambda > 0$ or $\lambda < 0$. This shows that the notion '$X \wedge Y$ does not change sign on Σ' is independent of the generator ρ.

DEFINITION IV.3.2. *We say that the operator L given by* (IV.49) *satisfies condition $(\mathcal{P})$ in Ω if and only if*

(1) *the orbits of L in Ω have dimension at most two;*
(2) *the orbits of L of dimension two are orientable and $\Re L \wedge \Im L$ does not change sign on any two-dimensional orbit of L.*

It is clear that the above definition is coordinate-free. We will now see that it is invariant under multiplication by a nonvanishing factor.

PROPOSITION IV.3.3. *Let L given by* (IV.49) *satisfy condition* $(\mathcal{P})$ *in* Ω *and let* $h \in C^\infty(\Omega)$ *be a complex nonvanishing function. Then* $L' = hL$ *satisfies* $(\mathcal{P})$ *in* Ω.

PROOF. Write $h = \alpha + i\beta$ with $\alpha, \beta \in C^\infty(\Omega)$ real. Then, $L' = X' + iY'$ with $X' = \alpha X - \beta Y$ and $Y' = \alpha Y + \beta X$. The orbits of L and L' are identical because both L and L' generate the same bundle, so L' has no orbits of dimension higher than two. Let Σ be an orbit of L' of dimension two. Since Σ is also an orbit of L, $X \wedge Y$ does not change sign on Σ and it follows that $X' \wedge Y' = (\alpha^2 + \beta^2) X \wedge Y$ does not change sign on Σ either. $\square$

If L is written in special coordinates in which it has the form (IV.51), condition $(\mathcal{P})$ may be expressed in a more concrete way that extends Definition IV.1.5.

PROPOSITION IV.3.4. *Let L be given by* (IV.51) *in* $\Omega = \{|x| < r\} \times (-T, T)$. *Then L satisfies* $(\mathcal{P})$ *in* Ω *if and only if the following holds:*

for every $x = (x_1, \ldots, x_n) \in \{|x| < r\}$ *and* $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$,

$$\text{the function } (-T, T) \ni t \mapsto \sum_{j=1}^{n} b_j(x, t)\xi_j \text{ does not change sign.} \tag{IV.52}$$

PROOF. We begin by showing that if L is given by (IV.51) in Ω the orbits of L of dimensions one and two have a simple description. Since $X = \partial_t$ the orbits of X in Ω are the vertical segments $\{x_0\} \times (-T, T)$. Thus, if (x_0, t_0) belongs to an orbit Σ it follows that $\{x_0\} \times (-T, T) \subset \Sigma$ and this implies that every orbit of L of any dimension may be written as a union of vertical segments. If Σ is a one-dimensional orbit, X and Y are linearly dependent at every point of Σ so $Y = \sum_j b_j \partial_{x_j}$ must vanish identically on Σ, leading to the conclusion that $\Sigma = \{x_0\} \times (-T, T)$ for some $x_0 \in \{|x| < r\}$ such that $b_j(x_0, t) = 0$ for all $1 \le j \le n$, $|t| < T$. Conversely, if $b_j(x_0, t) = 0$ for all $1 \le j \le n$, $|t| < T$ then $\{x_0\} \times (-T, T)$ is a one-dimensional orbit.

We may write $Y = (b_1, \ldots, b_n)$ and denote by $Y \cdot \xi$ the inner product in $\mathbb{R}^n$ of Y and $\xi = (\xi_1, \ldots, \xi_n)$. With this notation (IV.52) states that $t \mapsto Y(x, t) \cdot \xi$ does not change sign.

If Σ is an orbit of dimension ≥ 2 that contains the point (x_0, t_0) there must be a point $(x_0, t_1) \in \Sigma$ such that $Y(x_0, t_1) \ne 0$ for otherwise $(x_0, t_0) \times (-T, T)$ would be a one-dimensional orbit intersecting Σ, which is not possible. Consider the maximal integral curve γ in $\{|x| < r\}$ through the point x_0

of the vector field $Y(x, t_1)$, $x \in \{|x| < r\}$. Then $\gamma \times (-T, T)$ is a closed subset of Σ which is also a two-dimensional manifold. Thus, if the dimension of Σ is two we conclude by connectedness that $\Sigma = \gamma \times (-T, T)$, in particular every two-dimensional orbit of L is orientable. Observe that $Y(\cdot, t_1)$ does not vanish in γ (otherwise γ would reduce to a single point) and set $\mathbf{v}(x) = Y(x, t_1)$. Then $\rho = \mathbf{v} \wedge \partial_t \in \bigwedge^2(\Sigma)$ never vanishes.

Assume now that L satisfies $(\mathcal{P})$ and we wish to prove (IV.52) for some x_0 and ξ fixed. If (x_0, t_0) belongs to a one-dimensional orbit for some $t_0 \in (-T, T)$, then $Y(x_0, t) = 0$ for $|t| < T$ and obviously $t \mapsto Y(x_0, t) \cdot \xi$ cannot change sign. Hence we may assume that $Y(x_0, t_0) \neq 0$ for some $t_0 \in (-T, T)$, so $(x_0, t_0) \in \Sigma$ where Σ is an orbit of L of dimension two on which $X \wedge Y$ does not change sign. Let γ be the integral curve of $\mathbf{v}(x) = Y(x, t_0)$ in $\{|x| < r\}$ through the point x_0. Then $\Sigma = \gamma \times (-T, T)$ and $\rho = \mathbf{v} \wedge \partial_t$ generates $\bigwedge^2(\Sigma)$ at every point of Σ. Let $(x_0, t) \in \Sigma$. Since Y is a horizontal vector tangent to $\gamma \times (-T, T)$ we see that $Y(x_0, t) = \lambda(x_0, t)\mathbf{v}(x_0)$. Furthermore, $X \wedge Y(x_0, t) = \partial_t \wedge \lambda(x_0, t)\mathbf{v}(x_0) = \lambda(x_0, t)\rho(x_0, t)$, so either $\lambda(x_0, t) \geq 0$ on $(-T, T)$ or $\lambda(x_0, t) \leq 0$ on $(-T, T)$. This proves that the vector-valued map $(-T, T) \ni t \mapsto Y(x_0, t)$ does not change direction and $t \mapsto Y(x_0, t) \cdot \xi$ does not change sign for any $\xi \in \mathbb{R}^n$ and $|x_0| < r$.

Conversely, let us prove that (IV.52) implies condition $(\mathcal{P})$. Fix a point $(x_0, t_0) \in \{|x| < r\} \times (-T, T)$ and assume that it belongs to an orbit Σ of dimension ≥ 2. If $Y(x_0, t) = 0$ for all $|t| < T$ then the dimension of Σ would be one, so changing t_0 we may as well assume that $Y(x_0, t_0) \neq 0$. Let γ be the integral curve through x_0 of the vector field $\mathbf{v}(x) = Y(x, t_0)$ in $\{|x| < r\}$. Then, for every $x \in \gamma$, $Y(x, t) = \lambda(t, x)\mathbf{v}(x)$ with $\lambda \geq 0$. Indeed, if for some $x \in \gamma$ and $t_1 \in (-T, T)$ the vectors $Y(x, t_1)$ and $\mathbf{v}(x)$ were not parallel or were parallel but pointing in opposite directions, they would lie on different half-spaces determined by a hyperplane $\{\eta : \ \eta \cdot \xi = 0\}$, i.e., $Y(x, t_0) \cdot \xi$ and $Y(x, t_1) \cdot \xi$ would have opposite signs, contradicting (IV.52). In particular, this shows that both X and Y are tangent to $\gamma \times (-T, T)$, which makes $\gamma \times (-T, T)$ invariant under the flow of X and Y. This shows that $\Sigma \subset \gamma \times (-T, T)$ and since the orbit has dimension ≥ 2 and the latter set is connected we conclude that $\Sigma = \gamma \times (-T, T)$, which shows that there are no orbits of dimension > 2. Also, $X \wedge Y(x, t) = \lambda(x, t)\partial_t \wedge \mathbf{v}(x)$, $(x, t) \in \Sigma$, so $X \wedge Y$ does not change sign on Σ. $\square$

We are now able to extend Theorem IV.1.9 to any number of variables.

THEOREM IV.3.5. *Let L given by* (IV.49) *satisfy* (IV.50) *and condition* $(\mathcal{P})$ *in a neighborhood of the origin and fix $1 < p < \infty$. Then, there exist a* *neighborhood U of the origin and a constant $C > 0$ such that the following a priori estimate holds for every $\varphi \in C_c^\infty(U)$:*

$$\|\varphi\|_{L^p(\mathbb{R}^{n+1})} \leq C \operatorname{diam}(\operatorname{supp}\varphi)\, \|L\varphi\|_{L^p(\mathbb{R}^{n+1})}. \tag{IV.53}$$

Moreover, the constant C depends only on p and the L^∞ norms of the derivatives of order at most two of the coefficients of L. Furthermore, a similar inequality holds with ${}^tL$ in the place of L.

Proof. The proof of this theorem requires six steps. Since Theorem IV.3.5 follows from Theorem IV.1.9 when $n = 1$, we will assume in the proof that $n \geq 2$.

The first step. Renaming coordinates if necessary we may assume that $A(0,0) \neq 0$. Then, dividing by A in a neighborhood of the origin and applying Lemma IV.3.1 we put L in the form (IV.51). The new vector field thus obtained still satisfies condition $(\mathcal{P})$ by its invariance under multiplication by nonvanishing factors and change of coordinates. If φ is a test function supported in a small neighborhood of the origin and Φ is the diffeomorphism induced by the change of variables, the L^p norm of φ and the L^p norm of $\varphi \circ \Phi$ are comparable because the Jacobian determinant $\det(\Phi')$ satisfies $c_1 \leq |\det(\Phi')| \leq c_2$ in a neighborhood of the origin for some positive constants c_1, c_2. Note that the derivatives of order k of the coefficients b_j, $j = 1, \ldots, n$, may be estimated in terms of bounds for the derivatives of order up to $k+1$ of the original coefficients $A, B_1, \ldots, B_n$, as one extra derivative is consumed by the change of coordinates. Furthermore, by multiplying the coefficients b_j, $j = 1, \ldots, n$, by a non-negative cut-off function equal to 1 on a neighborhood of the origin, we may assume that $b_1, \ldots, b_n \in C_c^\infty(\mathbb{R}^{n+1})$. Hence, it is enough to prove the theorem when L is given by (IV.51) and its coefficients are compactly supported, provided that we prove that the constant C in (IV.53) depends only on p and the L^∞ norms of the derivatives of order at most *one* of the coefficients of L.

The second step. We assume that L is given by (IV.51) and its coefficients are compactly supported, then denote by $\vec{b}(x,t)$ the vector field in $\mathbb{R}^n$ given by $\sum_{j=1}^n b_j(x,t)\partial/\partial x_j$. In view of Proposition IV.3.4 and its proof, the fact that L verifies $(\mathcal{P})$ implies that there exists a unit vector field $\vec{v}(x)$ defined on $\mathbb{R}^n$ such that

$$\vec{b}(x,t) = |\vec{b}(x,t)|\vec{v}(x), \qquad x \in \mathbb{R}^n, t \in \mathbb{R}.$$

Note that $\vec{v}(x_0)$ may be defined arbitrarily if $\vec{b}(x_0,t) = 0$ for all t. Set

$$\mathbb{N} = \left\{ x \in \mathbb{R}^n : \qquad \vec{b}(x,t) = 0, \quad |t| < 1 \right\} \tag{IV.54}$$

and

$$\rho(x) = \sup_{|t|<1} |\vec{b}(x,t)|, \qquad x \in \mathbb{R}^n,$$

so that $\mathbb{N}$ is precisely the set where $\rho(x)$ vanishes. From now on we use the notations $\Omega = \mathbb{R}^n \times (-1,1)$ and $\Omega_T = \mathbb{R}^n \times (-T,T)$, $0 < T < 1$.

Lemma IV.3.6. *Let χ be the characteristic function of $\mathbb{N}$. Then $L\chi = 0$ in the sense of distributions.*

Proof. Let $\varphi \in C_c^\infty(\Omega)$. Then

$$\begin{aligned}\langle \chi, {}^tL\varphi\rangle &= -\int_{\mathbb{N}\times(-1,1)} \varphi_t + i\sum_{j=1}^n b_j\partial_{x_j}\varphi + i\varphi\sum_{j=1}^n \partial_{x_j} b_j \,\mathrm{d}x\mathrm{d}t \\ &= -\int_{\mathbb{N}}\int_{-1}^1 \varphi_t(x,t)\,\mathrm{d}t\,\mathrm{d}x = 0\end{aligned}$$

where we have used that $\sum_{j=1}^n \partial_{x_j} b_j$ vanishes a.e. on $\mathbb{N}\times(-1,1)$. Indeed, if $(\partial b_j/\partial x_j)(x_0,t_0) \neq 0$ for some $1 \le j \le n$ and $(x_0,t_0) \in \mathbb{N}\times(-1,1)$, by the implicit function theorem there is an $\epsilon > 0$ such that the set $\{x:\ b_j(x,t_0) = 0\} \cap \{|x-x_0| < \epsilon\}$ is a hypersurface. Thus, $\rho(x) > 0$ a.e. in $\{|x-x_0| < \epsilon\}$. This shows that $\{\rho = 0\} \cap \{\sum_j \partial_{x_j} b_j \neq 0\}$ has measure zero. $\square$

In view of Lemma IV.3.6, $[L,\chi] = 0$ so to obtain (IV.53) it is enough to prove separately the inequalities

$$\|\chi\varphi\|_{L^p(\mathbb{R}^{n+1})]} \le CT\|L\chi\varphi\|_{L^p(\mathbb{R}^{n+1})]}, \quad \varphi \in C_c^\infty(\Omega_T), \tag{IV.55}$$

$$\|(1-\chi)\varphi\|_{L^p(\mathbb{R}^{n+1})} \le CT\|L(1-\chi)\varphi\|_{L^p(\mathbb{R}^{n+1})}, \quad \varphi \in C_c^\infty(\Omega_T). \tag{IV.56}$$

The third step. We prove inequality (IV.55). The proof of (IV.55) is easy because $L\chi\varphi = \chi L\varphi = \chi\varphi_t$, so

$$\chi(x)\varphi(x,t) = \int_{-T}^t L(\chi\varphi)(x,s)\,\mathrm{d}s.$$

Hence,

$$\begin{aligned}\|\chi(\cdot)\varphi(\cdot,t)\|_{L^p(\mathbb{R}^n)} &\le \int_{-T}^t \|L(\chi\varphi)(\cdot,s)\|_{L^p(\mathbb{R}^n)}\,\mathrm{d}s \\ &\le (2T)^{1/p'}\|L(1-\chi)\varphi\|_{L^p(\mathbb{R}^{n+1})},\end{aligned}$$

with $p'^{-1} + p^{-1} = 1$. Raising both sides to the power p and integrating with respect to t between $-T$ and T we obtain (IV.55) with $C = 2$.

The fourth step. We introduce a partition of unity that reduces the proof of inequality (IV.56) to the proof of local estimates for test functions. Note that the function $(1-\chi)\varphi$ is not even continuous which, of course, is a source of trouble. The main idea to overcome this difficulty is to write $(1-\chi)$ as a series of convenient test functions supported in $\Omega\backslash\mathbb{N}$.

We start by proving some lemmas.

LEMMA IV.3.7. *Let $\rho(x)$ and $\mathbb{N}$ be as defined above.*

(1) *The function $\rho(x)$ is Lipschitz and*

$$\|\nabla\rho\|_{L^\infty} \le \|\nabla_x\vec{b}\|_{L^\infty}. \tag{IV.57}$$

(2) *Outside $\mathbb{N}$ the vector $\vec{v}(x)$ is locally Lipschitz and satisfies*

$$|\nabla\vec{v}(x)| \le \frac{2\|\nabla_x\vec{b}\|_{L^\infty}}{\rho(x)} \quad \textit{for } x \notin \mathbb{N}. \tag{IV.58}$$

PROOF. Let $x, y \in \mathbb{R}^n$ and let $t \in [-1, 1]$ such that $\rho(x) = |\vec{b}(x, t)|$. Then

$$\begin{aligned}\rho(x) = |\vec{b}(x,t)| &\le |\vec{b}(y,t)| + |\vec{b}(y,t) - \vec{b}(x,t)| \\ &\le \rho(y) + \|\nabla_x\vec{b}\|_{L^\infty}|x-y|.\end{aligned}$$

This shows that $\rho(x) - \rho(y) \le \|\nabla_x\vec{b}\|_{L^\infty}|x-y|$ and interchanging x and y we are led to $|\rho(x)-\rho(y)| \le \|\nabla_x\vec{b}\|_{L^\infty}|x-y|$ for all $x, y \in \mathbb{R}^n$. This implies (IV.57).

Next, given $x_0 \notin \mathbb{N}$ select $|t| \le 1$ such that $\rho(x_0) = |\vec{b}(x_0, t)| > 0$. Then $|\vec{b}(x,t)|$ is positive and differentiable in a neighborhood of x_0, so

$$|\nabla\vec{v}(x_0)| = \left|\nabla_x \frac{\vec{b}}{|\vec{b}|}(x_0, t)\right| \le \left|\frac{\nabla_x\vec{b}}{|b|} + \vec{b}\otimes\frac{\nabla_x|\vec{b}|}{|\vec{b}|^2}\right| \le \frac{2\|\nabla\vec{b}\|_{L^\infty}}{\rho(x)}$$

where we have used that $|\nabla_x|\vec{b}|| \le |\nabla_x b|$. This proves (IV.58). □

In the sequel, *cube* will mean a closed cube in $\mathbb{R}^n$, with sides parallel to the axes. Two such cubes will be said to be *disjoint* if their interiors are disjoint. If Q is a cube with side length ℓ and $\lambda > 0$ is a positive number, λQ will denote the cube with the same center as Q and side length equal to $\lambda\ell$.

LEMMA IV.3.8. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^+$ be a Lipschitz continuous function with Lipschitz constant $0 < \mu \le 1$, i.e., $|f(x) - f(y)| \le \mu|x-y|$, $x, y \in \mathbb{R}^n$. Assume that $F = f^{-1}\{0\}$ is not empty and set $\mathcal{O} = \{x \in \mathbb{R}^n : f(x) > 0\}$. There exists a collection of cubes $\mathcal{F} = \{Q_1, Q_2, \dots\}$ such that*

(1) $\bigcup_j Q_j = \mathcal{O} = \mathbb{R}^n\backslash F$;
(2) *the $Q_j \in \mathcal{F}$ are mutually disjoint;*
(3) $\operatorname{diam}(Q_j) \le \inf_{Q_j} f(x) \le \sup_{Q_j} f(x) \le 5\operatorname{diam}(Q_j)$.

Proof. Let $\mathcal{Q}_0$ denote the family of cubes with side length one and vertices with integral coordinates. For every integer k we define

$$\mathcal{Q}_k = \{2^{-k}Q : \quad Q \in \mathcal{Q}_0\}$$

so the cubes in $\mathcal{Q}_k$ form a mesh of cubes of side length 2^{-k} and diameter $\sqrt{n}2^{-k}$. Each cube $\in \mathcal{Q}_k$ gives rise to 2^n cubes $\in \mathcal{Q}_{k+1}$ by bisecting the sides. Set for any integer k

$$\mathcal{O}_k = \{x \in \mathbb{R}^n : \quad 2\sqrt{n}2^{-k} < f(x) \le 4\sqrt{n}2^{-k}\}.$$

Note that $\mathcal{O}_k \subset \mathcal{O}$ and $\mathcal{O} = \bigcup_k \mathcal{O}_k$.

We now define

$$\mathcal{F}_0 = \bigcup_k \{Q \in \mathcal{Q}_k : \quad Q \cap \mathcal{O}_k \ne \emptyset\}.$$

Let $Q \in \mathcal{F}_0 \cap \mathcal{Q}_k$. There exists $x \in Q$ such that $2\sqrt{n}2^{-k} < f(x) \le 4\sqrt{n}2^{-k}$. Given $y \in Q$ we have

$$f(x) - \mu|y-x| \le f(y) \le f(x) + \mu|y-x|,$$

so using that $\mu \le 1$ and $|y-x| \le \sqrt{n}2^{-k} = \operatorname{diam}(Q)$ we get

$$\operatorname{diam}(Q) \le \inf_Q f(x) \le \sup_Q f(x) \le 5\operatorname{diam}(Q).$$

Since $f(y) > 0$ on Q it follows that $Q \subset \mathcal{O}$. Also, given $y \in \mathcal{O}$ there exists a unique k such that $y \in \mathcal{O}_k$ and y also belongs to some $Q \in \mathcal{Q}_k$ because $\bigcup\{Q \in \mathcal{Q}_k\} = \mathbb{R}^n$, so $y \in Q$ and $Q \in \mathcal{F}_0$, which shows that $\bigcup\{Q \in \mathcal{F}_0\} = \mathcal{O}$. Thus, the cubes of $\mathcal{F}_0$ satisfy (1) and (3) although they may not be disjoint. To obtain the required collection $\mathcal{F}$ we must discard from $\mathcal{F}_0$ the superfluous cubes, which is easy because if two distinct cubes in $\mathcal{F}_0$ are not disjoint one contains the other. Namely, if $Q_1, Q_2 \in \mathcal{F}_0$ are not disjoint, then $Q_1 \in \mathcal{Q}_{k_1}$ and $Q_2 \in \mathcal{Q}_{k_2}$ with $k_1 \ne k_2$, so if, say, $k_1 > k_2$ it turns out that $Q_1 \subset Q_2$. Hence, if $Q \in \mathcal{F}_0$ is contained in some other cube $Q' \in \mathcal{F}_0$ we discard Q and apply the same procedure to Q', discarding it if it is contained in a bigger cube of $\mathcal{F}_0$ and keeping it in the opposite case. For a fixed cube Q, this process stops after a finite number of steps, otherwise the cubes $Q \subset Q' \subset Q'' \subset \cdots$ would fill $\mathbb{R}^n$, contradicting that $F \ne \emptyset$. Thus, each cube $Q \in \mathcal{F}_0$ is contained in a maximal cube of $\mathcal{F}_0$ and the collection $\mathcal{F}$ of those cubes of $\mathcal{F}_0$ which are maximal satisfies (1), (2), and (3). $\square$

We now need a more detailed discussion of the family $\mathcal{F}$ defined in the previous lemma. Although two distinct cubes Q_1 and $Q_2 \in \mathcal{F}$ are always disjoint in the sense that they have disjoint interior their intersection may be

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nonempty, as they could share a vertex, an edge, or some k-dimensional face, $k < n$. In this case we say that Q_1 and Q_2 *touch*.

Proposition IV.3.9. *If two cubes $Q_1, Q_2 \in \mathcal{F}$ touch, then*

$$\frac{1}{4}\,\mathrm{diam}\,(Q_2) \le \mathrm{diam}\,(Q_1) \le 4\,\mathrm{diam}\,(Q_2).$$

Proof. Let Q_1 and $Q_2 \in \mathcal{F}$ have a common point x in their boundaries and assume without loss of generality that $\mathrm{diam}\,(Q_1) \ge \mathrm{diam}\,(Q_2)$, so their respective sides ℓ_1 and ℓ_2 are related by $\ell_2 = 2^{-k}\ell_1$ for some integer $k \ge 0$. If $z \in Q_2$ we have

$$f(z) \le f(x) + \mu\sqrt{n}\ell_2 \le \sqrt{n}\ell_1(5+2^{-k}) \le 6\sqrt{n}\ell_1,$$

where we have used that Q_1 satisfies (3) of Lemma IV.3.8 to estimate $f(x)$. Now, (3) applied to Q_2 gives $\mathrm{diam}\,(Q_2) \le \sup_{z\in Q_2} f(z) \le 6\,\mathrm{diam}\,(Q_1)$. Since the quotient $\mathrm{diam}\,(Q_2)/\mathrm{diam}\,(Q_1)$ is a power of 2, the latter estimate implies that $\mathrm{diam}\,(Q_2)/\mathrm{diam}\,(Q_1) \le 4$. $\square$

Proposition IV.3.10. *If $Q \in \mathcal{F}$, less than 12^n cubes of $\mathcal{F}$ touch Q.*

Proof. Let $Q \in \mathcal{F}$ have side $\ell = 2^{-k}$. There are exactly $3^n - 1$ cubes in $\mathcal{Q}_k$ that touch Q and each one of them contains at most 4^{n-1} cubes that belong to $\mathcal{Q}_{k+2}$ and touch Q. Since by Proposition IV.3.9 the cubes of $\mathcal{F}$ that touch Q may only have the side lengths ℓ, $\ell/2$, or $\ell/4$ it is easily seen that the total number of cubes of $\mathcal{F}$ that touch Q is $\le (3^n-1)4^{n-1} < 12^n$. $\square$

The family $\mathcal{F}$ that disjointly fills up $\mathcal{O}$ with closed cubes gives rise to a cover by open cubes that has the bounded intersection property. We fix $0 < \varepsilon < 1/4$ and for any $Q \in \mathcal{F}$ denote by Q^* the cube with the same center as Q but with side dilated by the factor $1+\varepsilon$. Let Q_1 and $Q_2 \in \mathcal{F}$ do not touch. We claim that Q_1^* and Q_2 cannot intersect. Indeed, the union of Q_1 with all the cubes of $\mathcal{F}$ that touch Q_1 (among which Q_2 is not) contains, by Proposition IV.3.9, the cube $(5/4)Q_1$ whose interior contains Q_1^*. This shows that $Q_1^* \cap Q_2 = \emptyset$. Consider now a point $x \in \mathcal{O}$ and select $Q \in \mathcal{F}$ such that $x \in Q$. If $x \in Q_j^*$ for some $Q_j \in \mathcal{F}$ then $Q \cap Q_j^* \ne \emptyset$, which implies that Q and Q_j touch. Then Proposition IV.3.10 shows that x belongs to at most 12^n cubes Q_j^*. If $z \in Q^*$ then $f(z) \ge \inf_Q f - \mu\varepsilon\,\mathrm{diam}\,(Q) \ge (3/4)\mathrm{diam}\,(Q) \ge (3/5)\mathrm{diam}\,(Q^*)$. Similarly, $f(z) \le 5\,\mathrm{diam}\,(Q) + \mu\varepsilon\,\mathrm{diam}\,(Q) \le 5\,\mathrm{diam}\,(Q^*)$. Thus, for every $Q \in \mathcal{F}$ we have

$$\frac{1}{2}\mathrm{diam}\,(Q^*) \le \inf_{Q^*} f(x) \le \sup_{Q^*} f(x) \le 5\,\mathrm{diam}\,(Q^*). \tag{IV.59}$$

This estimate implies that $Q^* \subset \mathcal{O}$ and since the interior $\mathrm{Int}\,(Q^*) \supset Q$ we see that $\{\mathrm{Int}\,(Q^*)\}$ is an open cover of $\mathcal{O}$ with the bounded intersection property.

LEMMA IV.3.11. *Let $\mathbb{N} \subset \mathbb{R}^n$ be the closed set defined in* (IV.54) *and let $0 < \mu \leq 1$, $1 < p < \infty$. There exists a covering of $\mathbb{R}^n \backslash \mathbb{N}$ by open cubes with sides parallel to the coordinate axes $\{\mathrm{Int}\,(Q_j^*)\}$, $j = 1, 2, \ldots,$ such that the intersection of 12^n cubes of the family is always empty and for any $j = 1, 2, \ldots,$ we have the estimate:*

$$\frac{1}{2}\mathrm{diam}\,(Q_j^*) \leq \mu \inf_{Q_j^*} \rho(x) \leq \mu \sup_{Q_j^*} \rho(x) \leq 5\,\mathrm{diam}\,(Q_j^*). \tag{IV.60}$$

Furthermore, there are functions $\phi_j \in C_c^\infty(\mathbb{R}^n \backslash \mathbb{N})$ such that $\{\phi_j^p\}$ is a partition of unity in $\mathbb{R}^n \backslash \mathbb{N}$ subordinated to the covering $\{\mathrm{Int}\,(Q_j^*)\}$ and for a certain constant $C > 0$,

$$\|\nabla \phi_j\|_{L^\infty} \leq \frac{C}{\mathrm{diam}\,(Q_j^*)}, \qquad j = 1, 2, \ldots \tag{IV.61}$$

PROOF. From now on we assume without loss of generality that $\|\nabla_x \vec{b}\|_{L^\infty} \leq 1$. We apply Lemma IV.3.8 with $f(x) = \mu\rho(x)$ so $F = \mathbb{N}$. The hypotheses are satisfied because the Lipschitz constant of $\rho(x)$ is 1 by (IV.57) and the complement of $\mathbb{N}$ is bounded so $\mathbb{N} \neq \emptyset$. Thus we obtain the collection $\mathcal{F}$ of disjoint cubes $\{Q_j\}$ which, dilated by the factor $1 + \varepsilon$, yields the associated collection $\{Q_j^*\}$ of cubes whose interiors cover $\mathbb{R}^n \backslash \mathbb{N}$, have the bounded intersection property, and satisfy (IV.59). This proves (IV.60). Fix a function $0 \leq \psi \in C_c^\infty(\mathbb{R}^n)$ supported in $|x| < (1+\varepsilon)/2$ such that $\psi^p(x)$ is smooth and $\psi(x) = 1$ if $|x| \leq 1/2$ (such a function is easily constructed). If $Q_j \in \mathcal{F}$, denote by x_j its center and by ℓ_j its side length. Then $\psi_j(x) = \psi((x - x_j)/\ell_j) \in C_c^\infty(\mathrm{Int}\,(Q_j^*))$ and $\psi_j(x) = 1$ on Q_j. We have

$$\|\nabla \psi_j\|_{L^\infty} \leq \frac{\|\nabla \psi\|_{L^\infty}}{\ell_j} \leq \frac{C}{\mathrm{diam}\,(Q_j^*)}. \tag{IV.62}$$

Note that $\Psi = \sum_j \psi_j^p$ is smooth and ≥ 1 in $\mathbb{R}^n \backslash \mathbb{N}$. Let us estimate $\nabla\Psi(x)$ on the support of ψ_j. If $x \in Q_j^*$ and $\psi_k(x) \neq 0$ for some $k \in \mathbb{Z}^+$ it follows that $Q_j^* \cap Q_k^* \neq \emptyset$. We know that Q_j^* is contained in the union of Q_j with those cubes of $\mathcal{F}$ which touch it and the same can be said about Q_k. This implies that there are cubes $Q_{j'}$ and $Q_{k'}$ in $\mathcal{F}$ such that

(1) $Q_{j'}$ touches Q_j;
(2) $Q_{k'}$ touches Q_k;
(3) $Q_{j'} \cap Q_{k'} \neq \emptyset$ so they either coincide or touch.

Applying Proposition IV.3.9 three times we obtain that $\operatorname{diam}(Q_k) \geq 4^{-3} \operatorname{diam}(Q_j)$ and Proposition IV.3.10 tells us that there are less than $N = 12^{3n}$ integers k such that $Q_j^* \cap Q_k^* \neq \emptyset$. This shows that at most N terms $\psi_k^p(x)$ of the infinite sum that defines $\Psi(x)$ are not zero if $x \in \operatorname{supp}(\psi_j)$. Thus, using the analogue for ψ_k^p of (IV.62) we obtain

$$\sup_{Q_j^*} |\nabla\Psi(x)| \leq \sum_k \sup_{Q_j^*} |\nabla\psi_k^p(x)| \leq \sum_k \frac{C}{\operatorname{diam}(Q_k^*)} \leq \frac{4^3 NC}{\operatorname{diam}(Q_j^*)}. \tag{IV.63}$$

Since

$$|\nabla\Psi^{-1/p}(x)| \leq \frac{1}{p}\|\Psi^{-1-1/p}\|_{L^\infty}|\nabla\Psi(x)| \leq |\nabla\Psi(x)|,$$

because $\Psi \geq 1$, (IV.63) implies

$$\sup_{Q_j^*} |\nabla\Psi^{-1/p}(x)| \leq \frac{C}{\operatorname{diam}(Q_j^*)}. \tag{IV.64}$$

Set

$$\phi_j(x) = \frac{\psi_j(x)}{\Psi^{1/p}(x)}.$$

Then, $\{\phi_j^p\}$ is a partition of unity in $\mathbb{R}^n \backslash \mathbb{N}$ with the required properties. Indeed, to prove (IV.61) we use the Leibniz rule and invoke (IV.62) and (IV.64). $\square$

The fifth step. We prove estimate (IV.56) when $\varphi(x,t)$ is supported in $Q_j^* \times (-T,T)$, $Q_j \in \mathcal{F}$. Assume that φ is supported in $Q_k^* \times (-T,T)$ for a certain cube $\in \mathcal{F}$; the value of $T < 1$ will be chosen momentarily. Since we are assuming that $\|\nabla_x \vec{b}\|_{L^\infty} \leq 1$, (IV.58) yields

$$|\nabla\vec{v}(x)| \leq \frac{2}{\rho(x)} \qquad \text{for } x \notin \mathbb{N}.$$

This shows, in view of (IV.60), that $|\nabla\vec{v}(x)| \leq 4\mu/\operatorname{diam}(Q_j^*)$ on Q_j^*. Furthermore, $\mathbb{R}^n \backslash \mathbb{N}$ is bounded so $\operatorname{diam}(Q_j^*) \leq C$, $j \in \mathbb{Z}$. Hence, $\vec{v}(x)$ is approximately constant on Q_j^* if μ is small; this allows us to rectify its flow as follows. Since $\vec{v}$ is a unit vector, we may assume without loss of generality that at the center x_j of Q_j^* we have $v_1(x_j) \geq 1/\sqrt{n}$. Then, $|v_1(x_j) - v_1(x)| \leq 4\mu < 1/(2\sqrt{n})$ for μ fixed once for all, small but independent of j, and we may assume that $v_1(x) \geq 1/(2\sqrt{n})$ on Q_k^*. Solving the differential equations

$$\frac{\mathrm{d}x_j}{\mathrm{d}y_1} = \frac{v_j(x)}{v_1(x)}, \qquad x_j(0) = y_j, \qquad j = 2, \ldots, n \tag{IV.65}$$

we obtain a change of variables on a neighborhood of Q_k^* given by $x_1 = y_1$, $x_j = x_j(y_1; y_2, \ldots, y_n)$, $1 < j \leq n$, where the right-hand side denotes the solution of (IV.65). In the new coordinates $\vec{v}(x(y)) = v_1(x(y))\partial/\partial y_1$ and L assumes the form

$$\frac{\partial}{\partial t} - ib_1(x(y), t)\frac{\partial}{\partial y_1}$$

with $b_1 > 0$, since $(b_1(x(y), t), 0, \ldots, 0) = \vec{b}(x(y), t)$ implies $b_1(x(y), t) = |\vec{b}(x(y), t)|$. Set $B(y, t) = b_1(x(y), t)$. Then, by the chain rule,

$$\|\nabla_y B\|_{L^\infty} \leq C\|\nabla_x b_1\|_{L^\infty} \leq C'$$

because the Lipschitz constant of the change of variables $y \mapsto x(y)$ is bounded by a constant independent of j, as follows from the fact that the right-hand side of the ODE (IV.65) is bounded by $C\mu$. Now we apply Theorem IV.1.9 with $p = q$ to the vector field

$$L_1 = \frac{\partial}{\partial t} - iB(y, t)\frac{\partial}{\partial y_1} \tag{IV.66}$$

that we regard as a vector field in two variables depending on a parameter $y' = (y_2, \ldots, y_n)$. For some constants C and T_0 whose size only depends on $\|\nabla_x b_1\|_{L^\infty}$ we get for any $0 < T \leq T_0$

$$\|\varphi(\cdot, \cdot, y')\|_{L^p}^p \leq CT\|L_1\varphi(\cdot, \cdot, y')\|_{L^p}^p, \quad \varphi \in C_c^\infty(Q_j^\dagger \times (-T, T)),$$

where the L^p norms are taken in the variables (y_1, t) and the map $y \mapsto x(y)$ takes $Q_j^\dagger$ onto Q_j^*. Integrating this estimate with respect to y' we get

$$\|\varphi\|_{L^p}^p \leq CT\|L_1\varphi\|_{L^p}^p, \quad \varphi \in C_c^\infty(Q_j^\dagger \times (-T, T)).$$

Observing that the absolute value of the Jacobian determinant of $y \mapsto x(y)$ is close to 1 uniformly in $j \in \mathbb{Z}^+$, the latter estimate implies in the original variables (x, t)

$$\|\varphi\|_{L^p}^p \leq CT\|L\varphi\|_{L^p}^p, \quad \varphi \in C_c^\infty(Q_j^* \times (-T, T)), \tag{IV.67}$$

which may be regarded as estimate (IV.56) for $\varphi \in C_c^\infty(Q_j^* \times (-T, T))$.

The sixth step. We prove (IV.56) in general. Let $\varphi \in \Omega_T$ and set $\varphi_j = \phi_j\varphi$ where $\{\phi_j\}$ is the collection of functions described by Lemma IV.3.11. We have

$$(1 - \chi(x))|\varphi(x, t)|^p = \sum_j |\phi_j(1 - \chi(x))\varphi(x, t)|^p.$$

Integrating this identity and taking account of (IV.67),

$$\|(1 - \chi)\varphi\|_{L^p}^p = \sum_j \|\phi_j(1 - \chi)\varphi\|_{L^p}^p \leq CT\sum_j \|L(\phi_j\varphi)\|_{L^p}^p$$

$$\leq CT\|(1-\chi)L\varphi\|_{L^p}^p + CT\sum_j \|(L\phi_j)(1-\chi)\varphi\|_{L^p}^p$$

where we have used the Leibniz rule and the fact that $\sum_j \phi_j^p = 1$. The second term on the right-hand side is dominated by $CT\|(1-\chi)\varphi\|_{L^p}^p$. Indeed,

$$|L\phi_j(x)| = |\vec{b}(x,t)\cdot\nabla_x\phi_j(x)| \leq \sup_{Q_j^*}|\vec{b}|\,|\nabla_x\phi_j| \leq \frac{C}{\mu} \leq C_1$$

in view of the definition of ρ, (IV.60) and (IV.61). Hence, $|L\phi_j(x)|^p \leq C$ and since $|L\phi_j(x)|^p = 0$ except for at most 12^n values of j we also have $\sum_j |L\phi_j(x)|^p \leq C$. Thus,

$$\|(1-\chi)\varphi\|_{L^p}^p \leq CT\|(1-\chi)L\varphi\|_{L^p}^p + CT\|(1-\chi)\varphi\|_{L^p}^p$$

and the last term can be absorbed as soon as $CT < 1/2$. This proves (IV.56). We have already seen in steps 1 and 2 that (IV.53) follows in general once (IV.55) and (IV.56) are proved for L of the form (IV.51), so the proof of Theorem IV.3.5 is now complete for L and we may also replace L by $-L+c(x,t)$ in (IV.53) if $c(x,t)$ is any bounded function provided we shrink the neighborhood U of the origin, in particular, we may replace L by the transpose operator ${}^tL = -L - i\,\mathrm{div}_x\,\vec{b}$. □

As usual, we obtain by duality

Corollary IV.3.12. *Let L given by* (IV.49) *satisfy* (IV.50) *and condition* ($\mathcal{P}$) *in a neighborhood of the origin and fix $1 < p < \infty$. Then, there exist R_0 and $C > 0$ such that for every $0 < R < R_0$ and $f \in L^p(\mathbb{R}^{n+1})$ there exists $u \in L^p(\mathbb{R}^{n+1})$ with norm*

$$\|u\|_{L^p(\mathbb{R}^{n+1})} \leq C\,R\|f\|_{L^p(\mathbb{R}^{n+1})}$$

that satisfies the equation

$$Lu = f \quad \textit{for } |x|^2 + t^2 < R^2. \tag{IV.68}$$

Moreover, the constants C and R_0 depend only on p and the L^∞ norms of the derivatives of order at most two of the coefficients of L.

Let us assume now that we are dealing with a locally integrable vector field L in an open set of $\mathbb{R}^{n+1}$ that contains the origin. After an appropriate local change of coordinates (x,t) we may assume that there are functions $Z_j(x,t)$, $j = 1,\ldots,n$ defined on a neighborhood of the origin of the form

$$Z_j(x,t) = x_j + i\varphi_j(x,t), \quad j = 1,\ldots,n,$$

with $\varphi_j(x,t)$ smooth and real satisfying

$$\varphi_j(0,0) = \nabla_x \varphi_j(0,0) = 0, \quad j = 1, \dots, n,$$

such that

$$LZ_j = 0, \quad j = 1, \dots, n.$$

We denote by Z the function $Z = (Z_1, \dots, Z_n)$ with values in $\mathbb{C}^n$ and similarly write $\varphi = (\varphi_1, \dots, \varphi_n)$, so $Z(x,t) = x + i\varphi(x,t)$. The $n \times n$ matrix

$$\varphi_x = \begin{pmatrix} \partial\varphi_1/\partial x_1 & \cdots & \partial\varphi_1/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial\varphi_n/\partial x_1 & \cdots & \partial\varphi_n/\partial x_n \end{pmatrix}$$

vanishes at the origin and after modification of L outside a neighborhood of the origin we may assume that the functions $\varphi_j(x,t)$ are defined throughout $\mathbb{R}^{n+1}$, have bounded derivatives of all orders, and satisfy

$$\|\varphi_x(x,t)\| \leq \frac{1}{2}, \qquad (x,t) \in \mathbb{R}^{n+1}.$$

This implies that the matrix $Z_x = I + i\varphi_x$ is everywhere invertible and we write $Z_x^{-1}(x,t) = (\mu_{jk}(x,t))$. Then the vector fields

$$M_j = \sum_{k=1}^{n} \mu_{jk}(x,t) \frac{\partial}{\partial x_k}, \quad j = 1, \dots, n \tag{IV.69}$$

commute pairwise and the vector field

$$L_1 = \frac{\partial}{\partial t} - \sum_{k=1}^{n} \lambda_k(x,t) \frac{\partial}{\partial x_k}$$

commutes with $M_1, \dots, M_n$ and is proportional to L if

$$\lambda_k(x,t) = -i \sum_{j=1}^{n} \mu_{kj}(x,t) \frac{\partial \varphi_j}{\partial t}(x,t).$$

Furthermore, $M_1, \dots, M_n, L$ are linearly independent at every point and generate $T\mathbb{R}^{n+1}$. Multiplying L by a nonvanishing factor we may assume that $L = L_1$.

We now extend Theorem IV.2.3 to several variables.

THEOREM IV.3.13. *Assume that L is a smooth vector field defined in an open subset $\Omega \subset \mathbb{R}^{n+1}$ and let $c(x,t) \in C^\infty(\Omega)$. If L satisfies $(\mathcal{P})$ in Ω and is locally integrable then every point $p \in \Omega$ has a neighborhood U such that the equation*

$$Lu + cu = f, \quad f \in C_c^\infty(U)$$

may be solved with $u \in C^\infty(U)$. Conversely, if L is locally solvable in C^∞ then L is locally integrable.

Proof. The construction of smooth solutions is a straightforward extension of the two-dimensional case. We write

$$D = -L^2 - \lambda(M_1^2 + \cdots + M_n^2),$$

where $M_1, \ldots, M_n$ are given by (IV.69) and $\lambda > 0$ is a large parameter. Since $L = L_1$ and M_j commute, $j = 1, \ldots, n$, it follows that L and D commute. If $\ell(x, t, \xi, \tau)$ denotes the symbol of L, $m_j(x, t, \xi)$ denotes the symbol of M_j and $d(x, t, \xi, \tau) = -(\ell^2 + \lambda(m_1^2 + \cdots + m_n^2))(x, t, \xi, \tau)$ is the principal symbol of D, we have

$$\{\ell, d\}(x, t, \xi, \tau) = 0, \qquad (x, t, \xi, \tau) \in \mathbb{R}^{2(n+1)}.$$

For large $\lambda > 0$, D is a uniformly elliptic second-order differential operator. Consider, for fixed $s \in \mathbb{R}$, the pseudo-differential operator

$$B^\epsilon u(x, t) = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^{N+1}} \mathrm{e}^{i(x\cdot\xi + t\tau)}\, p(x, t, \xi, \tau)\, \widehat{u}(\xi, \tau)\, \mathrm{d}\xi \mathrm{d}\tau$$

with symbol

$$b^\epsilon(x, t, \xi, \tau) = \frac{\chi(t)(1 + d(x, t, \xi, \tau))^{s/2}}{(1 + \epsilon\, d(x, t, \xi, \tau))^{1/2}}$$

where $\chi(t) \in C_c^\infty(-T, T)$ and $\chi(t) = 1$ for $|t| \le (3/4)T$. Here we choose T so that the estimate

$$\|u(x, t)\|_{L^2(\mathbb{R}^{n+1})} \le C\|Lu(x, t)\|_{L^{n+1}(\mathbb{R}^2)} \tag{IV.70}$$

holds for every $u \in C_c^\infty(\mathbb{R}^n \times (-T, T))$ for some $C > 0$, as guaranteed by the proof of Theorem IV.3.5. The estimate can be extended to any $u \in L_c^2(\mathbb{R}^n$ $(-T, T) \times (-T, T))$ such that $Lu \in L_c^2(\mathbb{R}^n(-T, T) \times (-T, T)$ by Friedrich's lemma. It follows that $b^\epsilon \to b = \chi(1 + d)^{s/2}$ in the symbol space $S_{1,0}^s$ and that $\|u\|_s \sim \|Bu\|_{L^2}$ if B is the pseudo-differential with symbol b and $u \in H_c^s(\mathbb{R}^n \times (-T/2, T/2))$. Furthermore, $[L, B]$ has order $s - 1$ on $\mathbb{R} \times (-T/2, t/2)$. If $u \in H_c^{s-1}(\mathbb{R}^n \times (-T/2, T/2))$ is such that $Lu \in H_c^s(\mathbb{R}^n \times (-T/2, T/2))$ we may apply (IV.70) to $B^\epsilon u$. Letting $\epsilon \to 0$ we obtain

$$\|Bu\|_{L^2(\mathbb{R}^{n+1})} \le C(\|BLu\|_{L^2(\mathbb{R}^{n+1})} + \|[L, B]u\|_{L^2(\mathbb{R}^{n+1})})$$

which implies that $u \in H^s(\mathbb{R}^{n+1})$ and

$$\|u\|_{H^s(\mathbb{R}^{n+1})} \le C_s(\|Lu\|_{H^s(\mathbb{R}^{n+1})} + \|u\|_{H^{s-1}(\mathbb{R}^{n+1})}). \tag{IV.71}$$

Once (IV.71) is known, general arguments lead to an a priori estimate

$$\|u\|_{H^s(\mathbb{R}^{n+1})} \le C_s \|Lu\|_{H^s(\mathbb{R}^{n+1})}, \tag{IV.72}$$

if $u \in H_c^{s-1}(\mathbb{R}^n \times (-T/2, T/2))$ is such that $Lu \in H_c^s(\mathbb{R} \times (-T/2, T/2))$ and to the existence of local smooth solutions, as described in the proof of Theorem IV.2.3. We leave details to the reader.

While the method to obtain smooth solutions starting from the existence of L^2 solutions is essentially the same independently of the number of variables, the proof that smooth local solvability implies local integrability is rather different if $n = 1$ or $n > 2$. In the proof of Theorem IV.2.3 it was shown that, for $n = 1$, solving $Lu = f$ for a specific f obtained from the coefficients of L was enough to produce locally a smooth Z such that $LZ = 0$ and $\mathrm{d}Z \neq 0$. Nothing like this is available if $n > 1$ and we must proceed indirectly. Assume that L given by (IV.51) is locally solvable in C^∞ and we wish to find n first integrals with linearly independent differentials defined in a neighborhood of a given point p that we may as well assume to be the origin. The first step is to find a complete set of *approximate* first integrals, namely, n smooth functions $Z_j^\#$, $j = 1, \ldots, n$, such that $LZ_j^\# = f_j$ vanishes to infinite order at the origin—i.e., $f_j(x) = O(|x|^k)$, $k = 1, 2, \ldots$—and $\mathrm{d}Z_1^\#(0), \ldots, \mathrm{d}Z_n^\#(0)$ are linearly independent. To find $Z_j^\#$ we solve first the noncharacteristic Cauchy problem

$$\begin{cases} LU_j &= 0, \\ U_j(x, 0) &= x_j, \end{cases}$$

in the sense of formal power series. The coefficients of the formal series U_j corresponding to monomials that do not contain t are determined by the initial condition $U_j(x, 0)$, i.e., they are all zero with the exception of the coefficient of x_j which is 1. The coefficients of monomials of the form $t^\ell x^\alpha$ are determined from $LU_j = 0$ inductively on ℓ. Once the formal series U_j has been found we take as $Z_j^\#$ any smooth function that has U_j as its Taylor series at the origin (the existence of such a function is usually called Borel's lemma). By their very definition $Z_1^\#, \ldots, Z_n^\#$ are approximate first integrals. To obtain exact first integrals by correction of $Z_1^\#, \ldots, Z_n^\#$ we must solve the equations $Lu_j = f_j$, $j = 1, \ldots, n$, in a neighborhood of the origin and then define $Z_j = Z_j^\# - u_j$. Clearly, $LZ_j = 0$, so the problem is now to verify that $\mathrm{d}Z_1(0), \ldots, \mathrm{d}Z_n(0)$ are linearly independent. This will be guaranteed if we can make sure that $|\mathrm{d}u_j(0)|$ is small. Let K be a ball centered at the origin such that $LC^\infty(K) = C^\infty(K)$ and let $\mathcal{H}$ denote the subspace of $C^\infty(K)$ of the (equivalence classes of) functions h such that $Lh = 0$. Then L defines a continuous linear map from $C^\infty(K)/\mathcal{H}$ onto $C^\infty(K)$ which, by the open mapping theorem for Fréchet spaces, has a continuous inverse. This means, in particular, that given $\epsilon > 0$ there exists $\delta > 0$ and $m \in \mathbb{Z}^+$ such that for every $f \in C^\infty(K)$ such that $\|D^\beta f\|_{L^\infty(K)} < \delta$ for all $|\beta| \leq k$ there exist $u \in C^\infty(K)$ such that $Lu = f$ and $\|\mathrm{d}u\|_{L^\infty(K)} < \epsilon$. Let $\chi(x, t) \in C_c^\infty(\mathbb{R}^{n+1})$ be equal to 1 for $|x|^2 + t^2 < 1$ and set $f_{j,\rho}(x, t) = f_j(x, t)\chi(\rho x, \rho t)$. Since f_j vanishes to infinite order at the origin we see that, choosing ρ big enough, $\|D^\beta f_{j,\rho}\|_{L^\infty} < \delta$ for all $|\beta| \leq k$. Choose now u_j such that $Lu_j = f_{j,\rho}$ and $\|\mathrm{d}u_j\|_{L^\infty(K)} < \epsilon$. Since $f_{j,\rho} = f_j$ for $|x|^2 + t^2 < 1/\rho$ we see that the functions $Z_j = Z_j^\# - u_j$, $j = 1, \ldots, n$ form a complete set of first integrals in a neighborhood of the origin if ϵ is taken small enough. □

IV.4 Necessary conditions for local solvability

In this section we discuss the necessity of condition $(\mathcal{P})$ for the local solvability of a locally integrable vector field. Assume that L defined in $\Omega \subset \mathbb{R}^{n+1}$ by (IV.49) is locally solvable in the sense of Definition IV.1.2. We will show that L must satisfy condition $(\mathcal{P})$ in Ω. In doing so, due to the local nature of the problem, we may assume that L is given by (IV.51) and that $\Omega = B \times (-T, T)$ where $B \subset \mathbb{R}^n$ is a ball centered at the origin. We may also assume that there is a vector-valued function $Z(x, t) = (Z_1(x, t), \ldots, Z_n(x, t))$ defined in a neighborhood U of $\overline{\Omega}$ such that $LZ_j = 0$, $j = 1, \ldots, n$ and $\|I - Z_x\| < 1/2$ in Ω, where I denotes the identity matrix. In particular, the form $\mathrm{d}Z_1 \wedge \cdots \wedge \mathrm{d}Z_n$ does not vanish in Ω and the pairing

$$C_c^\infty(\Omega) \times C_c^\infty(\Omega) \ni (f, v) \mapsto \int fv \det(Z_x)\, \mathrm{d}x\mathrm{d}t$$

is nondegenerate. The formula

$$\int f\, Lv \det(Z_x)\, \mathrm{d}x\mathrm{d}t = -\int Lf\, v \det(Z_x)\, \mathrm{d}x\mathrm{d}t, \qquad v, f \in C_c^\infty(\Omega)$$

means that L and $-L$ are each other's formal transpose with respect to this pairing. The formula is also valid by continuity if $v \in \mathcal{D}'(\Omega)$ provided that we replace the integration by the standard duality between distributions and test function, i.e.,

$$\langle Lv, f \det(Z_x)\rangle = -\langle v, Lf \det(Z_x)\rangle, \qquad f \in C_c^\infty(\Omega),\ v \in \mathcal{D}'(\Omega). \tag{IV.73}$$

One of the basic tools in the study of necessary conditions for local solvability is Hörmander's lemma ([**H6**]), of which we give the following version.

Lemma IV.4.1. *Let L be as described above and suppose that for every $f \in C_c^\infty(\Omega)$ there exists $u \in \mathcal{D}'(\Omega)$ such that $Lu = f$. Then, for any compact set $K \subset \Omega$ there exist constants $C > 0$, $M \in \mathbb{Z}^+$ such that*

$$\left|\int fv \det(Z_x)\,\mathrm{d}x\mathrm{d}t\right| \le C \sum_{|\alpha|\le M} \|D^\alpha_{x,t} f\|_{L^\infty} \sum_{|\beta|\le M} \|D^\beta_{x,t} Lv\|_{L^\infty} \tag{IV.74}$$

for all $f, v \in C_c^\infty(K)$.

Proof. Let $K \subset\subset \Omega$ with nonempty interior be given and consider the bilinear form (IV.73) restricted to pairs $(f, v) \in C_c^\infty(K) \times C_c^\infty(K)$. Endow the first factor with the topology defined by the seminorms $\|D^\alpha_{x,t} f\|_{L^\infty}$—so it becomes a Fréchet space—and the second factor with the countable family of seminorms $\|D^\beta_{x,t} Lv\|_{L^\infty}$. Our solvability hypothesis implies that the latter topology is Hausdorff, indeed, if $v \in C_c^\infty(K)$ is such that $Lv = 0$ we may choose for any $f \in C_c^\infty(K)$ a distribution $u \in \mathcal{D}'(\Omega)$ such that $Lu = f$, so we have

$$\langle f, v \det(Z_x)\rangle = \langle Lu, v \det(Z_x)\rangle = -\langle u, Lv \det(Z_x)\rangle = 0$$

for any $f \in C_c^\infty(K)$, which implies that $v = 0$. For fixed v, the bilinear form clearly depends continuously on f. The solvability hypothesis implies that the dependence on v is also continuous for f fixed. Indeed, we may assume that $f = Lu$ for some $u \in \mathcal{D}'(\Omega)$. Hence

$$\int fv \det(Z_x)\,\mathrm{d}x\mathrm{d}t = \langle Lu, f \det(Z_x)\rangle = -\langle \det(Z_x)u, Lf\rangle$$

in view of (IV.73), which shows the continuity with respect to f for fixed v. A bilinear form defined on the product of a Fréchet space and a metrizable space which is separately continuous is continuous in both variables. This proves (IV.74). $\square$

The last lemma shows that in order to prove that L is not solvable it is enough to violate the a priori inequality (IV.74). We now describe a method to violate (IV.74) provided we find a solution h of the homogenous equation $Lh = 0$ with certain geometric property. Let $g \in C^0(\Omega)$ be a real function and $K \subset\subset \Omega$ be compact. We say that *g assumes a local minimum over K* if there exists $a \in \mathbb{R}$ and V open, $K \subset V \subset \Omega$ such that

(1) $g \equiv a$ on K;
(2) $g > a$ on $V \backslash K$.

Note that we may always replace the open set V with one of its open subsets with compact closure that contains K. In this case, still denoting the new set by V we have

$$\inf_{\partial V} g = a_1 > a.$$

Then, taking $a < b < a_1$ we see that the set $W = \{g < b\} \cap V$ has compact closure contained in V and $g \geq b > a$ on $\overline{V} \setminus W$.

The proof of the next lemma shows how (IV.74) may be violated.

LEMMA IV.4.2. *Assume that there exists $h \in C^\infty(\Omega)$ such that*

(i) $Lh = 0$;
(ii) $\Re h$ *assumes a local minimum over some* $K_1 \subset\subset \Omega$.

Then there exists $f \in C_c^\infty(\Omega)$ such that $Lu \neq f$ for all $u \in \mathcal{D}'(\Omega)$.

PROOF. By Lemma IV.4.1 it will be enough to show that for a convenient choice of $K \subset\subset \Omega$, (IV.74) cannot hold for all $f, v \in C_c^\infty(K)$ whatever the choice of $M \in \mathbb{Z}^+$ and $C > 0$. By hypothesis $\Re h$ assumes a local minimum over $K_1 \subset\subset \Omega$ for some homogeneous solution h. Subtracting a constant we may assume that $\Re h = 0$ on K_1 and $\Re h \geq \varepsilon > 0$ on $\overline{V} \setminus W$ for some open sets $V \supset W \supset K_1$ such that $K \doteq \overline{V} \subset\subset \Omega$. Select $\zeta \in C_c^\infty(K)$, $0 \leq \zeta \leq 1$, such that $\zeta = 1$ on W and set, for a large parameter $\rho > 0$,

$$v_\rho(x, t) = \zeta(x, t) \mathrm{e}^{-\rho h(x,t)}.$$

Since $\mathrm{e}^{-\rho h(x,t)}$ is a homogeneous solution, $Lv_\rho = \mathrm{e}^{-\rho h} L\zeta$. Furthermore, $L\zeta$ is supported in $K \setminus W$ so it follows that

$$\sum_{|\beta| \leq M} \|D_{x,t}^\beta L v_\rho\|_{L^\infty} \leq C\rho^M \mathrm{e}^{-\varepsilon\rho}. \tag{IV.75}$$

Next, choose $\psi \in C_c^\infty(V)$, $0 \leq \psi \leq 1$, such that $\psi = 1$ on K_1 and $\Re h(x, t) < \varepsilon/2$ on the support of ψ. Define

$$f_\rho(x, t) = \frac{\psi(x, t)}{\det(Z_x(x, t))} \mathrm{e}^{\rho h(x,t)}.$$

Then

$$\sum_{|\alpha| \leq M} \|D_{x,t}^\alpha f_\rho\|_{L^\infty} \leq C\rho^M \mathrm{e}^{\varepsilon\rho/2}. \tag{IV.76}$$

On the other hand, since ζ and ψ are positive in a neighborhood of K_1,

$$\int f_\rho v_\rho \det(Z_x)\, \mathrm{d}x\mathrm{d}t = \int \zeta(x, t)\psi(x, t)\, \mathrm{d}x\mathrm{d}t = c > 0$$

which together with (IV.75) and (IV.76) shows that (IV.74) cannot hold for the pair $(f_\rho, v_\rho) \in C_c^\infty(K) \times C_c^\infty(K)$ if ρ is large enough. $\square$

Our next task is to produce solutions of the homogeneous equation $Lh = 0$ whose real part assumes a local minimum over a compact set assuming that condition $(\mathcal{P})$ does not hold. We will first discuss this in the case $n = 1$, which is technically simpler and the geometric ideas involved are easier to spot. Suppose $n = 1$, $L = \partial_t - (Z_t/Z_x)\partial_x$, $Z = x + i\varphi(x, t)$, $(x, t) \in \mathbb{R}^2$. We know by Lemma IV.2.2 that if $(\mathcal{P})$ does not hold then $t \mapsto \varphi(x_0, t)$ is not monotone for some x_0, or equivalently that $t \mapsto \varphi_t(x_0, t)$ takes opposite signs and, in particular, vanishes for some t_0. The simplest situation occurs when $\varphi_t(x_0, t_0) = 0$ and $\varphi_{tt}(x_0, t_0) \neq 0$. If $\varphi_{tt}(x_0, t_0) \doteq A > 0$, $\varphi_{xt}(x_0, t_0) \doteq B$ and $\varphi_{xx}(x_0, t_0) \doteq C$ set, for $\lambda > 0$ to be chosen later,

$$\begin{aligned} w(x, t) &= \frac{x - x_0 + i(\varphi(x, t) - \varphi(x_0, t_0))}{1 + i\varphi_x(x_0, t_0)} \\ h(x, t) &= w^2(x, t) - i\lambda w(x, t). \end{aligned}$$

Note that $w(x_0, t_0) = 0$, $w_t(x_0, t_0) = 0$, $w_x(x_0, t_0) = 1$—which implies that $\Im w_x(x_0, t_0) = 0$—and it is also clear that $Lh = 0$. Let us write $u(x, t) \doteq \Re h(x, t)$, so

$$u(x, t) = (\Re w(x, t))^2 - (\Im w(x, t))^2 + \lambda\, \Im w(x, t)$$

and it follows that $u(x_0, t_0) = u_t(x_0, t_0) = u_x(x_0, t_0) = 0$. Then,

$$\begin{aligned} u_{xx}(x_0, t_0) &= 2\,(\Re w_x(x_0, t_0))^2 + c\lambda\, C = 2 + c\lambda\, C, \\ u_{tt}(x_0, t_0) &= \lambda\, \Im w_{tt}(x_0, t_0) = c\lambda\, \varphi_{tt}(x_0, t_0) = c\lambda\, A > 0, \\ u_{xt}(x_0, t_0) &= c\lambda\, \varphi_{xt}(x_0, t_0) = c\lambda\, B, \end{aligned}$$

where $c = (1 + \varphi_x^2(x_0, t_0))^{-1} > 0$, which shows that the Hessian of u at (x_0, t_0) is positive definite if $\lambda > 0$ is small enough. Then $\Re h$ has a strict local minimum at (x_0, t_0), i.e., the hypotheses of Lemma IV.4.2 are satisfied if we choose $K_1 = \{(x_0, t_0)\}$. If $\varphi_{tt}(x_0, t_0) = A < 0$ we reason similarly, taking $\lambda < 0$ and small.

The previous discussion shows that when looking for a homogeneous solution h whose real part assumes a local minimum over a compact set we may work under the assumption that

$$\varphi_t(x, t) = 0 \implies \varphi_{tt}(x, t) = 0. \tag{IV.77}$$

Assume that condition $(\mathcal{P})$ does not hold in any square centered at $(0, 0)$. Then given $\epsilon > 0$ we may find points (x_*, t_1), (x_*, t_2) in the cube Q centered at the origin with side length ϵ such that, say, $t_1 < t_2$, $\varphi_t(x_*, t_1) < 0$, and $\varphi_t(x_*, t_2) > 0$. We consider homogeneous solutions of the form

$$h(x, t; x_0) = (Z(x, t) - Z(x_0, 0))^2 - i\lambda\, \frac{Z(x, t) - Z(x_0, 0)}{Z_x(x_0, 0)}$$

and the difficulty is to show under assumption (IV.77) that for an appropriate choice of $|\lambda| \leq 1$ and $|x_0| \leq 1$ our function h assumes a local minimum over a compact set. Writing h in terms of its real and imaginary parts,

$$h(x, t; x_0) = u^{x_0}(x, t) + iv^{x_0}(x, t),$$

we obtain

$$\begin{aligned} u^{x_0}(x, t) = (x - x_0)^2 - [\varphi(x, t) - \varphi(x_0, 0)]^2 \\ + \lambda\, c[\varphi(x, t) - \varphi(x_0, 0) - \varphi_x(x_0, 0)(x - x_0)] \end{aligned} \tag{IV.78}$$

where $c = (1 + \varphi_x^2(x_0, 0))^{-1} > 0$. A straightforward computation shows that $\partial_x u^{x_0}(x_0, 0) = u_x^{x_0}(x_0, 0) = 0$. Since $u_{xx}^{x_0}(0, 0) = 2 + \lambda\varphi_{xx}(0, 0)$ we may assume, taking λ small but fixed and shrinking Q, that $u_{xx}^{x_0} > 0$ on $\overline{Q}$. Then the connected component γ_{x_0} that contains the point $(x_0, 0)$ of the level set

$$\{(x, t) : \quad u_x^{x_0}(x, t) = 0\}$$

is a smooth curve that intersects transversally the x-axis at $(x_0, 0)$. Hence, the curves γ_{x_0} foliate a neighborhood of the origin and shrinking $\epsilon > 0$ if necessary we may assume $\bigcup_{|x_0|\leq\epsilon} \gamma_{x_0} \supset Q$. From now on we will assume that $|x_0| \leq \epsilon$. Note that the vector field

$$\ell = \frac{\partial}{\partial t} - \frac{u_{xt}^{x_0}}{u_{xx}^{x_0}} \frac{\partial}{\partial x} \tag{IV.79}$$

is tangent to the curve γ^{x_0} along γ^{x_0} so this curve may be realized as the graph of a function $x = x^{x_0}(t)$, $|t| < \epsilon_0$. Let us take a closer look at the behavior of u^{x_0} on the curve γ_{x_0}. For any $(x', t') \in \gamma_{x_0}$ we have that $u_x^{x_0}(x', t') = 0$ and $u_{xx}^{x_0}(x', t') > 0$ so $x \mapsto u^{x_0}(x, t')$ attains a strict minimum precisely at $x = x'$ (geometrically, the graph of $x \mapsto u^{x_0}(x, t')$ looks like a parabola pointing upwards with vertex at x'). Hence, there is a tubular neighborhood V of γ_{x_0} such that

$$\min_V u^{x_0}(x, t) = \min_{\gamma_{x_0}} u^{x_0}(x, t).$$

Thus, if we can find points (x_1', t_1'), (x_0', t_0'), (x_2', t_2') in γ_{x_0} such that $t_1' < t_0' < t_2'$ and

$$\begin{aligned} u^{x_0}(x_1', t_1') > u^{x_0}(x_0', t_0'), \\ u^{x_0}(x_2', t_2') > u^{x_0}(x_0', t_0') \end{aligned}$$

it follows that there is a compact set $K \subset \gamma^{x_0}$ such that $u^{x_0}(x, t)$ assumes a local minimum over K. To study the variation of u^{x_0} along γ_{x_0} we consider the parameterization $\gamma_{x_0}(s) = (x^{x_0}(s), s)$ and differentiate

$$u^{x_0}(x^{x_0}(s), s)$$

with respect to s. Since $u_x^{x_0}(x^{x_0}(s), s) \equiv 0$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s} u^{x_0} \circ \gamma_{x_0}(s) = u_t^{x_0} \circ \gamma_{x_0}(s) = [\varphi_t\,(c\lambda - 2(\varphi - \varphi(x_0, 0))^2)] \circ \gamma_{x_0}(s).$$

Shrinking $\epsilon < \epsilon_0$ we may assume that $2|\varphi(x,t) - \varphi(x_0, 0)|^2 < c|\lambda|/2$. Thus, *$u^{x_0}$ is monotone along γ_{x_0} if and only if φ_t does not change sign on γ_{x_0}.* Hence, if for some curve γ_{x_0} we find points (x_1', t_1'), (x_2', t_2') in γ_{x_0} such that $t_1' < t_2'$, $\varphi_t(x_1', t_1') < 0$, $\varphi_t(x_2', t_2') > 0$, then for $\lambda > 0$ and small the curve γ_{x_0} will contain a compact subset K over which u^{x_0} assumes a local minimum; if, instead, $\varphi_t(x_1', t_1') > 0$ and $\varphi_t(x_2', t_2') < 0$ we take $\lambda < 0$ in the definition of h to achieve the desired homogeneous solution. To see that such γ_{x_0} exists, consider the quadrilateral $Q^\flat$ having as horizontal sides the segments $t = \pm\epsilon$ and as 'vertical' sides the curves γ_{x_0} with $x_0 = \pm\epsilon$. Then $Q^\flat$ is the union of the curves γ_{x_0}, $-\epsilon < x_0 < \epsilon$. Assume by contradiction that φ_t does not change sign along any of these curves. We may decompose $Q^\flat$ into three disjoint sets: the union $Q_+^\flat$ of the curves γ_{x_0} that contain at least one point on which $\varphi_t > 0$, the union $Q_-^\flat$ of the curves γ_{x_0} that contain at least one point on which $\varphi_t < 0$, and the union $Q_0^\flat$ of the curves γ_{x_0} on which φ_t vanishes identically. Observe that $Q_+^\flat$ and $Q_-^\flat$ are open sets and neither $Q_+^\flat$ nor $Q_-^\flat$ can be empty, for this would imply that φ_t does not change sign on some square containing the origin and condition $(\mathcal{P})$ would be satisfied in that square, contradicting our assumptions. Since $Q_+^\flat$ and $Q^\flat \backslash Q_+^\flat$ are invariant sets (i.e., they are a union of the curves γ_{x_0} that intersect them) so is the boundary of $Q_+^\flat$. Let p be a boundary point of $Q_+^\flat$ and let γ_{x_0} be the curve passing through p. We claim that γ_{x_0} is a vertical segment. Indeed, $\gamma_{x_0} \subset Q_0^\flat$ since it cannot meet $Q_+^\flat \cup Q_-^\flat$. So φ_t vanishes identically on γ_{x_0} and also does φ_{tt} because of (IV.77). Let $q \in \gamma_{x_0}$. If $\varphi_{xt}^{x_0}(q) \neq 0$ the set $S = \{\varphi_x = 0\}$ is a smooth curve in a neighborhood of q and since $\varphi_{tt} = 0$ on S we conclude that the intersection of S with a neighborhood of q must be a vertical segment, in particular, the tangent to γ_{x_0} at q is vertical. Assume now that $\varphi_{xt}^{x_0}(q) = 0$. Differentiating twice (IV.78), first with respect to x, then with respect to t and evaluating the result at q we get $u_{xt}^{x_0}(q) = 0$ because $\varphi_t(q) = \varphi_{xt}(q) = 0$. Then the vector field ℓ given by (IV.79) reduces to ∂_t at q. Thus the velocity vector of γ_{x_0} is always vertical and γ_{x_0} is itself the vertical segment $\{x_0\} \times (-\epsilon, \epsilon)$.

Let us return to the points (x_*, t_1), (x_*, t_2) in the cube Q centered at the origin with side length ϵ such that $t_1 < t_2$, $\varphi_t(x_*, t_1) < 0$ and $\varphi_t(x_*, t_2) > 0$. Then trivially $(x_*, t_1) \in Q_-^\flat$ and $(x_*, t_2) \in Q_+^\flat$ so there exists a point $(x_*, t_0) \in \partial Q_+^\flat$ such that $t_1 < t_0 < t_2$. But, as we have seen, this implies that $\gamma_{x_*} = \{x_*\} \times (-\epsilon, \epsilon)$ and $\varphi_t(x_*, t) = 0$ for $|t| < \epsilon$, which is a contradiction. Thus, for some $|x_0| < \epsilon$, φ_t assumes opposite signs on γ_{x_0}, u^{x_0} is not monotone on γ_{x_0}, and $h(x, t; x_0)$ is a homogeneous solution whose real part assumes a local minimum over a compact set.

Essentially the same approach works in a higher number of variables although the proofs are technically more involved. The following elementary lemma about real quadratic forms in $\mathbb{R}^2$ will be useful:

LEMMA IV.4.3. *Assume that the real quadratic form*

$$q_1(x, y) = Ax^2 + 2Bxy + Cy^2, \qquad (x, y) \in \mathbb{R}^2, A, B, C \in \mathbb{R}$$

has positive trace $A + C > 0$ and set

$$q_2(x, y) = \Re\left[\left(\frac{C-A}{2} + iB\right)(x+iy)^2\right] = \frac{C-A}{2}(x^2 - y^2) - 2Bxy.$$

Then

$$q_1(x, y) + q_2(x, y) = \frac{A+C}{2}(x^2 + y^2)$$

is diagonal and positive definite.

PROOF. The assertion is self-evident. $\square$

We consider a vector field L given by (IV.51) defined on

$$\Omega = B \times (-T, T) \subset \mathbb{R}^n \times \mathbb{R}, \quad B = \{x \in \mathbb{R}^n : \quad |x| < \delta\}$$

and assume that there exist n first integrals $Z_1, \dots, Z_n$, $LZ_j = 0$, $j = 1, \dots, n$, with $\mathrm{d}Z_1, \dots, \mathrm{d}Z_n$ linearly independent in Ω. We write

$$Z = (Z_1, \dots, Z_n)$$

and further assume that $\det(Z_x) \neq 0$ in Ω, $Z(0, 0) = 0$ and $Z_x(0, 0) = I$. We also use the notation

$$\vec{b}(x, t) = \big(b_1(x, t), \dots, b_n(x, t)\big).$$

LEMMA IV.4.4. *Assume that there exists $(x_0, t_0) \in \Omega$ and $\xi \in \mathbb{R}^n$ such that*

(i) $\vec{b}(x_0, t_0) \cdot \xi = 0$;
(ii) $\vec{b}_t(x_0, t_0) \cdot \xi \neq 0$.

Then there exists $f \in C_c^\infty(\Omega)$ such that $Lu \neq f$ for all $u \in \mathcal{D}'(\Omega)$.

PROOF. By Lemma IV.4.2 we need only show that there exists a solution h of $Lh = 0$ such that $\Re h$ assumes a local minimum at $p = (x_0, t_0)$. Set $Z' = (Z'_1, \dots, Z'_n) = Z_x^{-1}(x_0, t_0)[Z(x, t) - Z(x_0, t_0)]$. Then $LZ'_j = 0$, $j = 1, \dots, n$, $Z'(p) = 0$, $Z'_x(p) = I$. Then, the change of coordinates $x' = x - x_0$, $t' = t - t_0$, shows that there is no loss of generality in assuming from the start that $(x_0, t_0) = (0, 0)$. Write $\Phi_j(x, t) = Z_j(x, t) - x_j$, so $\Phi_j(0, 0) = \partial_x \Phi_j(0, 0) = 0$, $j = 1, \ldots, n$. Set

$$W(x, t) = Z(x, t) \cdot \xi = \sum_{j=1}^{n} \xi_j Z_j(x, t).$$

Then $LW = 0$ and in view of (i) we get

$$0 = LW(0, 0) = \sum_{j=1}^{n} \xi_j \left(\frac{\partial \Phi_j}{\partial t}(0, 0) + i b_j(0, 0) \right) = i \Phi_t(0, 0) \cdot \xi$$

where $\Phi = (\Phi_1, \ldots, \Phi_n)$. Hence, $\Phi(0, 0) = \Phi_t(0, 0) = \Phi_x(0, 0) = 0$. We distinguish two cases.

Case 1. $\vec{b}(0, 0) = 0$. Differentiating with respect to t the equation $LW = 0$ we obtain $\Phi_{tt}(0, 0) \cdot \xi + i\vec{b}_t(0, 0) \cdot \xi = 0$ and using (ii) we derive

$$\Im \Phi_{tt}(0, 0) \cdot \xi \neq 0.$$

Set

$$\begin{aligned} h(x, t) &= Z_1^2(x, t) + \cdots + Z_n^2(x, t) - i\lambda\, W(x, t), \\ u(x, t) &\doteq \Re h(x, t) = |x + \Re \Phi|^2 - |\Im \Phi(x, t)|^2 + \lambda\, \Im \Phi(x, t) \cdot \xi. \end{aligned}$$

Thus, $u(0, 0) = 0$, $u_t(0, 0) = 0$, $\nabla_x u(0, 0) = 0$ and if we choose λ with the same sign as $\gamma = \Im \Phi_{tt}(0, 0) \cdot \xi$ it follows that the Taylor series of u at the origin is

$$u(x, y) = x_1^2 + \cdots + x_n^2 + |\lambda \gamma| t^2 + \lambda \sum_{j=1}^{n} c_j x_j t + \cdots$$

where the dots indicate terms of order > 2. Thus, the Hessian of u at the origin with respect to (x, t) is positive definite and u has a strict local minimum at the origin for $|\lambda|$ small.

Case 2. $\vec{b}_j(0, 0) \neq 0$ for some $1 \leq j \leq n$. After a linear change in the x-variables we may assume that

$$\begin{cases} b_1(0, 0) &= 1, \\ b_j(0, 0) &= 0, \quad j = 2, \ldots, n, \\ \qquad \xi &= (0, \xi_2, \ldots, \xi_n). \end{cases}$$

Since (ii) implies that $\xi \neq 0$ this case can only occur if $n \geq 2$. Set

$$W(x, t) = iZ(x, t) \cdot \xi = i \sum_{j=2}^{n} \xi_j Z_j(x, t).$$

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Proceeding as in Case 1 we obtain $\Re W_t(0,0) = \Re W_{x_j}(0,0) = 0$, for $j = 1, \ldots, n$. Differentiating the equation $LW = 0$ with respect to t we obtain

$$\partial_{tt}^2 W(0,0) + i\partial_{tx_1}^2 W(0,0) = \vec{b}_t(0,0) \cdot \xi \neq 0$$

while differentiation with respect to x_1 gives

$$\partial_{tx_1}^2 W(0,0) + i\partial_{x_1x_1}^2 W(0,0) = 0.$$

Using both equations to eliminate the term $\partial_{tx_1}^2 W(0,0)$ and replacing ξ by $-\xi$ if necessary we obtain

$$\partial_{tt}^2 \Re W(0,0) + \partial_{x_1x_1}^2 \Re W(0,0) \doteq 2\gamma > 0.$$

Applying Lemma IV.4.3 to the quadratic form

$$q_1(x_1, t) = \partial_{tt}^2 \Re W(0,0) t^2 + 2\partial_{x_1x_1}^2 \Re W(0,0) t x_1 + \partial_{x_1x_1}^2 \Re W(0,0) x_1^2$$

we find a complex number ζ such that $q_1(x_1, t) + \Re[\zeta(x_1 + it)^2]$ is positive definite. Since $\partial_{x_1} Z_1(0,0) = 1$ and it follows from $LZ_1(0) = 0$ that $\partial_t Z_1(0,0) = i$ the Taylor expansion in the variables (x_1, t) of ζZ_1^2 is

$$\zeta Z_1^2(x_1, 0, \ldots, 0, t) = \Re[\zeta(x_1 + it)^2] + \cdots$$

Thus,

$$\Re(W + \zeta Z_1^2)(x_1, 0, \ldots, 0, t) = \gamma(t^2 + x_1^2) + \cdots$$

If we now set

$$\begin{aligned} h(x,t) &= Z_1^2(x,t) + \cdots + Z_{n-1}^2(x,t) + \lambda\,(W(x,t) + \zeta Z_1^2), \\ u(x,t) &\doteq \Re h(x,t) \end{aligned}$$

we may check as in case (i) that $Lh = 0$ and that for $\lambda > 0$ small $u = \Re h$ has a positive definite Hessian at the origin. $\square$

Remark IV.4.5. Lemma IV.4.4 has the following geometric interpretation. Writing $L = X + iY$ with X and Y real we have that $X = \partial_t$, $Y = \vec{b}$, and $[X, Y] = \vec{b}_t$. Then conditions (i) and (ii) at $p = (x_0, t_0)$ mean that $[X, Y](p)$, $X(p)$, and $Y(p)$ are not linearly dependent. Indeed, if $AX(p) + BY(p) + C[X, Y](p) = 0$, the obvious fact $\vec{b} \cdot X = \vec{b}_t \cdot X = 0$ implies that $A = 0$ so $[X, Y](p)$ and $Y(p)$ would be collinear, contradicting (i) and (ii). This implies that the orbit Σ of the pair of vectors $\{X, Y\}$ that passes through p cannot have dimension ≤ 2. In fact, the three vectors $[X, Y](p)$, $X(p)$, and $Y(p)$ belong to $T_p(\Sigma)$ so $\dim \Sigma \leq 2$ would force a linear relationship between them. Hence, (i) and (ii) of Lemma IV.4.4 imply that $\dim \Sigma \geq 3$, which violates (1) of condition $(\mathcal{P})$ in Definition IV.3.2.

In order to find a solution h of $Lh = 0$ with the property that its real part assumes a local minimum over a compact set we need only worry about those cases not covered by Lemma IV.4.4, i.e., we may always assume that

$$\varphi_t(x,t)\cdot\xi = 0 \Longrightarrow \varphi_{tt}(x,t)\cdot\xi = 0, \quad (x,t)\in\Omega, \quad \xi\in\mathbb{R}^n. \tag{IV.80}$$

Let us assume that L does not satisfy condition $(\mathcal{P})$ in any cube centered on the origin and let us try to produce the required homogeneous solution h. As in the case of two variables we will look for solutions $h = u + iv$ such that the Hessian matrix u_{xx} is everywhere positive definite and the critical points of $x \mapsto u(x,t)$ are located on a certain curve γ so that when looking for a local minimum of u we only need to direct our attention to the restriction $u|_\gamma$. Then, assuming by contradiction that u is monotone on γ and that this happens for all the functions u of this type, we must conclude that L is forced to satisfy $(\mathcal{P})$ in some neighborhood of the origin. The first step is then to show the abundance of solutions of this type, which is taken care of by the next lemma that describes a family of solutions depending on two parameters, $x_0 \in B$ and $\eta \in \mathbb{R}^n$. The general form of these solutions is based on the function h introduced in case (i) of Lemma IV.4.4.

LEMMA IV.4.6. *If T and δ are small enough there exists a smooth function $h \in C^\infty(\Omega\times B\times\mathbb{R}^n)$,*

$$h(x,t;x_0,\eta) = u(x,t;x_0,\eta) + iv(x,t;x_0,\eta)$$

with u and v real such that

(i) *$Lh = 0$ in Ω for all $(x_0,\eta)\in B\times\mathbb{R}^n$;*
(ii) *$u_x(x_0,0;x_0,\eta) = 0$ and $v_x(x_0,0;x_0,\eta) = \eta$;*
(iii) *$u_{xx}(x,t;x_0,\eta)$ is positive definite at all points $(x,t)\in\Omega$ for all $(x_0,\eta)\in B\times\mathbb{R}^n$.*

PROOF. Set

$$h(x,t;x_0,\eta) = \lambda\,(1+|\eta|^2)^{1/2}\sum_{j=1}^{k}(Z(x,t)-Z(x_0,0))^2 + i\eta\cdot Z_x^{-1}(x_0,0)[Z(x,t)-Z(x_0,0)].$$

Since h is a polynomial in $Z_1,\ldots,Z_n$ it is apparent that (i) holds. Differentiating h with respect to x and evaluating the result at $(x,t) = (x_0,0)$ we get $h_x(x_0,0,x_0,\eta) = i\eta$ which shows (ii). Finally, write $(1+|\eta|^2)^{-1/2}u = F = \lambda f + g$. Then f is independent of η and $f_{xx}(0,0,0,\eta) = 2\lambda I$, $I =$ identity matrix, so f_{xx} has n eigenvalues $\geq \lambda > 0$ on $\Omega\times B\times\mathbb{R}^n$ if T and δ are chosen small.

Since g_{xx} is uniformly bounded in $\Omega \times B \times \mathbb{R}^n$, taking λ large we obtain that F_{xx} is positive definite in $\Omega \times B \times \mathbb{R}^n$, which implies the positivity of u_{xx}. $\square$

We regard the function h defined in Lemma IV.4.6 primarily as a function in the variables (x, t) that depends on the parameters (x_0, η), whose geometric meaning is furnished by (ii). To the function h we associate the real vector field V defined for $(x, t, x_0, \eta, \xi) \in \Omega \times B \times \mathbb{R}^n \times \mathbb{R}^n$ by

$$V = \frac{\partial}{\partial t} - A(x, t, x_0, \eta) \cdot \frac{\partial}{\partial x} + B(x, t, x_0, \eta) \cdot \frac{\partial}{\partial \xi}$$

where $A = (A_1, \dots, A_n)$, $B = (B_1, \dots, B_n)$ are defined by

$$A = u_{xx}^{-1} u_{xt},$$
$$B = v_{tx} - v_{xx} A.$$

Note that the jth component of Vu_x is $(Vu_x)_j = u_{tx_j} - (u_{xx}A)_j = u_{tx_j} - u_{tx_j} = 0$, $j = 1, \dots, n$ so u_x is constant along the integral curves of V. A similar computation shows that $(V(\xi - v_x))_j = 0$, $j = 1, \dots, n$ so $\xi - v_x$ is also constant along the integral curves of V. It follows that V is tangent to the submanifold of $\Omega \times B \times \mathbb{R}^n \times \mathbb{R}^n$ of dimension $2n+1$

$$\Sigma = \{(x, t, x_0, \eta, \xi) : \quad u_x(x, t; x_0, \eta) = 0, \quad \xi = v_x(x, t; x_0, \eta)\}.$$

Since $(x_0, 0, x_0, \eta, \eta) \in \Sigma$ by (ii) of Lemma IV.4.3 the partial derivative of

$$(x, t, x_0, \eta, \xi) \mapsto (u_x(x, t, x_0, \eta), \xi - v_x(x, t, x_0, \eta))$$

with respect to (x_0, η) at $(0, 0, 0, 0, 0, 0)$ is the identity. Thus, Σ may be parameterized by (x, t, ξ) for $|x| < \delta_1$ $|t| < T_1$, $|\xi| < \delta_1$ as the graph of a smooth map

$$(x, t, \xi) \mapsto (x_0(x, t, \xi), \eta(x, t, \xi))$$

with values in $\{|x_0| < \delta_2\} \times \{|\eta| < \delta_2\}$. We may assume, if δ and T are further shrunken, that the image of $|x| < \delta_1$ $|t| < T_1$, $|\xi| < \delta_1$ by the map

$$(x, t, \xi) \mapsto (x_0(x, t, \xi), t, \eta(x, t, \xi))$$

covers $\Omega \times B \times \{|\xi| < \delta\}$. Thus, the vector field

$$V_* = \frac{\partial}{\partial t} - \alpha(x, t, \xi) \cdot \frac{\partial}{\partial x} + \beta(x, t, \xi) \cdot \frac{\partial}{\partial \xi} \tag{IV.81}$$

where

$$\alpha(x, t, \xi) = A(x, t, x_0(x, t, \xi), \eta(x, t, \xi)),$$
$$\beta(x, t, \xi) = B(x, t, x_0(x, t, \xi), \eta(x, t, \xi))$$

agrees with V on Σ—in particular, V_* is tangent to Σ—and its coefficients do not depend on x_0 and η. Fix $x_0 \in B$ and $|\eta| < \delta$ and consider the function of $u(x, t, x_0, \eta)$ as a function of (x, t). By (iii) of Lemma IV.4.6 the roots of the equations $u_x(x, t; x_0, \eta) = 0$, $\xi - v_x(x, t; x_0, \eta) = 0$ determine a smooth curve $\tilde{\gamma}_{x_0\eta}$ in (x, t, ξ)-space contained in Σ that passes through the point $(x_0, 0, \eta)$. The curves $\tilde{\gamma}_{x_0\eta}$ may be parameterized as $\tilde{\gamma}_{x_0\eta}(s) = (x(s; x_0, \eta), s, \xi(s; x_0, \eta))$ and they foliate Σ as x_0, η vary. The vector field V_* is tangent to $\tilde{\gamma}_{x_0\eta}$ at every point of $\tilde{\gamma}_{x_0\eta}$ so we may parameterize $\tilde{\gamma}_{x_0\eta}$ so that its velocity vector is V_*. The projection of $\tilde{\gamma}_{x_0\eta}$ on (x, t)-space gives a curve $\gamma_{x_0\eta}$ passing through $(x_0, 0)$ on which u_x vanishes and u_{xx} is positive definite. Hence, there is a tubular neighborhood V of $\gamma_{x_0\eta}$ such that

$$\min_V u(x, t; x_0, \eta) = \min_{\gamma_{x_0\eta}} u(x, t; x_0, \eta).$$

Thus, if the restriction of u to $\gamma_{x_0\eta}$ assumes a local minimum over a compact subarc K of $\gamma_{x_0\eta}$ we will also have that u itself assumes a local minimum over K. In order for the restriction of u to $\gamma_{x_0\eta}$ to assume a local minimum over a compact subarc K we must find points $t_1 < t_2$ such that

$$\frac{\mathrm{d}}{\mathrm{d}s}[u(x(s; x_0, \eta), s)](t_1) < 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}s}[u(x(s; x_0, \eta), s)](t_2) > 0.$$

Now, writing $x(s; x_0, \eta) = x(s)$ and $\xi(s; x_0, \eta) = \xi(s)$ to simplify the notation,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}u(x(s), s) &= u_x(x(s), s; x_0, \eta)\frac{\mathrm{d}}{\mathrm{d}s}x(s) + u_t(x(s), s; x_0, \eta)\\ &= u_t(x(s), s; x_0, \eta)\\ &= \vec{b}(x(s), s) \cdot v_x(x(s), s; x_0, \eta)\\ &= \vec{b}(x(s), s) \cdot \xi(s).\end{aligned}$$

Note that the identity $u_x = \vec{b} \cdot v_x$ is just the real part of the equation $Lh = 0$. This reduced the problem of finding a homogeneous solution $h = u + iv$ whose real part assumes a local minimum over a compact set for an appropriate choice of (x_0, η) to the problem of finding a curve $\tilde{\gamma}_{x_0\eta}$ such that the function $q(x, t, \xi) = \vec{b}(x, t) \cdot \xi$ changes from negative to positive along $\tilde{\gamma}_{x_0\eta}$. Thus, from the fact that $(\mathcal{P})$ is not satisfied in any neighborhood of the origin—which amounts to saying that any cube centered at the origin contains an integral curve of $X = \partial_t$ along which $q(x, t, \xi)$ changes sign—we must derive that there exists an integral curve of V_* along which $q(x, t, \xi)$ changes sign. The tool to compare the changes of sign of a function along the integral curves of two different vector fields is provided by

LEMMA IV.4.7. *Let $U \subset \mathbb{R}^N$ be an open set, X and V_* Lipschitz vector fields in U and $q \in C^1(U)$ a real function such that*

(1) $q(x) = 0$ implies $Xq(x) \leq 0$;
(2) $q(x) = 0$ and $\mathrm{d}q(x) = 0$ imply that $X(x) = V_*(x)$.

Assume that the integral curves γ of V_* have the following property:

($\bullet$) *if $q(x) < 0$ for some $x \in \gamma$ then $q(y) \leq 0$ for all points $y \in \gamma$ that lie ahead of x in the order determined by the flow.*

Then, the integral curves of X also satisfy property ($\bullet$).

We postpone the proof of Lemma IV.4.7 and continue our reasoning. We apply the lemma with U given by $|x| < \delta_1$, $|t| < T_1$, $|\xi| < \delta_1$, $|x_0| < \delta$, $|\eta| < \delta$, $N = 4n+1$, $X = \partial_t$, V_* given by (IV.81) and $q(x, t, \xi) = \vec{b}(x, t) \cdot \xi$. Let us check that hypotheses (1) and (2) in the lemma are satisfied. From (IV.80) we get (1). Assume now that $q(x, t, \xi) = \mathrm{d}q(x, t, \xi) = 0$ at some point (x, t, ξ). Since q is independent of (x_0, η) we may say q and $\mathrm{d}q$ vanish at $p = (x, t, x_0, \eta, \xi) \in \Sigma$, $x_0 = x_0(x, t, \xi)$, $\eta = \eta(x, t, \xi)$ and since $V_* = V$ on Σ and X and V_* do not depend on (x_0, η) we need only prove that $V(p) = X(p)$. From $q(x, t, \xi) = \mathrm{d}q(x, t, \xi) = 0$ we derive that $\vec{b}(x, t) = 0$, $\vec{b}_t(x, t) \cdot \xi = 0$, $\vec{b}_{x_j}(x, t) \cdot \xi = 0$, $j = 1, \ldots, n$. The real part of $Lh = 0$ is $u_t = \vec{b} \cdot \nabla v$ which, differentiated with respect to x_j, gives $u_{tx_j}(x, t, \xi) = 0$, so the coefficient A_j of $\partial/\partial x_j$ in V satisfies $A_j(x, t, x_0, \eta) = 0$. Similarly, differentiating $v_t + \vec{b} \cdot \nabla u = 0$ we get that $B = v_{xt} - v_{xx}A = v_{xt} = -\vec{b}_x \cdot u_x - u_{xx}(\vec{b}) = 0$ at (x, t, ξ) so $V_*(p) = V(p) = \partial_t = X(p)$ which proves (2). Since L does not satisfy $(\mathcal{P})$ there is an integral curve of X contained in U on which q changes sign from minus to plus. Then, by Lemma IV.4.7, V_* cannot possess property ($\bullet$) showing the existence of a curve $\tilde{\gamma}_{x_0\eta}$ along which $\vec{b} \cdot \xi$ changes sign from minus to plus as required to show that $u(x, t; x_0, \eta)$ assumes a local minimum over a compact set of Ω, which, by Lemma IV.4.2 implies that L is not solvable in Ω. Summing up,

THEOREM IV.4.8. *Assume that L, given by* (IV.49)*, is locally solvable in Ω. Then every point $p \in \Omega$ has a neighborhood U such that L satisfies condition $(\mathcal{P})$ in U.*

To complete the proof of the theorem we must prove Lemma IV.4.7.

We start by recalling that if $f : (a, b) \to \mathbb{R}$ is a continuous function we define

$$D^+ f(x) = \limsup_{\epsilon \searrow 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

which may vary in the range $[-\infty, \infty]$. The mean value inequality states that if $f \in C^0[a, b]$ there exists $c \in (a, b)$ such that $f(b) - f(a) \le D^+ f(c)(a-b)$. If $f(a) = f(b)$ it is enough to choose $c \in (a, b)$ so that $f(c) = \inf f(x)$ and the general case is reduced to this one by subtracting the affine function $f(a) + (x-a)(f(b) - f(a))/(b-a)$. It follows that if $D^+ f(x) \le 0$, $x \in (a, b)$, then $f(x)$ is monotone nonincreasing.

Let V be a Lipschitz vector field in $U \subset \mathbb{R}^N$, that is, $|V(x) - V(y)| \le K|x-y|$, $x, y \in U$. We denote by $\Phi_t(x)$, the forward flow of V stemming from x, i.e., the solution $\Phi_t(x)$ defined in a maximal interval $0 \le t < T(x)$ of the ODE

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(x) &= V(\Phi_t(x)), \\ \quad \Phi_t(0) &= x. \end{cases}$$

Let $F \subset U$ be a closed set. We say that F is positively V-invariant, or just V-invariant for brevity, if

$$x \in F \implies \Phi_t(x) \in F \quad \text{for all} \quad t \in [0, T(x)).$$

The characterization of V-invariant sets given below is due to Brézis ([**Br**]).

The following properties are equivalent:

(i) F is positively V-invariant;

(ii) $\forall x \in F, \quad \lim\limits_{\varepsilon \searrow 0} \dfrac{\operatorname{dist}(x + \varepsilon V(x), F)}{\varepsilon} = 0.$

Indeed, assume (i). Then

$$\begin{aligned} \frac{\operatorname{dist}(x + \varepsilon V(x), F)}{\varepsilon} &\le \frac{|x + \varepsilon V(x) - \Phi_\varepsilon(x)|}{\varepsilon} \\ &= \left| V(x) - \frac{\Phi_\varepsilon(x) - x}{\varepsilon} \right| \end{aligned}$$

and the right-hand side converges to 0 as $\varepsilon \searrow 0$.

Conversely, assume that (ii) holds. To prove (i) it is enough to show that the Lipschitz continuous function $f : [0, T(x)) \to [0, \infty)$ defined by

$$f(t) = \operatorname{dist}(\Phi_t(x), F)$$

vanishes identically. This will follow if we prove that $\mathrm{e}^{-At} f(t)$ is nonincreasing for some $A > 0$, since $f(0) = 0$. Thus, it is enough to show that

$$D_t^+(\mathrm{e}^{-At} f)(t) \le \mathrm{e}^{-At}(D_t^+ f(t) - Af(t)) \le 0,$$

which in turn is implied by $D_t^+ f(t) \le Af(t)$. Fix $t \in (0, T(x))$ and choose $z_t \in F$ such that $f(t) = |\Phi_t(x) - z_t|$. For small $\varepsilon > 0$ we have

$$f(t + \varepsilon) = \operatorname{dist}(\Phi_{t+\varepsilon}(x), F)$$

$$\le |\Phi_{t+\varepsilon}(x) - \Phi_\varepsilon(z_t)| + |\Phi_\varepsilon(z_t) - z_t - \varepsilon V(z_t)| \\ + \operatorname{dist}(z_t + \varepsilon V(z_t), F).$$

Now $|\Phi_{t+\varepsilon}(x) - \Phi_\varepsilon(z_t)| = |\Phi_\varepsilon(\Phi_t(x)) - \Phi_\varepsilon(z_t)|$, so by Gronwall's inequality,

$$|\Phi_\varepsilon(\Phi_t(x)) - \Phi_\varepsilon(z_t)| \le e^{K\varepsilon}|\Phi_t(x) - z_t| = e^{K\varepsilon} f(t),$$

for $\varepsilon > 0$ small, where K is the Lipschitz constant of V. Thus,

$$\frac{f(t+\varepsilon) - f(t)}{\varepsilon} \le \frac{(e^{K\varepsilon} - 1) f(t)}{\varepsilon} \\ + \left| \frac{\Phi_\varepsilon(z_t) - z_t}{\varepsilon} - V(z_t) \right| + \frac{\operatorname{dist}(z_t + \varepsilon V(z_t), F)}{\varepsilon}$$

and letting $\varepsilon \searrow 0$ we get $D_t^+ f(t) \le K f(t)$, since the right-hand side's middle term obviously $\to 0$ and the last one also does because we are assuming that (ii) holds. This shows that $e^{-Kt} f(t)$ is nonincreasing and proves (i).

We now prove Lemma IV.4.7.

Proof. Let U^- be the V_*-flow out of the set $\{x \in U : q(x) < 0\}$, i.e., a point $x \in U^-$ if $x = \Phi_t(y)$ for some $y \in U$ with $q(y) < 0$ and $0 \le t < T(y)$, where Φ_t is the flow of V_*. Hence, U^- is an open set and $\{q(x) < 0\} \subset U^- \subset \{q(x) \le 0\}$ because of ($\bullet$). By its very definition, U^- is positively V_*-invariant and so is its closure $F = \overline{U^-}$. Indeed, if $x \in F$ there exist a sequence $(x_j) \subset U^-$ such that $x_j \to x$. If $0 < t < T(x)$, then $0 < t < T(x_j)$ for large j because $s \mapsto T(x)$ is lower semicontinuous. Then $\Phi_t(x_j) \in U^-$ by the V_*-invariance of U^- and $\Phi_t(x) = \lim_j \Phi_t(x_j) \in \overline{U^-}$.

To prove the lemma we will show that F is X-invariant, which clearly implies that X has property ($\bullet$) because $F \subset \{q(x) \le 0\}$. We must show that

$$\lim_{\varepsilon \searrow 0} \frac{\operatorname{dist}(x + \varepsilon X(x), F)}{\varepsilon} = 0, \qquad x \in F. \tag{IV.82}$$

If $q(x) < 0$ this is trivially true, since $x + \varepsilon X(x) \in F$ for small $\varepsilon > 0$. If $q(x) = dq(x) = 0$, (2) implies that $X(x) = V_*(x)$ so

$$\frac{\operatorname{dist}(x + \varepsilon X(x), F)}{\varepsilon} = \frac{\operatorname{dist}(x + \varepsilon V_*(x), F)}{\varepsilon}$$

and the right-hand side $\to 0$ as $\varepsilon \searrow 0$ because F is V_*-invariant.

If $q(x) = 0$ and $dq(x) \ne 0$, the set $\{q(y) = 0\} \cap W$ is a C^1 manifold where W is a convenient ball centered at x. It is easy to find a smooth unit vector field $N(y)$ that meets $\{q(y) = 0\}$ transversally and points toward $\{q(y) < 0\}$, so $Nq < 0$ on $W \cap \{q(y) = 0\}$. Let $\Psi_t(y, \lambda)$ denote the flow of the vector $X^\lambda \doteq X + \lambda N$, $\lambda \ge 0$. Then, (1) implies that $X^\lambda q(y) < 0$ on $W \cap \{q(y) = 0\}$ for any $\lambda > 0$.

Note that no integral curve of X^λ, $\lambda > 0$, that stems from a point in $W \cap \{q(y) < 0\}$ can cross $W \cap \{q(y) = 0\}$ (this would amount to traveling against the flow at $W \cap \{q(y) = 0\}$) and this implies that $q(\Psi_t(x,\lambda)) < 0$ for $\lambda > 0$, $t > 0$ small, in particular $\Psi_t(x,\lambda) \in U^-$. Hence, $\Psi_t(x,0) = \lim_{\lambda \searrow 0} \Psi_t(x,\lambda) \in F$ where the limit holds by the continuous dependence on the parameter λ. Thus, the flow $\Psi_t(x,0)$ of X does not exit F for small values of $t > 0$, which easily implies (IV.82), as in the proof of '(i) $\Longrightarrow$ (ii)' of the characterization of flow-invariant sets. $\square$

Notes

A few years after the publication of Hans Lewy's example [**L1**], Hörmander ([**H6**], [**H7**]) shed new light on the nonsolvability phenomenon explaining it in a novel way. Although his results are set in the framework of general order operators of principal type we will describe its consequences for vector fields. He proved that if a (nonvanishing) vector field L is locally solvable in Ω then the principal symbol of the commutator $[L, \overline{L}]$ between L and its conjugate must vanish at every zero of the principal symbol $\ell(x,\xi)$ of L. A vector field with this property is said to satisfy condition $(\mathcal{H})$. For the Lewy operator condition $(\mathcal{H})$ is violated at every point. If the coefficients of L are real or constant $[L, \overline{L}]$ vanishes identically. This was a most remarkable advance because it explained a phenomenon that had appeared as an isolated example in terms of very general geometric properties of the symbol, an invariantly defined object. However, it turns out that condition $(\mathcal{H})$ does not tell apart the solvable vector fields from the nonsolvable ones among some examples considered by Mizohata ([**M**]), which we now describe. Let k be a positive integer and consider the vector field in $\mathbb{R}^2$ defined by

$$M_k = \frac{\partial}{\partial y} - iy^k \frac{\partial}{\partial x}.$$

If $k = 1$ condition $(\mathcal{H})$ is violated at all points of the x-axis so, in particular, M_1 is not locally solvable at the origin. For $k \geq 2$ condition $(\mathcal{H})$ is satisfied everywhere. On the other hand, it follows from relatively simple arguments that M_k is locally solvable at the origin if and only if k is even ([**Gr**], [**Ga**]). The principal symbol of M_k is $m_1 = -i(\eta - iy^k\xi)$. The crucial difference between k odd and k even is that in the first case the function y^k changes sign and in the second case it doesn't. Nirenberg and Treves ([**NT**]) elaborated these examples and identified a property that turned out to be the right condition for local solvability of vector fields, i.e., condition $(\mathcal{P})$. When L

satisfies $(\mathcal{P})$ the arguments in [**NT**] allow $Lu = f$ to be solved locally with u in the Sobolev space $L^{2,-1}$ for $f \in L^2$. This result was improved by Treves ([**T2**]) to L^2 solvability, i.e., u can be taken in L^2. Concerning the regularity of the coefficients, it was shown in [**Ho1**] that if L is in the canonical form

$$Lu = \frac{\partial u}{\partial t} + i \sum_{j=1}^{n} b_j(x, t) \frac{\partial u}{\partial x_j}, \tag{a}$$

with b_j real-valued and Lipschitz and satisfies $(\mathcal{P})$ then it is locally solvable in L^2. Since there is loss of one derivative in the process of obtaining coordinates in which L has this form one must require, in general, that derivatives up to order one of the coefficients of L be Lipschitz. However, in two variables (i.e., when $n = 1$) it is possible to prove L^2 solvability directly without assuming that L is in the special form (a) ([**HM1**]). Hence, planar vector fields with Lipschitz coefficients that satisfy $(\mathcal{P})$ are locally solvable in L^2. This result is essentially sharp in the sense that there are counterexamples to L^2 solvability and to the existence of L^2 a priori estimates if the coefficients are only restricted to belong to the Hölder class C^α for any $0 < \alpha < 1$ ([**J1**], [**HM1**], [**HM2**]). Whether any vector field with Lipschitz coefficients that satisfies $(\mathcal{P})$ in three or more variables is locally solvable in L^2 is an open problem at the time of this writing.

It is a characteristic feature of locally solvable operators of order one that the L^2 a priori estimates that they satisfy can be extended to L^p estimates for $1 < p < \infty$, a fact that turns out to be false for second-order operators in three or more variables (for results in that direction see [**Li**], [**K**], [**KT1**], [**KT2**], [**Gu**], [**Ch1**]). Solvability in L^p for vector fields was first considered in [**HP**], where the method involved pseudo-differential operators and demanded smooth coefficients. On the other hand, using the method of H. Smith ([**Sm**]), L^p a priori estimates in the range $1 < p < \infty$ can be proved in one stroke under the same regularity hypothesis on the coefficients initially known to guarantee just L^2 estimates ([**HM2**]). This is the point of view used in the presentation of a priori estimates in this book, although for simplicity we have not included the proof that in two variables L^p estimates for vector fields with Lipschitz coefficients are valid without assuming they are in the canonical form (a) ([**HM2**]). The proof of a priori estimates in several variables is reduced, thanks to the geometry of $(\mathcal{P})$ that prevents the existence of orbits of dimension higher than 2, to two-dimensional a priori estimates that are glued by a partition of unity associated with a convenient Whitney decomposition in cubes. The presentation in this chapter owes much to the discussion in [**S1**] about decomposition of open sets in cubes.

While it is true that for any locally solvable vector field L and $1 < p < \infty$ the equation $Lg = f$ can locally be solved in L^p if f is in L^p, this is false, in general, for $p = \infty$ as we saw in the example after Remark IV.1.12 that was taken from [**HT2**]. This difficulty can be dealt with by introducing the space $X = L^\infty(\mathbb{R}_t; \mathrm{bmo}(\mathbb{R}_x))$ of measurable functions $u(x, t)$ such that, for almost every $t \in \mathbb{R}$, $x \mapsto u(x, t) \in \mathrm{bmo}(\mathbb{R})$ and $\|u(t, \cdot)\|_{\mathrm{bmo}} \leq C < \infty$ for a.e. $t \in \mathbb{R}$, where $\mathrm{bmo}(\mathbb{R})$ is a space of bounded mean oscillation functions, dual to the semilocal Hardy space $h^1(\mathbb{R})$ of Goldberg. This was first observed in [**BHS**], where it is proved that for a substantial subclass of the class of locally solvable vector fields L, the equation $Lu = f$ can be locally solved with $u \in X$ if $f \in L^\infty$. This result was later improved by showing that for any locally solvable vector field L the equation $Lu = f$ can be locally solved with $u \in X$ for any $f \in X$ ([**daS**], [**HdaS**]) which can be regarded as an ersatz for $p = \infty$ of the L^p local solvability valid for $1 < p < \infty$. The presentation in Section IV.1.2 follows closely [**HdaS**] but replaces lemma 4.5 of that paper—which is true but incorrectly proved—by Lemma IV.1.17 which is sharper.

A priori estimates in L^2 easily give a priori estimates in $L^{2,s}$ for any $s \in \mathbb{R}$ but the absorption of lower-order terms requires shrinking of the neighborhood in which the estimate holds in a way that makes its diameter tend to zero when $|s| \to \infty$. Therefore, the technique of a priori estimates gives solutions of arbitrary high but finite regularity for smooth right-hand sides. Using a different approach, Hörmander ([**H9**]) proved solvability for differential operators of arbitrary order that satisfy $(\mathcal{P})$ by studying the propagation of singularities of the equation $Pu = 0 \mod (C^\infty)$, showing the existence of semiglobal solutions, i.e., solutions defined on a full compact set under the geometric assumption that bicharacteristics do not get trapped in the given compact set. Furthermore, the solutions can be taken smooth if f is smooth. In Sections IV.2 and IV.3 of this chapter, the construction of smooth solutions is simplified by the assumption that the vector fields are locally integrable. Since vector fields that satisfy $(\mathcal{P})$ are indeed locally integrable, the local integrability hypothesis is superfluous, however this fact depends on the difficult and long theorems on smooth solvability by Hörmander ([**H9**], [**H5**]). Thus, it would be interesting to have a shorter *ad hoc* proof of the local existence of smooth solutions for vector fields that satisfy $(\mathcal{P})$ without invoking local integrability.

Concerning the necessity of $(\mathcal{P})$, Nirenberg and Treves had shown in their seminal paper [**NT**] that local solvability implies $(\mathcal{P})$ for vector fields with real-analytic coefficients and conjectured the same implication should hold for smooth coefficients. This state of affairs remained unchanged for 15 years

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until Moyer ([**Mo**]) removed in 1978 the analyticity hypothesis for operators in two variables in a never published manuscript. His ideas, however, were applied by Hörmander [**H4**] to extend the result for operators in any number of variables with smooth coefficients. The discussion of the necessity of $(\mathcal{P})$ in Section IV.4 of this chapter is again simplified by the assumption of local integrability and follows the presentation in [**T3**] (see also [**T5**] and [**CorH2**]).