Can. J. Math.Vol. 45 (5), 1993 pp. 1032–1066

DIXMIER-DOUADY CLASSES OF DYNAMICAL SYSTEMS AND CROSSED PRODUCTS

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ABSTRACT Continuous-trace C^*-algebras A with spectrum T can be characterized as those algebras which are locally Morita equivalent to $C_0(T)$ The Dixmier-Douady class $\delta(A)$ is an element of the Čech cohomology group $\check{H}^3(T,\mathbb{Z})$ and is the obstruction to building a global equivalence from the local equivalences Here we shall be concerned with systems (A,G,α) which are locally Morita equivalent to their spectral system $(C_0(T),G,\tau)$, in which G acts on the spectrum T of A via the action induced by α Such systems include locally unitary actions as well as N-principal systems Our new Dixmier-Douady class $\delta(A,G,\alpha)$ will be the obstruction to piecing the local equivalences together to form a Morita equivalence of (A,G,α) with its spectral system Our first main theorem is that two systems (A,G,α) and (B,G,β) are Morita equivalent if and only if $\delta(A,G,\alpha)=\delta(B,G,\beta)$ In our second main theorem, we give a detailed formula for $\delta(A\rtimes_\alpha G)$ when (A,G,α) is N-principal

1. **Introduction.** The Dixmier-Douady class of a continuous-trace C^*-algebra A with spectrum T is a class $\delta(A)$ in the Čech cohomology group $\check{H}^3(T,\mathbb{Z})$, which for separable algebras determines A up to spectrum-preserving stable isomorphism. Since every class arises, there is for each $\delta\in\check{H}^3(T,\mathbb{Z})$ an essentially unique stable continuous-trace C^*-algebra, and this realization of cohomology classes has found a variety of uses (*e.g.*, [21, 22]). The class $\delta(A)$ also determines A up to spectrum-preserving Morita equivalence; while this interpretation does not isolate a unique representative for each class, it does avoid the use of non-canonical stable isomorphisms, and hence gives a more natural theory.

It is tempting to extend the Dixmier-Douady theory to cover dynamical systems (A,G,α) involving continuous-trace algebras, and indeed this has already been done in [9] for $G=\mathbb{Z}/2\mathbb{Z}$, and in [8] for G discrete. Here we want to discuss a cocycle-based theory which is particularly well-suited to the N-principal systems (Definition 4.4) studied in [13, §2] and [17], where the main examples involve locally compact groups rather than discrete ones; our invariant classifies a system up to the Morita equivalence of [3, 2]. We shall then use it to describe the crossed product $A\rtimes_\alpha G$ of an N-principal system (A,G,α), thus satisfactorily completing our analysis of these systems in [17].

The continuous-trace C^*-algebras with spectrum T can be characterized as the algebras which are locally Morita equivalent to the commutative algebra $C_0(T)$, and the Dixmier-Douady class is the obstruction to building a global equivalence with $C_0(T)$ from the local equivalences. We shall be concerned with systems (A,G,α) which are locally Morita equivalent to their *spectral system* $\big(C_0(T),G,\tau\big)$, in which G acts on the spectrum T of A via the action induced by α: this means there are local equivalences

Received by the editors January 13, 1992
AMS subject classification 46L55, 46L05

between A and $C_0(T)$ carrying actions of G compatible with the actions on A and $C_0(T)$. These systems include most of the ones we have been studying over the past ten years. For example, if G acts trivially on T, (A, G, α) is locally Morita equivalent to $\big(C_0(T), \tau\big)$ exactly when α is locally unitary as in [10, 15]; if T is a locally trivial principal bundle for some quotient G/N of G, then (A, G, α) is locally Morita equivalent to $\big(C_0(T), \tau\big)$ if and only if it is N-principal as in [16, 17]. If (A, G, α) is locally Morita equivalent to $\big(C_0(T), \tau\big)$, our new Dixmier-Douady class $\delta(A, G, \alpha)$ will be the obstruction to piecing the local equivalences together to form a Morita equivalence of (A, G, α) with its spectral system $\big(C_0(T), \tau\big)$.

The obstruction $\delta(A, G, \alpha)$ necessarily involves a cocycle ν representing the usual Dixmier-Douady class $\delta(A)$, and data λ codifying the action of G: we realize $\delta(A)$ via a 2-cocycle ν with values in the sheaf $\mathcal{S}$ of circle-valued functions (using the isomorphism $\check{H}^2(T, \mathcal{S}) \cong \check{H}^3(T, \mathbb{Z})$), and think of the combination (λ, ν) as an element of an equivariant cohomology group $\check{H}^2_G(T, \mathcal{S})$. Our first main theorem identifies this equivariant cohomology group with the Morita equivalence classes of systems locally Morita equivalent to $\big(C_0(T), \tau\big)$. We intend to discuss the topological properties of this group elsewhere, but include a few brief comments in our final section. For example, when G acts trivially on T, $\check{H}^2_G(T, \mathcal{S})$ is easy to compute directly, and our classification theorem quickly yields the results of [10] and [9].

If (A, G, α) is an N-principal system with spectrum $p \colon T \longrightarrow T/G$, the spectrum of $A \rtimes_\alpha G$ is a principal $\hat{N}$-bundle over T/G, which fits into a commutative diagram

$$\begin{array}{ccccc} & & (A \times_\alpha N)^\wedge & & \\ & \text{ind} \swarrow & & \searrow \text{Res} & \\ (A \rtimes_\alpha G)^\wedge & & & & T \\ & q \searrow & & \swarrow p & \\ & & T/G & & \end{array}$$

of principal bundles [13, Theorem 2.2]. In [17], we characterized the $\hat{N}$-bundles which could arise as q, and our constructions imposed restrictions on the possible values of $\delta(A \rtimes_\alpha G)$; we were not, however, able to determine $\delta(A \rtimes_\alpha G)$ itself. In our second main theorem, we shall give a detailed formula for a cocycle representing $\delta(A \rtimes_\alpha G)$. In fact, we shall do better: the dual system $(A \rtimes_\alpha G, \hat{G}, \hat{\alpha})$ is $N^\perp$-principal, and we have written down a cocycle representing $\delta(A \rtimes_\alpha G, \hat{G}, \hat{\alpha})$.

We begin with a short section on preliminaries, in which we review the basic properties of imprimitivity bimodules and Morita equivalence. We then discuss the Morita equivalence approach to the Dixmier-Douady theory. Unfortunately, although this was worked out over ten years ago by several different mathematicians (*e.g.*, [7, 1]), the details have never appeared. In Section 3, we have not repeated arguments which are later provided in the equivariant case, but have otherwise tried to be complete.

Our main program starts in Section 4, where we investigate the local Morita equivalence of systems. Things are not quite as straightforward as in Section 3: in particular, we need to know that two Morita equivalences of the same systems are locally isomorphic, and we have to do some work to show that this is always true for N-principal systems. We tackle this by isolating a topological property of the transformation group (T, G) which implies the required local uniqueness, which will appear as a hypothesis in our classification theorem, and which is automatically satisfied if $T \longrightarrow T/G$ is a principal G/N-bundle. After these technicalities have been dealt with, and we have introduced our equivariant cohomology groups in Section 5, we discuss our Dixmier-Douady invariant in Section 6. Our procedure is similar to that of [4, Chapter 10], except that we use techniques for constructing imprimitivity bimodules developed in [11] in place of the C^*-bundle constructions of [4], and use [14] to produce systems with arbitrary Dixmier-Douady class. The main classification theorem is Theorem 6.3.

Section 7 contains our results on N-principal systems, and Theorem 7.3 our calculation of $\delta(A \rtimes_\alpha G)$. We verify that our formula is consistent with the restrictions imposed in [17], and that it gives the answer obtained in [12] for the special case in which $\hat{A}$ is Y/N for some principal G-bundle Y.

ACKNOWLEDGMENT. This research was supported by the Australian Research Council. Part of the work was done while we were both visiting Denmark in 1990, and we thank all our colleagues there for their warm hospitality.

2. **Preliminaries.** We begin by reviewing the basic properties of the imprimitivity bimodules of Rieffel [19, 20]. If A and B are C^*-algebras, an $A - B$-*imprimitivity bimodule* is an $A - B$-bimodule $\mathcal{X}$ equipped with A- and B-valued inner products, denoted ${}_A\langle \cdot, \cdot \rangle$, $\langle \cdot, \cdot \rangle_B$ respectively, and if there is such an $\mathcal{X}$ we say A and B are *Morita equivalent*. Modulo a slight change of notation for the A-valued inner product, we use the list of axioms given in [11, pp. 184–185, Equations (1)–(6)]. As in [11], we shall further assume that the seminorm $\|x\| = \|\langle x, x\rangle_B\|^{1/2} = \|{}_A\langle x, x\rangle\|^{1/2}$ is actually a norm, and that $\mathcal{X}$ is complete in this norm.

There are two key examples which help to fix the ideas. First, a Hilbert space H is a $\mathcal{K}(H) - \mathbb{C}$-imprimitivity bimodule, with the natural left action of $\mathcal{K} = \mathcal{K}(H)$, the usual $\mathbb{C}$-valued inner product, and ${}_{\mathcal{K}}\langle h, k\rangle$ the rank-one operator $h \otimes \bar{k}$. Second, a C^*-algebra A is itself an $A - A$-imprimitivity bimodule, with A acting by left and right multiplication, and ${}_A\langle a, b\rangle = ab^*$, $\langle a, b\rangle_A = a^*b$.

The actions of A, B on an $A - B$ imprimitivity bimodule $\mathcal{X}$ extend to actions of the multiplier algebras $M(A), M(B)$, characterized by

$$m(a \cdot x) = (ma) \cdot x, \quad \text{and} \quad (x \cdot b)n = x \cdot (bn).$$

The action of $M(A)$ induces an isomorphism of $M(A)$ onto the C^*-algebra $\mathcal{L}(\mathcal{X}_B)$ of bounded B-linear operators on $\mathcal{X}$ [6, Lemma 16]; if $T \in \mathcal{L}(\mathcal{X}_B)$, the corresponding multiplier $m_T \in M(A)$ is characterized by

$$m_T({}_A\langle x, y\rangle) \cdot z = T(x) \cdot \langle y, z\rangle_B \quad \text{for } x, y, z \in \mathcal{X}.$$

Of course, it is standard practice to confuse m_T and T. We shall need to know later that the *-strong topology on $\mathcal{L}(\mathcal{X}_B)$ and the strict topology on $M(A)$ agree on bounded subsets. The boundedness here is crucial: in general, strict convergence implies *-strong convergence, but not vice-versa. To see this, suppose $m_\alpha \longrightarrow m$ strictly and $x \in \mathcal{X}$; without loss of generality, $m = 0$. As ${}_A\langle x, x\rangle$ is positive in A, there exists $a \in A$ such that ${}_A\langle x, x\rangle = aa^*$, and then

$$\|m_\alpha(x)\|^2 = \|m_\alpha({}_A\langle x, x\rangle)m_\alpha^*\| = \|(m_\alpha a)(m_\alpha a)^*\| = \|m_\alpha a\|^2 \longrightarrow 0.$$

Because $m_\alpha \longrightarrow m$ strictly exactly when $m_\alpha^* \longrightarrow m^*$ strictly, this proves that $m_\alpha \longrightarrow m$ *-strongly. On the other hand, it is easy to see that if $m_\alpha \longrightarrow m$ *-strongly, then $m_\alpha a \longrightarrow ma$ for all a which are linear combinations of elements of the form ${}_A\langle x, y\rangle$. Such a are dense in A, but to make the necessary approximation argument work, we need to know that $\|m_\alpha\|$ is bounded.

We are interested in classifying algebras with given spectrum T, and actions on these algebras which induce a given action of G on T. Many of our arguments involve localizing, and for this to work, our imprimitivity bimodules must respect these identifications of spectra. The mechanism for this was worked out in [11], and goes roughly as follows.

It was shown by Rieffel in [19] that an $A - B$-imprimitivity bimodule $\mathfrak{X}$ induces a homeomorphism $h_{\mathfrak{X}}$ of $\hat{A}$ onto $\hat{B}$. If $\hat{A}$ and $\hat{B}$ have both been identified with a fixed locally compact Hausdorff space T, and $h_{\mathfrak{X}}$ is the identity, we shall call $\mathfrak{X}$ an *$A -_T B$-imprimitivity bimodule*; this is equivalent to saying that the left and right actions of $C_b(T)$ on $\mathfrak{X}$ obtained by embedding $C_b(T)$ in $M(A)$ and $M(B)$ coincide [11, Proposition 1.11]. If there is such a bimodule $\mathfrak{X}$, we say A and B are *Morita equivalent over T*. We stress that this is stronger than ordinary Morita equivalence: for example, suppose h is an orientation-reversing homeomorphism of S^3, such as a reflection of determinant -1, and A is a continuous-trace algebra whose Dixmier-Douady class $\delta(A)$ generates $\check{H}^3(S^3, \mathbb{Z}) \cong \mathbb{Z}$. Then the pull-back h^*A satisfies $\delta(h^*A) = h^*\big(\delta(A)\big) = -\delta(A)$ [15, 1.4], and hence is not Morita equivalent to A over S^3 (see Theorem 3.5 below). But the C^*-algebras A and h^*A are isomorphic: by definition, h^*A is the balanced tensor product $C(S^3) \otimes_{C(S^3)} A$, where the right action of $C(S^3)$ on itself is given by $f \cdot g = f(g \circ h)$, and $h \otimes \mathrm{i}$ is an isomorphism of h^*A onto $A = C(S^3) \otimes_{C(S^3)} A$ (where $C(S^3)$ acts normally on itself).

If $\mathfrak{X}$ is an $A -_T B$-imprimitivity bimodule, a closed subset F of T determines ideals A_F, B_F in A, B such that the spectra of the quotients $A^F = A/A_F$, $B^F = B/B_F$ can be naturally identified with F. The subspace

$$\mathfrak{X}_F = \{x \in \mathfrak{X} : \langle x, x\rangle_B \in B_F\}$$

of the $A -_T B$-bimodule $\mathfrak{X}$ is then an $A_F -_{T\setminus F} B_F$-imprimitivity bimodule, and the quotient $\mathfrak{X}^F = \mathfrak{X}/\mathfrak{X}_F$ is then naturally an $A^F -_F B^F$-imprimitivity bimodule [20, §3; 11, Proposition 1.7]. We use the obvious notation x^F for the image of x in $\mathfrak{X}^F$, and similarly $a^F \in A^F$, $b^F \in B^F$; thus, almost by definition, we have

$$a^F \cdot x^F = (a \cdot x)^F, \quad {}_{A^F}\langle x^F, x^F\rangle = ({}_A\langle x, x\rangle)^F, \ etc.$$

If $t \in T$, we write $A(t)$, $\mathfrak{X}(t)$, $x(t)$ for $A^{\{t\}}$, $\mathfrak{X}^{\{t\}}$, $x^{\{t\}}$. We remark that it seems to be substantially more convenient to localize by passing to the quotients associated to compact neighborhoods in T rather than the ideals associated to open neighborhoods, even though for continuous-trace algebras the two approaches are technically equivalent.

Next, we recall some constructions involving imprimitivity bimodules. If $\mathfrak{X}$ and $\mathfrak{Y}$ are, respectively, $A -_T B$- and $B -_T C$-imprimitivity bimodules, then there are well-defined pairings on the algebraic tensor product $\mathfrak{X} \odot \mathfrak{Y}$ satisfying

$$\langle\langle x \otimes y, x' \otimes y' \rangle\rangle_C = \big\langle \langle x', x\rangle_B y, y' \big\rangle_C$$
$$_A\langle\langle x \otimes y, x' \otimes y' \rangle\rangle = {}_A\big\langle x, x'\, {}_B\langle y', y\rangle \big\rangle,$$

and completing gives an $A -_T C$-imprimitivity bimodule $\mathfrak{X} \otimes_B \mathfrak{Y}$ [11, 1 3] The notation $\mathfrak{X}\otimes_B\mathfrak{Y}$ (rather than $\mathfrak{X}\otimes\mathfrak{Y}$) is chosen to remind us that completing $\mathfrak{X}\odot\mathfrak{Y}$ involves modding out lots of vectors of length 0, and in particular those of the form $x \cdot b \otimes y - x \otimes b \cdot y$, so that we can think of $\mathfrak{X} \otimes_B \mathfrak{Y}$ as a balanced tensor product We stress that $\mathfrak{X} \otimes_B \mathfrak{Y}$ is not a Banach space tensor product in the usual sense, although it does follow from the Cauchy-Schwartz inequality [19, 2 9] that

$$(2\ 1) \qquad \|x \otimes y\| \le \|x\|\|y\|,$$

and we shall use this frequently

If $\mathfrak{X}$ is an $A -_T B$-imprimitivity bimodule, the dual $\tilde{\mathfrak{X}}$ of $\mathfrak{X}$ is the set $\{\tilde{x} : x \in \mathfrak{X}\}$, made into a $B -_T A$-imprimitivity bimodule as follows

$$b \cdot \tilde{x} = (x \cdot b^*)^\sim \quad \tilde{x} \cdot a = (a^* \cdot x)^\sim$$
$$_B\langle \tilde{x}, \tilde{y}\rangle = \langle x, y\rangle_B \quad \langle \tilde{x}, \tilde{y}\rangle_A = {}_A\langle x, y\rangle,$$

for $x, y \in \mathfrak{X}$, $a \in A$, and $b \in B$ The idea is that $\tilde{\mathfrak{X}}$ is an inverse for $\mathfrak{X}$ formally, the map $x \otimes \tilde{y} \longrightarrow {}_A\langle x, y\rangle$ is an $A - A$ imprimitivity bimodule isomorphism of $\mathfrak{X} \otimes_B \tilde{\mathfrak{X}}$ onto A, and similarly $\tilde{\mathfrak{X}} \otimes_A \mathfrak{X} \cong B$

Finally, we shall need to use some basic sheaf cohomology, we adopt the conventions of [23, §5 33], and view cohomology classes in terms of Čech cocycles Throughout, $\mathcal{S}$ and $\mathcal{R}$ denote the sheaves of germs of continuous $\mathbb{T}$- and $\mathbb{R}$-valued functions, respectively Since the sheaf $\mathcal{R}$ is fine, the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{R} \longrightarrow \mathcal{S} \longrightarrow 1$$

of sheaves induces isomorphisms $\check{H}^p(T, \mathcal{S}) \cong \check{H}^{p+1}(T, \mathbb{Z})$, which we shall use without comment Thus, for example, we view the Dixmier-Douady class $\delta(A)$ of a continuous-trace algebra A as lying in $\check{H}^2(T, \mathcal{S})$ or $\check{H}^3(T, \mathbb{Z})$, according to convenience In general, if a Roman letter, such as G, denotes a locally compact group, we use the corresponding script letter $\mathcal{G}$ for the sheaf of germs of continuous G-valued functions

3 **Morita equivalence of continuous-trace C^*-algebras.** A C^*-algebra A with Hausdorff spectrum T is a continuous-trace algebra if for each $t_0 \in T$, there is a neighborhood N of t_0 and an element a of A such that $a(t)$ is a rank-one projection for all $t \in N$ [4, §4 5 4] Equivalently

PROPOSITION 3 1 *A C^*-algebra with Hausdorff spectrum T has continuous trace if and only if A is locally Morita equivalent to $C_0(T)$, in the sense that each point $t \in T$ has a closed neighborhood F such that A^F is Morita equivalent to $C_0(F)$ over F*

LEMMA 3 2 *Suppose X is an $A -_T C_0(T)$ imprimitivity bimodule, and $x \in X$ satisfies $\langle x, x\rangle_{C_0(T)}(t) = 1$ for all t in a closed set K Then ${}_A\langle x, x\rangle(t)$ is a rank one projection for all $t \in K$*

PROOF Note first that if we localize to the set $\{t\}$, then the map $f \longrightarrow f(t)$ induces an isomorphism of $C_0(T)(t)$ onto $\mathbb{C}$, and the quotient $\mathfrak{X}(t)$ is a Hilbert space, the action of $A(t)$

on $\mathfrak{X}(t)$ identifies $A(t)$ with the algebra $\mathcal{K}\big(\mathfrak{X}(t)\big)$, and provides a concrete representation realizing the class $t \in \hat{A}$. But since $x(t)$ is a unit vector in the Hilbert space $\mathfrak{X}(t)$ for all $t \in K$, ${}_{A(t)}\langle x(t), x(t)\rangle = ({}_{A}\langle x, x\rangle)(t)$ is a rank-one projection for $t \in K$. ∎

PROOF OF PROPOSITION 3.1. Since any $A -_T C_0(T)$-imprimitivity bimodule is *a fortiori* a $C_0(T)$-module, it is easy to construct elements x such that $\langle x, x\rangle_{C_0(T)} \equiv 1$ throughout any given compact set, and hence the lemma gives one direction. Conversely, suppose that A has continuous trace, and fix $t_0 \in T$. Choose a compact neighborhood F of t_0 and $p \in A$ such that $p(t)$ is a rank-one projection for all $t \in F$. Then p^F is a projection in A^F, and $A^F p^F$ is an $A^F - p^F A^F p^F$-imprimitivity bimodule with what we claim are the obvious module actions and

$$ {}_{A^F}\langle a^F p^F, b^F p^F\rangle = a^F p^F (b^F)^*, \quad \langle a^F p^F, b^F p^F\rangle_{p^F A^F p^F} = p^F (a^F)^* b^F p^F. $$

Note that the map $f \longrightarrow f p^F$ is an isomorphism of $C(F)$ *into* $p^F A^F p^F$. On the other hand, if $a \in A$ and $t \in F$, then $p(t)$ is a rank-one projection in the algebra $A(t)$ of compact operators, and $(pap)(t) = p(t)a(t)p(t)$ must be a scalar multiple $f_a(t)p(t)$ of $p(t)$. We claim that f_a is continuous, so that $f \longrightarrow f p^F$ is an isomorphism of $C(F)$ *onto* $p^F A^F p^F$. Well, for $t, s \in F$ we have

$$ |f_a(t) - f_a(s)| = \big\|\big(f_a(t) - f_a(s)\big)p(t)\big\| = \big\|\big(pap - f_a(s)p\big)(t)\big\|. $$

Since $\hat{A}$ is Hausdorff, for fixed s the right-hand side is a continuous function of t which vanishes at s, and hence the left-hand side goes to 0 as $t \longrightarrow s$—in other words, f_a is continuous, as claimed, and $A^F p^F$ is an $A^F - C(F)$-imprimitivity bimodule. Because the actions of $C(F)$ on the left and right of $A^F p^F$ clearly coincide, it is actually an $A^F -_F C(F)$-imprimitivity bimodule. ∎

Proposition 3.1 both identifies the continuous-trace C^*-algebras as those *locally* Morita equivalent to commutative algebras, and raises the key structural question for these algebras: when is a continuous-trace algebra A with spectrum T *globally* Morita equivalent to $C_0(T)$? The Dixmier-Douady invariant $\delta(A) \in \check{H}^2(T, \mathcal{S})$ completely solves this problem: $\delta(A) = 0$ exactly when A is Morita equivalent to $C_0(T)$. We shall now outline how this works, omitting detailed proofs of those parts which are special cases of results in Section 4.

Although we shall use imprimitivity bimodules rather than the continuous fields of C^*-algebras which are fundamental in Dixmier's treatment [4, Chapter 10], we should point out that the two approaches are equivalent. For if $\mathcal{H}$ is a continuous field of Hilbert spaces over T, the space $\Gamma_0(\mathcal{H})$ of sections implements a Morita equivalence between $C_0(T)$ and the C^*-algebra $A = \Gamma_0\big(\mathcal{A}(\mathcal{H})\big)$ defined by $\mathcal{H}$ [11, 1.1]; further, every $A -_T C_0(T)$-imprimitivity bimodule $\mathfrak{X}$ gives rise to a continuous field $\mathcal{H}$ with fibres $\mathcal{H}(t) = \mathfrak{X}(t)$ and $\mathfrak{X} = \Gamma_0(\mathcal{H})$, and hence they all arise this way. While we feel the algebraic approach is more elegant, and potentially more powerful, than the original, we do find it helpful to think in terms of bundles; thus, for example, one can profitably think of the quotient map $\mathfrak{X} \longrightarrow \mathfrak{X}^F$ as restriction of sections to the subset F of T.

The key observations for the construction of the Dixmier-Douady class are that the bimodules implementing the local Morita equivalences are locally isomorphic, and that these local isomorphisms are unique up to multiplication by functions in $C(T, \mathbb{T})$.

LEMMA 3.3 (*cf.* [4, 10.7.10]). *Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ are $A -_T C_0(T)$-imprimitivity bimodules. Then each $t_0 \in T$ has a a closed neighborhood F such that there is an isomorphism $g: \mathfrak{X}^F \longrightarrow \mathfrak{Y}^F$ of $A^F -_F C_0(F)$-imprimitivity bimodules.*

PROOF. We begin by choosing a compact neighborhood K of t_0 and $x \in \mathfrak{X}$ such that $\langle x, x\rangle_{C_0(T)}(t) = 1$ for $t \in K$. Then by Lemma 3.2, ${}_A\langle x, x\rangle(t)$ is a rank-one projection for all $t \in K$. We consider the action of ${}_A\langle x, x\rangle$ on $\mathfrak{Y}$. As in the proof of Lemma 3.2, for each $t \in K$, the action of A on $\mathfrak{Y}(t)$ identifies $A(t)$ with $\mathcal{K}\big(\mathfrak{Y}(t)\big)$, and gives a concrete realization of the representation t; thus we can find $y \in \mathfrak{Y}$ such that

$$({}_A\langle x, x\rangle \cdot y)(t_0) = \big({}_A\langle x, x\rangle(t_0)\big) \cdot y(t_0) \neq 0.$$

Since ${}_A\langle x, x\rangle(t)$ is a projection, we have

$${}_A\langle x, x\rangle(t)({}_A\langle x, x\rangle \cdot y)(t) = ({}_A\langle x, x\rangle \cdot y)(t) \quad \text{for } t \in K,$$

and we can therefore multiply ${}_A\langle x, x\rangle \cdot y$ by an appropriate continuous function to find an element z of $\mathfrak{Y}$ satisfying $\|z(t)\| = 1$ and $({}_A\langle x, x\rangle \cdot z)(t) = z(t)$ for t in a neighborhood F of t_0. The last condition implies that $z(t)$ is a unit vector in the range of the rank-one projection ${}_A\langle x, x\rangle(t)$; hence ${}_A\langle x, x\rangle(t)$ and ${}_A\langle z, z\rangle(t)$ are equal, being orthogonal projections with the same range. Thus we have ${}_{A^F}\langle x^F, x^F\rangle = {}_{A^F}\langle z^F, z^F\rangle$. For notational convenience, we selectively drop the superscript F.

We claim that we can define $g: \mathfrak{X}^F \longrightarrow \mathfrak{Y}^F$ by $g(a \cdot x) = a \cdot z$ for $a \in A^F$. Since every $w \in \mathfrak{X}^F$ can be written ${}_{A^F}\langle w, x\rangle \cdot x$, to see that g is well-defined it is enough to show $\|a \cdot x\| = \|a \cdot z\|$. But this is easy: for $a \in A^F$ we have

$$\|a \cdot z\|^2 = \|{}_{A^F}\langle a \cdot z, a \cdot z\rangle\| = \|a\, {}_{A^F}\langle z, z\rangle a^*\| = \|a\, {}_{A^F}\langle x, x\rangle a^*\| = \|a \cdot x\|^2.$$

Exactly the same calculation shows that g preserves the inner products, and since g is clearly a bimodule homomorphism, g is the required isomorphism of imprimitivity bimodules. ∎

LEMMA 3.4. *Suppose $\mathfrak{X}$ is an $A -_T C_0(T)$-imprimitivity bimodule, and $g: \mathfrak{X} \longrightarrow \mathfrak{X}$ is an imprimitivity bimodule isomorphism. Then there is a function $\phi \in C(T, \mathbb{T})$ such that $g(x) = x \cdot \phi = \phi \cdot x$. (For the last equality, we view ϕ as a multiplier of A.)*

PROOF. We fix $t_0 \in T$, $x \in \mathfrak{X}$ such that $\langle x, x\rangle_{C_0(T)}(t_0) \neq 0$, and define $\phi_x(t)$ for t near t_0 by

$$\phi_x(t)\langle x, x\rangle_{C_0(T)}(t) = \langle x, g(x)\rangle_{C_0(T)}(t).$$

In fact ϕ_x is independent of the choice of x: if y also satisfies $\langle y, y\rangle_{C_0(T)}(t_0) \neq 0$, then

$$\begin{aligned}\langle x, g(x)\rangle_{C_0(T)}\langle g(y), y\rangle_{C_0(T)} &= \big\langle x, g(x) \cdot \langle g(y), y\rangle_{C_0(T)}\big\rangle_{C_0(T)} \\ &= \big\langle x, {}_A\langle g(x), g(y)\rangle \cdot y\big\rangle_{C_0(T)} \\ &= \big\langle x, {}_A\langle x, y\rangle \cdot y\big\rangle_{C_0(T)} \\ &= \langle x, x\rangle_{C_0(T)}\langle y, y\rangle_{C_0(T)},\end{aligned}$$

which implies both that $|\phi_x| = 1$ for t near t_0 (take $y = x$), and that $\phi_x(t) = \phi_y(t)$ whenever both are defined. Thus the ϕ_x combine to give a continuous function $\phi: T \longrightarrow \mathbb{T}$ such that

$$\langle x, x\cdot\phi\rangle_{C_0(T)}(t) = \phi(t)\langle x, x\rangle_{C_0(T)}(t) = \langle x, g(x)\rangle_{C_0(T)}(t) \quad \text{for all } t,$$

and the polarisation identity implies that $g(x) = x\cdot\phi$ for all $x \in \mathfrak{X}$. ∎

Now to define the Dixmier-Douady class of a continuous-trace algebra A, we use Proposition 3.1 to find a cover $\{N_\iota\}$ of T such that, for $F_\iota = \overline{N}_\iota$, there are $A^{F_\iota} -_{F_\iota} C_0(F_\iota)$-imprimitivity bimodules $\mathfrak{X}_\iota$. Using Lemma 3.3 and some standard trickery, we can refine the cover to ensure that on each intersection $F_{\iota\jmath} = F_\iota \cap F_\jmath$, there is an isomorphism $g_{\iota\jmath}: \mathfrak{X}_\jmath^{F_{\iota\jmath}} \longrightarrow \mathfrak{X}_\iota^{F_{\iota\jmath}}$ of $A^{F_{\iota\jmath}} -_{F_{\iota\jmath}} C_0(F_{\iota\jmath})$-imprimitivity bimodules (*cf.* [4, 10.7.11] and Lemma 6.1 below). On a triple intersection $F_{\iota\jmath k}$ we have two isomorphisms $g_{\iota\jmath}^{F_{\iota\jmath k}} \circ g_{\jmath k}^{F_{\iota\jmath k}}$ and $g_{\iota k}^{F_{\iota\jmath k}}$ of $\mathfrak{X}_k^{F_{\iota\jmath k}} = (\mathfrak{X}_k^{F_{\jmath k}})^{F_{\iota\jmath k}}$ onto $\mathfrak{X}_\iota^{F_{\iota\jmath k}}$, and Lemma 3.4 says there is a continuous function $\nu_{\iota\jmath k}: F_{\iota\jmath k} \longrightarrow \mathbb{T}$ such that

$$g_{\iota\jmath}^{F_{\iota\jmath k}} \circ g_{\jmath k}^{F_{\iota\jmath k}}(x) = g_{\iota k}^{F_{\iota\jmath k}}(x)\cdot\nu_{\iota\jmath k} \quad \text{for } x \in \mathfrak{X}_k^{F_{\iota\jmath k}}.$$

The functions $\{\nu_{\iota\jmath k}\}$ form a 2-cocycle with values in the sheaf $\mathcal{S}$ of germs of continuous $\mathbb{T}$-valued functions on T, and the class of $\{\nu_{\iota\jmath k}\}$ is independent of any of the choices we have made (*cf.* [4, 10.7.12] and Lemma 6.2 below). This class is the *Dixmier-Douady class* of A, and is denoted $\delta(A)$. In this setting, the Dixmier-Douady classification theorem of [5] becomes:

THEOREM 3.5. *Let A and B be continuous-trace C^*-algebras with spectrum T. Then A is Morita equivalent to B over T if and only if $\delta(A) = \delta(B)$ in $\check{H}^2(T, \mathcal{S})$. Every class in $\check{H}^2(T, \mathcal{S})$ is the Dixmier-Douady class of some continuous-trace algebra with spectrum T.*

This is the special case of Theorem 6.3 in which the group G is trivial. However, we point out that the necessity of $\delta(A) = \delta(B)$ is quite easy, and that the remaining parts are essentially in the literature already: for the sufficiency, one just borrows from [11, §2] the construction of imprimitivity bimodules from local data, and the last part is done explicitly in [14]. We stress that this result is well-known to the experts (*cf.*, *e.g.*, [1, §2.7]).

4. **Morita equivalence of systems.** We recall from [2] that two dynamical systems (A, G, α) and (B, G, β) are *Morita equivalent* if there is an $A - B$-imprimitivity bimodule $\mathfrak{X}$ and an action u of G on $\mathfrak{X}$ by linear transformations, which is strongly continuous (*i.e.*, $s \longmapsto u_s(x)$ is norm-continuous for all x) and satisfies

$$\alpha_s({}_A\langle x, y\rangle) = {}_A\langle u_s(x), u_s(y)\rangle \tag{4.1}$$
$$\beta_s(\langle x, y\rangle_B) = \langle u_s(x), u_s(y)\rangle_B.$$

If in addition $\mathfrak{X}$ is an $A -_T B$-imprimitivity bimodule, we say $(\mathfrak{X}, u)$ implements a *Morita equivalence of* (A, G, α) *and* (B, G, β) *over* T, or that $(\mathfrak{X}, u)$ is an $(A, \alpha) -_T (B, \beta)$-*imprimitivity bimodule.* (As in [3], it follows from Equation (4.1) that each u_s is isometric and that $u_s(a \cdot x) = \alpha_s(a) \cdot u_s(x)$, *etc.* Although checking the strong continuity of u is a nuisance, it is necessary: if we multiply $s \longrightarrow u_s(x)$ by a discontinuous character of G, we destroy the continuity of the action but Equation (4.1) is unaffected.)

If (A, G, α) is a system and A has Hausdorff spectrum T, we denote by τ the action of G on $C_0(T)$ induced by the action of G on T, and refer to $\big(C_0(T), G, \tau\big)$ as the *spectral system* of (A, G, α). Our Dixmier-Douady class will be the obstruction to piecing together local Morita equivalences of systems with the same spectral system to form a global Morita equivalence. As in Section 3, we first have to know how to localize: again, we prefer to localize to closed subsets F of T, but now they have to be G-invariant to ensure that there are local systems (A^F, G, α^F).

LEMMA 4.1. *Suppose that F is a closed G-invariant subset of T, and that (A, α) is Morita equivalent to $\big(C_0(T), \tau\big)$ over T via $(\mathfrak{X}, u)$. There is an action u^F of G on $\mathfrak{X}^F$ characterized by $u^F_s(x^F) = u_s(x)^F$, and then (A^F, α^F) is Morita equivalent to $\big(C_0(F), \tau\big)$ over F via $(\mathfrak{X}^F, u^F)$.*

PROOF. Since $u_s(x \cdot \phi) = u_s(x) \cdot \tau_s(\phi)$ for all $x \in \mathfrak{X}$ and $\phi \in C_0(T)$, it is immediate that $u_s(\mathfrak{X}_F) \subseteq \mathfrak{X}_F$. Hence $u^F_s(x^F) = u_s(x)^F$ uniquely defines u^F_s. The remainder of the lemma follows from [20; Corollaries 3.1 and 3.2] as in [11; §1]. ■

DEFINITION 4.2. Suppose that A is a C^*-algebra with Hausdorff spectrum T, and that (A, G, α) is a dynamical system with spectral system $\big(C_0(T), \tau\big)$. We say that (A, α) is *locally Morita equivalent* to $\big(C_0(T), \tau\big)$ if each point in T has a closed G-invariant neighborhood F such that (A^F, α^F) is Morita equivalent to $\big(C_0(F), \tau\big)$ over F.

Before we begin our study of systems which are locally Morita equivalent to their spectral systems, we point out that, for particular transformation groups (T, G), these systems turn out to be precisely the ones considered in [10, 13, 17]:

PROPOSITION 4.3. *Let A be a continuous-trace C^*-algebra with spectrum T, and suppose that G acts trivially on T. Then a system (A, G, α) is locally Morita equivalent to $\big(C_0(T), \tau\big) = \big(C_0(T), \mathrm{id}\big)$ if and only if α is locally unitary in the sense of* [10].

PROOF. Since the problem is local, we can suppose that there is an $(A, \alpha) -_T \big(C_0(T), \mathrm{id}\big)$-imprimitivity bimodule $(\mathfrak{X}, u)$, and prove that α is unitary. For each $s \in G$, u_s is $C_0(T)$-linear, and therefore belongs to the algebra $\mathcal{L}(\mathfrak{X}_{C_0(T)})$ of adjointable $C_0(T)$-linear operators on $\mathfrak{X}$; hence there is a multiplier $v_s \in M(A)$ characterized by

$$(v_s\, {}_A\langle x, y\rangle) \cdot z = u_s(x) \cdot \langle y, z\rangle_{C_0(T)}. \tag{4.2}$$

The condition

$$\langle u_s \cdot x, u_s \cdot y\rangle_{C_0(T)} = \mathrm{id}_s(\langle x, y\rangle_{C_0(T)}) = \langle x, y\rangle_{C_0(T)}$$

gives $u_s^* u_s = 1$, which, since u_s has u_{-s} as an inverse in $\mathcal{L}(\mathfrak{X})$, implies that both u_s and v_s are unitary. Because the group homomorphism u is strongly continuous, and $u_{-s} = u_s^*$, it is *-strongly continuous; since it is also bounded, we deduce that v is a strictly continuous homomorphism of G into $UM(A)$. Finally,

$$\begin{aligned}\big(\alpha_s(a)v_s\big)\cdot({}_A\langle x,y\rangle\cdot z) &= \alpha_s(a)(v_s\,{}_A\langle x,y\rangle)\cdot z\\ &= \alpha_s(a)u_s(x)\cdot\langle y,z\rangle_{C_0(T)}\\ &= u_s(a\cdot x)\cdot\langle y,z\rangle_{C_0(T)}\\ &= (v_s\,{}_A\langle a\cdot x,y\rangle)\cdot z\\ &= v_s a\cdot({}_A\langle x,y\rangle\cdot z),\end{aligned}$$

so that $\alpha_s(a)v_s = v_s a$, and v implements α. ∎

Since it is convenient to have a name for the type of systems studied in [16, 17], we recall the following definition.

DEFINITION 4.4. Suppose A is a continuous-trace algebra with spectrum T, and N is a closed subgroup of a locally compact abelian group G. We say that a dynamical system (A, G, α) is N-principal if all the isotropy groups for the induced action of G on T are equal to N, the quotient map $T \longrightarrow T/G$ is a principal G/N-bundle, and $\alpha|_N$ is locally unitary.

COROLLARY 4.5. *Let A be a continuous-trace algebra with spectrum T, and N be a closed subgroup of a locally compact abelian group G. Suppose that $p\colon T \longrightarrow Z$ is a locally trivial principal G/N-bundle, and view T as a G-space. Then a system (A, G, α) is locally Morita equivalent to $\big(C_0(T), G, \tau\big)$ if and only if (A, G, α) is N-principal. Indeed, if $(\mathfrak{X}, u)$ implements a Morita equivalence of (A^F, α^F) and $\big(C_0(F), \tau\big)$, then identifying $\mathcal{L}(\mathfrak{X}_{C_0(F)})$ with $M(A^F)$ allows us to view $u|_N$ as a strictly continuous homomorphism $u\colon N \longrightarrow UM(A^F)$ satisfying*

(1) $\alpha_n(a)^F = \mathrm{Ad}v_n(a^F)$ for $a \in A$, $n \in N$, and

(2) $\alpha_s(v_n) = v_n$ in $UM(A^F)$ for $s \in G$, $n \in N$;

in other words, v is a local Green twist for α over N in the sense of [16, 17].

PROOF. Since we already know that $T \longrightarrow T/G$ is a principal G/N-bundle, α is N-principal if and only if $\alpha|_N$ is locally unitary. Thus the first assertion follows immediately from the proposition. In the proof, we showed that $v\colon N \longrightarrow UM(A)$ was a strictly continuous homomorphism satisfying (1), and hence it remains to verify (2): as above, we may well suppose $F = T$. Using the characterization (4.2) of v_n, and the commutativity of G, we have

$$\begin{aligned}\alpha_s(v_n)({}_A\langle x,y\rangle)\cdot z &= \alpha_s\big(v_n\alpha_s^{-1}({}_A\langle x,y\rangle)\big)\cdot u_s\big(u_s^{-1}(z)\big)\\ &= u_s\Big(\big(v_n\,{}_A\langle u_s^{-1}(x), u_s^{-1}(y)\rangle\big)\cdot u_s^{-1}(z)\Big)\\ &= u_s\Big(u_n\big(u_s^{-1}(x)\big)\cdot\langle u_s^{-1}(y), u_s^{-1}(z)\rangle_{C_0(T)}\Big)\\ &= u_{sns^{-1}}(x)\cdot\tau_s\big(\langle u_s^{-1}(y), u_s^{-1}(z)\rangle_{C_0(T)}\big)\\ &= u_n(x)\cdot\langle y,z\rangle_{C_0(T)}\\ &= (v_n\,{}_A\langle x,y\rangle)\cdot z,\end{aligned}$$

which implies (2). ∎

https://doi.org/10.4153/CJM-1993-057-8 Published online by Cambridge University Press

We saw in the previous section that $A -_T C_0(T)$-imprimitivity bimodules are locally unique, and this uniqueness was crucial for the development of the Dixmier-Douady theory. For systems, the obvious analogue is false: two $(A,\alpha) - \big(C_0(T),\tau\big)$ Morita equivalences $(\mathfrak{X},u)$, $(\mathfrak{Y},v)$ need not be equivariantly isomorphic, even locally. In fact, there are two levels at which this can fail. First of all, we can only localize the *systems* to G-saturated subsets of T, and, while the bimodules $\mathfrak{X}$ and $\mathfrak{Y}$ are necessarily locally isomorphic by Lemma 3.3, they need not be isomorphic over G-invariant neighborhoods. And second, even if each point of T has a closed G-invariant neighborhood F such that $\mathfrak{X}^F \cong \mathfrak{Y}^F$, there is an algebraic obstruction to finding an equivariant isomorphism (see Lemma 4.13 below), and this obstruction is a fundamental ingredient in our Dixmier-Douady class of (A,α). Whether or not the first of these problems can be solved turns out to be a property of the topological transformation group (T,G), which fortunately holds automatically in the cases of interest to us.

DEFINITION 4.6. Let (T,G) be a locally compact transformation group. We shall say that *equivariant line bundles over* (T,G) *are locally trivial over* T/G if, whenever $\pi\colon L \longrightarrow T$ is a Hermitian complex line bundle with a unitary action of G satisfying $\pi(s\cdot l) = s\cdot\pi(l)$, each point of T has a G-invariant neighborhood F such that L is trivial over F. (We do *not* insist that the trivialisation is equivariant.)

Before discussing this property, we want to show that it does imply the local uniqueness of imprimitivity bimodules, as claimed above.

LEMMA 4.7. *Suppose that* $(\mathfrak{X},u)$, $(\mathfrak{Y},v)$ *implement Morita equivalences between* (A,α) *and* $\big(C_0(T),\tau\big)$, *and that equivariant line bundles over* (T,G) *are locally trivial over* T/G. *Then each point of* T *has a* G*-invariant neighborhood* F *such that* $\mathfrak{X}^F$ *and* $\mathfrak{Y}^F$ *are isomorphic as* $A^F -_F C_0(F)$*-imprimitivity bimodules.*

PROOF. We begin by reducing to the case where $(A,\alpha) = \big(C_0(T),\tau\big)$ by considering the $C_0(T) - C_0(T)$-imprimitivity bimodule $\tilde{\mathfrak{Y}} \otimes_A \mathfrak{X}$. From the calculation

$$\begin{aligned}
{}_{C_0(T)}\langle\!\langle v_s(y_1)^\sim \otimes u_s(x_1), v_s(y_2)^\sim \otimes u_s(x_2)\rangle\!\rangle &= \big\langle v_s(y_1)^\sim, v_s(y_2)^\sim \cdot {}_A\langle u_s(x_2), u_s(x_1)\rangle\big\rangle_{C_0(T)}\\
&= \langle v_s(y_1), v_s({}_A\langle x_1, x_2\rangle \cdot y_2)\rangle_{C_0(T)}\\
&= \tau_s\big({}_{C_0(T)}\langle\!\langle \tilde{y}_1 \otimes x_1, \tilde{y}_2 \otimes x_2\rangle\!\rangle\big),
\end{aligned}$$

we deduce that $w_s(\tilde{y}\otimes x) = v_s(y)^\sim \otimes u_s(x)$ defines an action w of G on $\tilde{\mathfrak{Y}} \otimes_A \mathfrak{X}$ such that $(\tilde{\mathfrak{Y}} \otimes_A \mathfrak{X}, w)$ is a $\big(C_0(T),\tau\big) -_T \big(C_0(T),\tau\big)$-imprimitivity bimodule (the strong continuity follows from Equation (2.1)).

By [11, Proposition A1], there is a Hermitian line bundle L over T such that $\tilde{\mathfrak{Y}} \otimes_A \mathfrak{X}$ is isomorphic to $\Gamma_0(L)$. We claim that the action w of G on $\Gamma_0(L)$ induces an action of G on L. To see this, we deduce from the condition $w_s(f\cdot\phi) = w_s(f)\cdot\tau_s(\phi)$ that $f(t) = 0$ implies $w_s(f)(s\cdot t) = 0$, and hence there is a well-defined pairing: $G \times L \longrightarrow L$ given by

$$s\cdot\big(f(t)\big) = w_s(f)(s\cdot t). \tag{4.3}$$

It follows quite easily from the algebraic properties of the bimodule $\Gamma_0(L)$, and the continuity of w, that this pairing is a jointly continuous action of G on L which is unitary on the fibres. Since Equation (4.3) implies immediately that the bundle projection is equivariant, each point of T has a G-saturated neighborhood F such that L is trivial over F—or, equivalently, that there is a section $\sigma\colon F \longrightarrow L$ satisfying $|\sigma(t)| = 1$ for $t \in F$.

We now claim that $\mathfrak{X}^F \cong \mathfrak{Y}^F$. First, observe that the section σ induces an isomorphism $f \longrightarrow f \cdot \sigma$ of $C_0(F) -_F C_0(F)$-imprimitivity bimodules. Next, recall that there are natural imprimitivity bimodule isomorphisms of $(\tilde{\mathfrak{Y}} \otimes_A \mathfrak{X})^F$ onto $(\mathfrak{Y}^F)^\sim \otimes_{A^F} \mathfrak{X}^F$ [11, 1.10] and $\mathfrak{Y}^F \otimes_{C_0(F)} (\mathfrak{Y}^F)^\sim$ onto A^F [19, 6.22]. Then we have

$$\begin{aligned}\mathfrak{Y}^F &\cong \mathfrak{Y}^F \otimes_{C_0(F)} C_0(F) \cong \mathfrak{Y}^F \otimes_{C_0(F)} \Gamma_0(L)^F \\ &\cong \mathfrak{Y}^F \otimes_{C_0(F)} \big((\mathfrak{Y}^F)^\sim \otimes_{A^F} \mathfrak{X}^F\big) \cong A^F \otimes_{A^F} \mathfrak{X}^F \cong \mathfrak{X}^F,\end{aligned}$$

as claimed. ∎

REMARK 4.8. We have shown that if equivariant line bundles over (T, G) are locally trivial over T/G, then local Morita equivalences are unique in the sense we want. The proof shows that this condition on (T, G) is also necessary: if $(\mathfrak{X}, u)$ is an $(A, \alpha) - \big(C_0(T), \tau\big)$-imprimitivity bimodule, and (L, G) is an equivariant Hermitian line bundle, then the imprimitivity bimodules $\big(\mathfrak{X} \otimes_{C_0(T)} \Gamma_0(L)\big)^F$ and $\Gamma_0(L)^F$ are isomorphic only if $L|_F$ is trivial (to get a trivialisation, just tensor the isomorphism with $\tilde{\mathfrak{X}}$).

We now want to discuss transformation groups (T, G) where equivariant line bundles are always locally trivial over T/G. First of all, since there exist nontrivial line bundles over Z only if $H^2(Z, \mathbb{Z}) \neq 0$, it is easy to find spaces (T, G) with this property. For example, it holds whenever $p\colon T \longrightarrow T/G$ is a locally trivial fibre bundle over a locally contractible space and the fibre F has $H^2(F, \mathbb{Z}) = 0$. But, more surprisingly, when $p\colon T \longrightarrow T/G$ is a locally trivial principal bundle, it doesn't matter what $H^2(F, \mathbb{Z})$ is: we can trivialise the line bundle L over the image $c(W)$ of a local section $c\colon W \longrightarrow T$, and use the action of G on L to extend the trivialisation to $G \cdot W$. This applies in particular to the spectra of N-principal systems (Proposition 4.9 below). While not all transformation groups have the property (see Example 4.12), it does seem likely that equivariant line bundles over (T, G) will be locally trivial over T/G whenever T/G is reasonable. As evidence, we prove this is the case for any action of a compact group.

PROPOSITION 4.9. *Suppose G is a locally compact abelian group, and N is a closed subgroup such that $\hat{G} \longrightarrow \hat{N}$ has local sections. If $p\colon T \longrightarrow Z$ is a locally trivial principal G/N-bundle, then G-equivariant line bundles over (T, G) are locally trivial over $T/G = T/(G/N)$.*

PROOF. We begin with the special case in which $N = \{e\}$. Since $T \longrightarrow T/G$ is locally trivial, and this is a local problem, we may as well suppose that $T = Z \times G$, and fix a point $t = (z_0, s)$. The bundle $L|_{Z \times \{s\}}$ is locally trivial, so we can find a neighborhood V of z_0 and a section $\sigma\colon V \times \{s\} \longrightarrow L$ such that $|\sigma(z, s)| = 1$ for $z \in V$. We now define $\sigma(z, r) = rs^{-1} \cdot \sigma(z, s)$; this is easily seen to be continuous, and it is a section of L over $F = V \times G$ because

$$\pi\big(\sigma(z, r)\big) = \pi\big(rs^{-1} \cdot \sigma(z, s)\big) = rs^{-1} \cdot \pi\big(\sigma(z, s)\big) = rs^{-1} \cdot (z, s) = (z, r).$$

Since the action preserves the Hermitian metric, we have $|\sigma(z, s)| = 1$ for all (z, s), so the bundle is trivial over $V \times G$.

For the general case, the idea is to modify the action of G on L to give an action of G/N. If $n \in N$, the equivariance condition $\pi(n \cdot l) = n \cdot \pi(l) = \pi(l)$ implies that the transformation $l \longrightarrow n \cdot l$ preserves fibres, and hence there is a continuous map $\lambda: L \times N \longrightarrow \mathbb{T}$ such that $n \cdot l = \lambda(l, n)l$. Since the map $l \longrightarrow n \cdot l$ is unitary, λ is constant on G-orbits, and there is a continuous map $\delta: T/G \longrightarrow \hat{N}$ such that $\lambda(t, n) = \delta(G \cdot t)(n)$. Because $\hat{G} \longrightarrow \hat{N}$ is locally trivial and our problem is local in T/G, we may as well assume there is a continuous map $\epsilon: T/G \longrightarrow \hat{G}$ such that $\lambda(t, \cdot) = \epsilon(G \cdot t)$ on N. We can now define a continuous action of G/N on L by $(sN) \cdot l = \epsilon\big(\pi(l)\big)(s)^{-1}(s \cdot l)$, which covers the original action of G/N on T. Hence we can apply the special case to deduce that L is locally trivial over G-invariant neighborhoods. ∎

LEMMA 4.10. *Suppose (T, G) is a transformation group with G compact abelian and T paracompact. Then every equivariant line bundle over (T, G) is locally trivial over T/G.*

PROOF. Let $\pi: L \longrightarrow T$ be an equivariant line bundle, and fix $t \in T$. Since G is compact, the map $s \longrightarrow s \cdot t$ is a homeomorphism of G/G_t onto the orbit $G \cdot t$, and the argument of the previous proposition says that L is trivial over $G \cdot t$. If $\lambda_{ij}: M_{ij} \longrightarrow \mathbb{T}$ are transition functions for L, then the class of cocycle $\{\lambda_{ij}\}$ is trivial in $\check{H}^1(T, \mathcal{S})$ exactly when L is. Thus the result follows from the next lemma. ∎

LEMMA 4.11. *Suppose that G is compact and that T is a paracompact locally compact G-space. If $\lambda \in \check{H}^1(T, \mathcal{S})$ is such that $\lambda|_{G t} = 0$ in $\check{H}^1(G \cdot t, \mathcal{S})$, then there is a G-invariant neighborhood N of t such that $\lambda|_N = 0$ in $\check{H}^1(N, \mathcal{S})$.*

PROOF. Let $\{M_i'\}_{i \in A}$ be a locally finite cover of T such that there is a cocycle $\{\lambda_{ij}\} \in Z^1(\{M_i'\}, \mathcal{S})$ which represents λ. Also let $\{M_i\}_{i \in A}$ be an open cover of T such that $\overline{M}_i \subseteq M_i'$ for each i. We may assume that there are functions $\mu_i': M_i' \cap G \cdot t \longrightarrow \mathbb{T}$ such that $(\partial \mu')_{ij} = \lambda_{ij}|_{G t}$ for all i and j.

Let M be a compact neighborhood of $G \cdot t$. Since $\{M_i'\}$ is locally finite, $B = \{i : M_i' \cap M \neq \emptyset\}$ is finite. Furthermore, if $i \in B$, then we claim there is a neighborhood V_i of $G \cdot t$ with $V_i \subseteq M$ and a function $\mu_i: M_i \cap V_i \longrightarrow \mathbb{T}$ so that $\mu_i(y) = \mu_i'(y)$ for all $y \in M_i \cap G \cdot t$. In fact the Tietze theorem implies that there is a function $\tilde{\mu}_i: T \longrightarrow \mathbb{C}$ which extends $\mu_i': \overline{M}_i \cap G \cdot x \longrightarrow \mathbb{T} \subset \mathbb{C}$. Then we can put

$$V_i = \{y : |\tilde{\mu}_i(y)| > \tfrac{1}{2}\} \cup T \setminus \overline{M}_i,$$

and define $\mu_i(y) = |\tilde{\mu}_i(y)|^{-1}\tilde{\mu}_i(y)$ $(y \in M_i \cap V_i)$; this establishes the claim.

Let $V = \bigcap_{i \in B} V_i$. Then V is a neighborhood of $G \cdot t$ and we may view $\mu = \{\mu_i\}$ as a cochain in $C^0(\{M_i \cap V\}, \mathcal{S})$ such that $\{\lambda_{ij} \cdot (\partial \mu)_{ij}^{-1}\}$ is a cocycle which is identically one on $G \cdot t$. Since $G \cdot t$ is compact, there is a neighborhood V' of $G \cdot t$ such that for all i and j and all $y \in V' \cap M_{ij}$,

$$|\lambda_{ij}(y)(\partial \mu)_{ij}^{-1}(y) - 1| < \sqrt{2}.$$

(There are only finitely many i and j to consider.) Thus $\log\big(\lambda(\partial \mu)^{-1}\big)$ is a cocycle in $Z^1(\{M_i \cap V'\}, \mathcal{R})$ and must be equal to $\partial \nu$ for some cochain $\nu = \{\nu_i\} \in C^0(\{M_i \cap V'\}, \mathcal{R})$. Thus, $\lambda|_{V'} = \partial \mu \cdot \exp(\partial \nu) = \partial(\mu \cdot \exp \nu)$.

The result follows as V' must contain a G-invariant neighborhood N of the compact set $G \cdot t$. ∎

EXAMPLE 4.12. Let $T = \mathbb{T}^2$, viewed as $(\mathbb{R}/\mathbb{Z}) \times \mathbb{T}$, and let $\mathbb{R}$ act by the flow at an irrational angle, which is the quotient of the action on $\mathbb{R} \times \mathbb{T}$ given by $r \cdot (t, z) = (t + r, e^{2\pi i\theta r}z)$. Let L be the Hermitian line bundle over T obtained by taking the quotient of $\mathbb{R} \times \mathbb{T} \times \mathbb{C}$ by the equivalence relation in which

$$(t, z, \lambda) \sim (t - 1, z, z\lambda). \tag{4.4}$$

This bundle is nontrivial: if it were trivial, it would have a nonvanishing section, given by a function $f\colon [0,1] \times \mathbb{T} \longrightarrow \mathbb{C}\backslash\{0\}$ satisfying $f(1, z) = zf(0, z)$, and then $h_t = f(t, \cdot)f(0, \cdot)^{-1}$ would be a homotopy joining 1 to the identity function $z \longrightarrow z$. The action of $\mathbb{R}$ on T lifts to one on L via the formula

$$r \cdot (t, z, \lambda) = (t + r, e^{2\pi i\theta r}z, \lambda e^{-2\pi i\theta(rt+r^2/2)});$$

the $r^2/2$ term is there to ensure this defines an action of $\mathbb{R}$ on $\mathbb{R} \times \mathbb{T} \times \mathbb{C}$, and then one can easily verify that it respects the equivalence relation Equation (4.4), and hence induces a continuous action of $\mathbb{R}$ on L. But since the action of $\mathbb{R}$ on $\mathbb{T}^2$ is minimal, the only nonempty $\mathbb{R}$-invariant open set is the whole of $\mathbb{T}^2$, so L is not trivial over any $\mathbb{R}$-invariant neighborhoods. ∎

So far we have shown that, for the systems (T, G) of interest to us, the bimodules $\mathfrak{X}$ involved in Morita equivalences $(\mathfrak{X}, u)$ of systems are locally unique over equivariant neighborhoods. We now want to look at the obstruction to extending this uniqueness to cover the action u.

LEMMA 4.13. *Suppose $(\mathfrak{X}, u)$ and $(\mathfrak{Y}, v)$ are $(A, \alpha) -_T \big(C_0(T), \tau\big)$-imprimitivity bimodules, and $g\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ is an isomorphism of $A -_T C_0(T)$-imprimitivity bimodules. Then there is a continuous map $\lambda\colon T \times G \longrightarrow \mathbb{T}$ such that*

(1) $u_s\big(g(y)\big) = g\big(v_s(y)\big) \cdot \lambda(\cdot, s)$,

(2) $\lambda(t, rs) = \lambda(t, r)\lambda(r^{-1} \cdot t, s)$ for $t \in T$, $r, s \in G$.

PROOF. If we set $w_s = v_s^{-1} \circ g^{-1} \circ u_s \circ g$, then w is a strongly continuous map of G into the group of isometries of $\mathfrak{Y}$. Because the actions u, v are both compatible with the actions α, τ of A, $C_0(T)$, w_s is an $A - C_0(T)$-bimodule isomorphism for each s, and hence by Lemma 3.4 there is a continuous function $\rho(\cdot, s) \in C(T, \mathbb{T})$ such that

$$w_s(y) = y \cdot \rho(\cdot, s) \quad \text{for } y \in \mathfrak{Y},\ s \in G.$$

This equation is equivalent to

$$u_s\big(g(y)\big) = g\Big(v_s\big(y \cdot \rho(\cdot, s)\big)\Big) = g\big(v_s(y)\big) \cdot \tau_s\big(\rho(\cdot, s)\big),$$

and thus if we define $\lambda(t, s) = \rho(s^{-1} \cdot t, s)$, we have (1). To see that ρ and λ are continuous on $T \times G$, just note that both $\langle y, y\rangle_{C_0(T)}$ and the map $(t, s) \longmapsto \langle y, w_s(y)\rangle_{C_0(T)}(t)$ are continuous. Finally, we calculate

$$\begin{aligned} g\big(v_{rs}(y)\big)\lambda(\cdot, rs) &= u_{rs}\big(g(y)\big) = u_r\Big(u_s\big(g(y)\big)\Big) \\ &= u_r\Big(g\big(v_s(y)\big)\lambda(\cdot, s)\Big) \\ &= u_r\Big(g\big(v_s(y)\big)\Big)\tau_r\big(\lambda(\cdot, s)\big) \\ &= g\Big(v_r\big(v_s(y)\big)\Big)\lambda(\cdot, r)\tau_r\big(\lambda(\cdot, s)\big), \end{aligned}$$

and this implies (2). ∎

COROLLARY 4.14. *If G is abelian and N is a subgroup of G which acts trivially on T, then*

(1) $t \longrightarrow \lambda(t, \cdot)$ is a continuous map of T into $\hat{N}$;
(2) for each $n \in N$, $\lambda(\cdot, n)$ is constant on orbits.

PROOF. The cocycle identity Lemma 4.13(2) immediately implies that $\lambda(t, \cdot)$ is a homomorphism on N. A standard compactness argument shows that if $t_\iota \longrightarrow t$ in T, then $\lambda(t_\iota, \cdot) \longrightarrow \lambda(t, \cdot)$ uniformly on compact subsets of N, and hence (1) follows. Since G is abelian, we have $\lambda(t, sn) = \lambda(t, ns)$, and two applications of Lemma 4.13(2) give

$$\lambda(t, s)\lambda(s^{-1} \cdot t, n) = \lambda(t, n)\lambda(t, s),$$

which implies (2). ∎

5. **The equivariant cohomology group $\check{H}^2_G(T, \mathcal{S})$.** Suppose that T is a G-space with orbit map $p: T \longrightarrow T/G$. (Although, in this article, we will only be interested in the case where T is a principal G/N-bundle, for the following discussion T may be any G-space.) Let $\mathfrak{N} = \{N_\iota\}_{\iota \in I}$ be a cover of T/G by open sets. We define $Z^2\big(p^{-1}(\mathfrak{N}), \mathcal{S}\big)$ to be the collection of pairs (λ, ν) where $\lambda = \{\lambda_{\iota\jmath}\}$ is a 1-cochain consisting of continuous functions $\lambda_{\iota\jmath}: p^{-1}(N_{\iota\jmath}) \times G \longrightarrow \mathbb{T}$, and $\nu = \{\nu_{\iota\jmath k}\}$ is a 2-cocycle consisting of continuous functions $\nu_{\iota\jmath k}: p^{-1}(N_{\iota\jmath k}) \longrightarrow \mathbb{T}$ such that

$$\lambda_{\iota\jmath}(t, sr) = \lambda_{\iota\jmath}(t, s)\lambda_{\iota\jmath}(s^{-1} \cdot t, r), \tag{5.1}$$

while

$$\lambda_{\iota\jmath}(t, s)\lambda_{\jmath k}(t, s)\nu_{\iota\jmath k}(t) = \lambda_{\iota k}(t, s)\nu_{\iota\jmath k}(s^{-1} \cdot t). \tag{5.2}$$

Of course, $Z^2\big(p^{-1}(\mathfrak{N}), \mathcal{S}\big)$ becomes an abelian group when equipped with the usual pointwise multiplication of cocycles.

We define $B^2\big(p^{-1}(\mathfrak{N}), \mathcal{S}\big)$ to be the subgroup of pairs (λ, ν) in $Z^2\big(p^{-1}(\mathfrak{N}), \mathcal{S}\big)$ for which there exist continuous functions $\phi_{\iota\jmath}: p^{-1}(N_{\iota\jmath}) \longrightarrow \mathbb{T}$ and $\sigma_\iota: p^{-1}(N_\iota) \times G \longrightarrow \mathbb{T}$, such that

$$\sigma_\iota(t, rs) = \sigma_\iota(t, r)\sigma_\iota(r^{-1} \cdot t, s) \tag{5.3}$$

$$\lambda_{\iota\jmath}(t, s) = \overline{\phi_{\iota\jmath}(t)}\phi_{\iota\jmath}(s^{-1} \cdot t))\sigma_\iota(t, s)\overline{\sigma_\jmath(t, s)}, \text{ and} \tag{5.4}$$

$$\nu_{\iota\jmath k}(t) = \phi_{\iota\jmath}(t)\phi_{\jmath k}(t)\overline{\phi_{\iota k}(t)}. \tag{5.5}$$

It may be comforting to notice that if $(\{\lambda_{ij}\},\{\nu_{ijk}\})$ are defined by continuous functions as in Equations (5.3), (5.4), and (5.5), then $(\{\lambda_{ij}\},\{\nu_{ijk}\})$ belongs to $Z^2_G\big(p^{-1}(\mathfrak{U}),\mathcal{S}\big)$.

We define $\check{H}^2_G\big(p^{-1}(\mathfrak{U}),\mathcal{S}\big)$ to be the quotient of $Z^2_G\big(p^{-1}(\mathfrak{U}),\mathcal{S}\big)$ by $B^2_G\big(p^{-1}(\mathfrak{U}),\mathcal{S}\big)$.

Now suppose that $\mathfrak{V}=\{N_\alpha\}_{\alpha\in A}$ is a refinement of $\mathfrak{U}$ by open sets. Let $r: A\longrightarrow I$ be a refinement map: *i.e.*, any map such that $N_\alpha\subseteq N_{r(\alpha)}$ for all $\alpha\in A$. Then given $(\lambda,\nu)\in Z^2_G\big(p^{-1}(\mathfrak{U}),\mathcal{S}\big)$, we obtain an element $\tilde{r}(\lambda,\nu)=\big(r(\lambda),r(\nu)\big)$ in $Z^2_G\big(p^{-1}(\mathfrak{V}),\mathcal{S}\big)$ in the expected way:

$$r(\lambda)_{\alpha\beta}(t,s)=\lambda_{r(\alpha)r(\beta)}(t,s),\text{ and}$$
$$r(\nu)_{\alpha\beta\gamma}(t)=\nu_{r(\alpha)r(\beta)r(\gamma)}(t).$$

In this way, we obtain a homomorphism $\tilde{r}: Z^2_G\big(p^{-1}(\mathfrak{U}),\mathcal{S}\big)\longrightarrow Z^2_G\big(p^{-1}(\mathfrak{V}),\mathcal{S}\big)$ which takes $B^2_G\big(p^{-1}(\mathfrak{U}),\mathcal{S}\big)$ into $B^2_G\big(p^{-1}(\mathfrak{V}),\mathcal{S}\big)$. Therefore we obtain a homomorphism r^*: $\check{H}^2_G\big(p^{-1}(\mathfrak{U}),\mathcal{S}\big)\longrightarrow \check{H}^2_G\big(p^{-1}(\mathfrak{V}),\mathcal{S}\big)$. We claim that r^* is independent of our choice of refining map. If s is another such map, then let $(\tilde{\lambda},\tilde{\nu})=\big(r(\lambda),r(\nu)\big)\big(s(\lambda),s(\nu)\big)^{-1}=\big(r(\lambda)\overline{s(\lambda)},r(\nu)\overline{s(\nu)}\big)$. Then standard arguments such as in [23; §5.33] imply that $\tilde{\nu}\in B^2\big(p^{-1}(\mathfrak{V}),\mathcal{S}\big)$ and hence that there are continuous functions $\phi_{\alpha\beta}: p^{-1}(N_{\alpha\beta})\longrightarrow\mathbb{T}$ such that $\tilde{\nu}_{\alpha\beta\gamma}(t)=\phi_{\alpha\beta}(t)\phi_{\beta\gamma}(t)\overline{\phi_{\alpha\gamma}(t)}$. In fact, it follows from Equation (10) of [23; §5.33] that we may take

$$\phi_{\alpha\beta}(t)=\nu_{s(\alpha)r(\alpha)r(\beta)}(t)\overline{\nu_{s(\alpha)s(\beta)r(\beta)}(t)}.$$

Furthermore, we may define

$$\sigma_\alpha(t,g)=\overline{\lambda_{s(\alpha)r(\alpha)}(t,g)}.$$

Then σ_α satisfies Equation (5.3), and one can compute that

$$\tilde{\lambda}_{\alpha\beta}(t,g)\overline{\sigma_\alpha(t,g)}\sigma_\beta(t,g)=\lambda_{s(\alpha)r(\alpha)}(t,g)\lambda_{r(\alpha)r(\beta)}(t,g)\overline{\lambda_{s(\alpha)s(\beta)}(t,g)\lambda_{s(\beta)r(\beta)}(t,g)},$$

which using (5.2) is

$$\begin{aligned}&=\lambda_{s(\alpha)r(\beta)}(t,g)\overline{\nu_{s(\alpha)r(\alpha)r(\beta)}(t)}\nu_{s(\alpha)r(\alpha)r(\beta)}(g^{-1}\cdot t)\cdot\\&\qquad\cdot\overline{\lambda_{s(\alpha)r(\beta)}(t,g)}\nu_{s(\alpha)s(\beta)r(\beta)}(t)\overline{\nu_{s(\alpha)s(\beta)r(\beta)}(g^{-1}\cdot t)}\\&=\overline{\phi_{\alpha\beta}(t)}\phi_{\alpha\beta}(g^{-1}\cdot t).\end{aligned}$$

In other words, $(\tilde{\lambda},\tilde{\nu})\in B^2_G\big(p^{-1}(\mathfrak{V}),\mathcal{S}\big)$ as required. Therefore we may regard

$$\big\{\check{H}^2_G\big(p^{-1}(\mathfrak{U}),\mathcal{S}\big)\big\}_{\mathfrak{U}},$$

where $\mathfrak{U}$ is allowed to run over all open coverings of T/G, as a system of abelian groups directed by refinement. Thus, we can define

$$\check{H}^2_G(T,\mathcal{S})=\varinjlim_{\mathfrak{U}}\check{H}^2_G\big(p^{-1}(\mathfrak{U}),\mathcal{S}\big).$$

Elsewhere [18], we will present a more general treatment in which we define a family of groups $\check{H}^n_G(T,\mathcal{S})$ which will fit into a generalized Gysin sequence (See also [17; §5a]). Here we have restricted our discussion to the case $n=2$ as that is all we require for the sequel.

6. **The Dixmier-Douady class of a system.** We are now ready to construct the Dixmier-Douady class of a system (A, G, α). Throughout, A will be a continuous-trace C^*-algebra with spectrum T. For technical reasons, we shall assume that T is paracompact and T/G Hausdorff; in our main application, $T \longrightarrow T/G$ will actually be a locally trivial G/N-bundle for some quotient G/N of G. We think of the Dixmier-Douady class of the system as the obstruction to piecing together local Morita equivalences with the spectral system to form a global Morita equivalence with the spectral system. In order to produce a scalar-valued cocycle from these local equivalences, we need to do some standard shuffling with open covers, and we obtain the necessary lemmas by modifying [4, 10.7.11 and 10.7.12].

LEMMA 6.1. *Suppose that A has paracompact spectrum T, that (A, G, α) is locally Morita equivalent to $\big(C_0(T), \tau\big)$, that T/G is Hausdorff, and that equivariant line bundles over (T, G) are locally trivial over T/G. Then there is an open cover $\mathfrak{N} = \{N_i\}_{i\in I}$ of T/G such that, writing $F_i = \overline{p^{-1}(N_i)}$, there is an $(A^{F_i}, \alpha^{F_i}) -_{F_i} \big(C_0(F_i), \tau\big)$-imprimitivity bimodule $(\mathfrak{X}_i, u^i)$ for each $i \in I$, and there is an isomorphism $g_{ij}\colon \mathfrak{X}_j^{F_{ij}} \longrightarrow \mathfrak{X}_i^{F_{ij}}$ for each $i,j \in I$.*

PROOF. By assumption there are closed G-invariant sets $\{E_\gamma\}_{\gamma\in\Gamma}$, whose interiors U_γ cover T, and $(A^{E_\gamma}, \alpha^{E_\gamma}) -_{E_\gamma} \big(C_0(E_\gamma), \tau\big)$-imprimitivity bimodules $(\mathfrak{X}_\gamma, u^\gamma)$. Since T/G is paracompact, and in view of Lemma 4.1, we may suppose that $\{U_\gamma\}_{\gamma\in\Gamma}$ is locally finite. Therefore we can assume that there are closed G-invariant sets $C_\gamma \subset p^{-1}(U_\gamma)$, whose interiors V_γ cover T.

For the moment, fix $\gamma \in \Gamma$ and $t \in V_\gamma$. Using local finiteness, there is a neighborhood W of t such that $\Gamma' = \{\beta \in \Gamma : W \cap C_\beta \neq \emptyset\}$ is finite. We claim that for each $\beta \in \Gamma'$ there exist closed G-invariant neighborhoods W_β of t such that $W_\beta \subseteq W$ and such that there exists an imprimitivity bimodule isomorphism of $\mathfrak{X}_\beta^{W_\beta\cap C_\beta}$ onto $\mathfrak{X}_\gamma^{W_\beta\cap C_\beta}$. For if $t \in C_\beta$, then $t \in p^{-1}(U_\gamma) \cap p^{-1}(U_\beta)$, and the claim follows by Lemma 4.7; if $t \notin C_\beta$, then we can simply choose W_β so that $W_\beta \cap C_\beta = \emptyset$. Either way, $W_{\gamma,t} = \bigcap_{\beta\in\Gamma'} W_\beta$ is a closed G-invariant neighborhood of t with the property that there is an imprimitivity bimodule isomorphism of $\mathfrak{X}_\beta^{W_{\gamma,t}\cap C_\beta}$ onto $\mathfrak{X}_\gamma^{W_{\gamma,t}\cap C_\beta}$.

Finally, we let $I = \{(\gamma, t) \in \Gamma \times T : t \in C_\gamma\}$. For $i = (\gamma, t)$, we put $F_i = W_i$, $\mathfrak{X}_i = \mathfrak{X}_\gamma^{W_i}$, and $u^i = (u^\gamma)^{W_i}$, and the existence of the g_{ij} follows from the previous paragraph. ∎

LEMMA 6.2. *We maintain the assumptions and notations of Lemma 6.1.*

(1) For each $i,j \in I$, there is a continuous function $\lambda_{ij}\colon F_{ij} \times G \longrightarrow \mathbb{T}$ such that, for all $x \in \mathfrak{X}_j^{F_{ij}}$ and $s \in G$,

$$(u_s^i)^{F_{ij}}\big(g_{ij}(x)\big) = \lambda_{ij}(\cdot, s)\cdot\Big(g_{ij}\big((u_s^j)^{F_{ij}}(x)\big)\Big). \tag{6.1}$$

(2) For each $i,j,k \in I$, there is a continuous function $\nu_{ijk}\colon F_{ijk} \longrightarrow \mathbb{T}$ such that, for each $x \in \mathfrak{X}_k^{F_{ijk}}$,

$$g_{ij}^{F_{ijk}}\big(g_{jk}^{F_{ijk}}(x)\big) = \nu_{ijk}\cdot g_{ik}^{F_{ijk}}(x). \tag{6.2}$$

(3) The pair $(\lambda, \nu) = (\{\lambda_{ij}\}, \{\nu_{ijk}\})$ is a cocycle in $Z_G^2\big(p^{-1}(\mathfrak{N}), \mathcal{S}\big)$.

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(4) *The class* $\delta(A,\alpha)$ *of* (λ,ν) *in* $\check{H}^2_G(T,\mathcal{S})$ *depends only on* (A,G,α), *and not on any of the choices we have made.*

PROOF. After localizing the Morita equivalences to F_{ij} using Lemma 4.1, we apply Lemma 3.3 to the isomorphisms g_{ij}: this gives us the continuous functions λ_{ij} satisfying Equations (6.1) and (5.1). Similarly, applying Lemma 3.3 to the imprimitivity bimodule automorphisms $(g_{ik}^{F_{ijk}})^{-1}\circ g_{ij}^{F_{ijk}}\circ g_{jk}^{F_{ijk}}$ of $\mathfrak{X}_k^{F_{ijk}}$ gives the function ν_{ijk} satisfying Equation (6.2). We can verify that ν_{ijk} is a cocycle by comparing the two sides of

$$(\nu_{jkl}\nu_{ikl}^{-1}\nu_{ijl})\cdot x=\nu_{ijk}\cdot x,$$

exactly as in [4, p. 235]. To complete the proof of (3), we verify (5.2):

$$\begin{aligned}(u_s^i)^{F_{ijk}}\circ g_{ik}^{F_{ijk}}&=(u_s^i)^{F_{ijk}}\circ(\overline{\nu_{ijk}}\cdot g_{ij}^{F_{ijk}}\circ g_{jk}^{F_{ijk}})\\&=\tau_s(\overline{\nu_{ijk}})\cdot\left((u_s^i)^{F_{ijk}}\circ g_{ij}^{F_{ijk}}\circ g_{jk}^{F_{ijk}}\right)\\&=\tau_s(\overline{\nu_{ijk}})\lambda_{ij}(\cdot,s)\lambda_{jk}(\cdot,s)\nu_{ijk}\cdot g_{ik}^{F_{ijk}}\circ(u_s^k)^{F_{ijk}}.\end{aligned}$$

Finally, suppose that $\mathfrak{B}=\{F_\gamma\}_{\gamma\in\Lambda}$, $(\mathfrak{X}_\gamma,u^\gamma)$, and $g_{\gamma\beta}$ have been chosen as in Lemma 6.1, and let (λ',ν') be the associated 2-cocycle. As in the proof of Lemma 6.1, we may replace $\mathfrak{A}$ and $\mathfrak{B}$ by refinements so that there are imprimitivity bimodule isomorphisms $g_{i\gamma}\colon\mathfrak{X}_\gamma^{F_{i\gamma}}\longrightarrow\mathfrak{X}_i^{F_{i\gamma}}$. Let $A=I\cup\Lambda$. Then the family $\{F_a\}_{a\in A}$, $(\mathfrak{X}_a,u^a)$, and g_{ab} satisfy conditions (1), (2), and (3) of Lemma 6.1. Let (λ'',ν'') be the corresponding cocycle. Since $\{F_a\}_{a\in A}$ refined by both $\mathfrak{A}$ and $\mathfrak{B}$, the classes of (λ,ν) and (λ',ν') coincide with the class of (λ'',ν'') in $\check{H}^2_G(T,\mathcal{S})$. This concludes the proof. ■

Of course, we call $\delta(A,\alpha)$ the *Dixmier-Douady class* of the system (A,G,α). Our main theorem says that, under mild hypotheses on $\hat{A}/G$, this is a complete invariant for systems which are locally Morita equivalent to their spectral system. We stress that the hypotheses on (T,G) are automatically satisfied when $T\longrightarrow T/G$ is a principal bundle for some quotient of G (Proposition 4.9).

THEOREM 6.3. *Let* (T,G) *be a locally compact transformation group such that* T *is paracompact,* T/G *is Hausdorff, and equivariant line bundles over* (T,G) *are locally trivial over* T/G. *Suppose* (A,G,α) *and* (B,G,β) *are both locally Morita equivalent to* $\big(C_0(T),\tau\big)$. *Then* (A,α) *and* (B,β) *are Morita equivalent over* T *if and only if* $\delta(A,\alpha)=\delta(B,\beta)$ *in* $\check{H}^2_G(T,\mathcal{S})$. *Further, every class in* $\check{H}^2_G(T,\mathcal{S})$ *is the Dixmier-Douady class of some system* (A,G,α) *which is locally Morita equivalent to* $\big(C_0(T),\tau\big)$.

Since the proof of this result is quite complicated—even without the group action (*cf.* [4, §10.7–9])—we shall break it up into 3 propositions.

PROPOSITION 6.4. *Suppose that* A *and* B *are* C^**-algebras with paracompact spectrum* T, *that* (A,G,α) *and* (B,G,β) *are both locally Morita equivalent to* $\big(C_0(T),\tau\big)$, *that* T/G *is Hausdorff, and that equivariant line bundles over* (T,G) *are locally trivial over* T/G. *Then if* (A,α) *is Morita equivalent to* (B,β) *over* T, *we have* $\delta(A,\alpha)=\delta(B,\beta)$ *in* $\check{H}^2_G(T,\mathcal{S})$.

PROOF. Let $\mathfrak{X}$ be an $A-_T B$-imprimitivity bimodule and let u be an action of G on $\mathfrak{X}$ such that $(A,\alpha)\sim_{X,u}(B,\beta)$. Apply Lemma 6.1 to (B,β) to produce closed G-invariant sets $\{F_i\}_{i\in I}$, equivalences $(\mathfrak{Y}_i, w_i)$, and imprimitivity bimodule isomorphisms $\{h_{ij}\}$. Let (λ, ν) be the associated 2-cocycle. For each i, $\mathfrak{X}_i = \mathfrak{X}^{F_i} \otimes_{B^{F_i}} \mathfrak{Y}_i$ is an $A^{F_i} -_{F_i} C_0(F_i)$-imprimitivity bimodule, and, as in the proof of Lemma 4.7, $u^i_s = u_s \otimes w^i_s$ defines an action of G on $\mathfrak{X}_i$, giving us $(A^{F_i}, \alpha^{F_i}) -_{F_i} \big(C_0(F_i), \tau\big)$ imprimitivity bimodules $(\mathfrak{X}_i, u^i)$. Moreover, on F_{ij} we have

$$\begin{aligned}\langle\!\langle x \otimes h_{ij}(y), x' \otimes h_{ij}(y')\rangle\!\rangle_{C_0(F_i)} &= \big\langle \langle x', x\rangle_{B^{F_i}} \cdot h_{ij}(y), h_{ij}(y')\big\rangle_{C_0(F_i)} \\ &= \big\langle h_{ij}(\langle x', x\rangle_{B^{F_i}} \cdot y), h_{ij}(y')\big\rangle_{C_0(F_i)} \\ &= \big\langle \langle x', x\rangle_{B^{F_j}} \cdot y, y'\big\rangle_{C_0(F_j)} \\ &= \langle\!\langle x \otimes y, x' \otimes y'\rangle\!\rangle_{C_0(F_j)},\end{aligned}$$

and similarly for the A-valued inner products; thus there are isomorphisms $g_{ij} = 1 \otimes h_{ij}$ of $\mathfrak{X}_j^{F_{ij}}$ onto $\mathfrak{X}_i^{F_{ij}}$. But one can immediately verify the formulas

$$\begin{aligned} g_{ij}^{F_{ijk}} \circ g_{jk}^{F_{ijk}} &= \nu_{ijk} \cdot g_{ik}^{F_{ijk}}, \text{ and} \\ (u^i_s)^{F_{ij}}\big(g_{ij}(x)\big) &= \lambda_{ij}(\cdot, s) \cdot g_{ij}\big((u^j_s)^{F_{ij}}(x)\big),\end{aligned}$$

on elementary tensors, and hence it follows that (λ, ν) also represents $\delta(A, \alpha)$. ∎

PROPOSITION 6.5. *Suppose (A, G, α) and (B, G, β) are dynamical systems which are locally uniquely Morita equivalent to $\big(C_0(T), \tau\big)$, that T is paracompact, that T/G is Hausdorff, and that equivariant line bundles over (T, G) are locally trivial over T/G. If $\delta(A, \alpha) = \delta(B, \beta)$ in $\check{H}^2_G(T, \mathcal{S})$, then (A, α) and (B, β) are Morita equivalent over T.*

PROOF. Using Lemma 6.1, we can find closed G-invariant sets $\{F_i\}$ whose interiors form a cover $\mathfrak{U}$ of T and such that: for each $i \in I$, there are $(A^{F_i}, \alpha^{F_i}) -_{F_i} \big(C_0(F_i), \tau\big)$- and $(B^{F_i}, \beta^{F_i}) -_{F_i} \big(C_0(F_i), \tau\big)$-imprimitivity bimodules $(\mathfrak{E}_i, u^i)$ and $(\mathfrak{D}_i, v^i)$; and for each $i,j \in I$, isomorphisms $g_{ij}: \mathfrak{E}_j^{F_{ij}} \longrightarrow \mathfrak{E}_i^{F_{ij}}$ and $h_{ij}: \mathfrak{D}_j^{F_{ij}} \longrightarrow \mathfrak{D}_i^{F_{ij}}$. Let (λ, ν) and (λ', ν') be the associated cocycles in $Z^2_G(\mathfrak{U}, \mathcal{S})$. By assumption, these cocycles represent the same class in $\check{H}^2_G(T, \mathcal{S})$. Thus, refining $\mathfrak{U}$ if necessary, we may assume that there are continuous functions $\phi_{ij}: F_{ij} \longrightarrow \mathbb{T}$, $\sigma_i: F_i \times G \longrightarrow \mathbb{T}$ satisfying

$$\begin{aligned}\sigma_i(t, sr) &= \sigma_i(t, s)\sigma_i(s^{-1} \cdot t, r) \\ \lambda_{ij}(t, s) &= \overline{\phi_{ij}(t)}\phi_{ij}(s^{-1} \cdot t)\sigma_i(t, s)\overline{\sigma_j(t, s)}\lambda'_{ij}(t, s), \text{ and} \\ \nu_{ijk}(t) &= \phi_{ij}(t)\phi_{jk}(t)\overline{\phi_{ik}(t)}\nu'_{ijk}(t).\end{aligned}$$

Consequently we may replace v^i_s by $\sigma_i(\cdot, s) \cdot v^i_s$, h_{ij} by $\phi_{ij} \cdot h_{ij}$, and assume from here on that $(\lambda, \nu) = (\lambda', \nu')$.

As in the proof of Lemma 4.7, $\mathfrak{Y}_i = \mathfrak{E}_i \otimes_{C_0(F_i)} \tilde{\mathfrak{D}}_i$ is an $A^{F_i} -_{F_i} B^{F_i}$-imprimitivity bimodule, and there are isomorphisms $k_{ij}: \mathfrak{Y}_j^{F_{ij}} \longrightarrow \mathfrak{Y}_i^{F_{ij}}$ such that

$$k_{ij}(e \otimes \tilde{d}) = g_{ij}(e) \otimes h_{ij}(d)^{\sim}$$

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(recall that $\mathfrak{Y}_j^{F_{ij}} \cong \mathfrak{E}_j^{F_{ij}} \otimes_{C_0(F_{ij})} \tilde{\mathfrak{D}}_j^{F_{ij}}$ by [11; Lemma 1.10]). Since we have arranged that $(\lambda, \nu) = (\lambda', \nu')$, we have $k_{ij} \circ k_{jk} = k_{ik}$. Now we follow the construction in [11; Proposition 2.3]. Let $\mathfrak{Y}'$ be the collection of $(y_i) \in \prod_{i \in I} \mathfrak{Y}_i$ with the property that

$$k_{ij}(y_j^{F_{ij}}) = y_i^{F_{ij}}$$

for all $i, j \in I$. Then for $(x_i), (y_i) \in \mathfrak{Y}'$, and $t \in F_{ij}$,

$$\begin{aligned} {}_{A^{F_i}}\langle x_i, y_i\rangle(t) &= \big({}_{A^{F_i}}\langle x_i^{F_{ij}}, y_i^{F_{ij}}\rangle\big)^{F_{ij}}(t) \\ &= \Big({}_{A^{F_i}}\langle k_{ij}(x_j^{F_{ij}}), k_{ij}(y_j^{F_{ij}})\rangle\Big)^{F_{ij}}(t) \\ &= \big({}_{A^{F_j}}\langle x_j^{F_{ij}}, y_j^{F_{ij}}\rangle\big)^{F_{ij}}(t) \\ &= {}_{A^{F_j}}\langle x_j, y_j\rangle(t). \end{aligned}$$

A similar formula holds for $\langle \cdot, \cdot \rangle_{B^{F_i}}$. Thus if $x = (x_i)$ and $y = (y_i)$ are elements of $\mathfrak{Y}'$, and $t \in F_i$, we can define

$${}_A\langle x, y\rangle(t) = {}_{A^{F_i}}\langle x_i, y_i\rangle(t), \quad \text{and} \quad \langle x, y\rangle_B(t) = \langle x_i, y_i\rangle_{B^{F_i}}(t),$$

and the left-hand sides depend only on x, y, and t. Just as in [11], we see that when restricted to

$$\mathfrak{Y} = \{y \in \mathfrak{Y}' : t \longmapsto \| {}_A\langle y, y\rangle(t)\| \text{ vanishes at } \infty\},$$

${}_A\langle \cdot, \cdot\rangle$ and $\langle \cdot, \cdot\rangle_B$ define A- and B-valued inner products making $\mathfrak{Y}$ into an $A -_T B$-imprimitivity bimodule. Next, we deduce from the usual calculations that $u_s^i \otimes \tilde{v}_s^i$ induces an an action w^i of G on $\mathfrak{Y}_i$ such that $(A^{F_i}, \alpha^{F_i}) \sim_{\mathfrak{Y}_i, w^i} (B^{F_i}, \beta^{F_i})$. Then for $e \otimes \tilde{d} \in \mathfrak{Y}_i$, we have

$$\begin{aligned} k_{ij}\big(w_s^j(e \otimes \tilde{d})\big) &= g_{ij}\big(u_s^j(e)\big) \otimes \Big(h_{ij}\big(v_s^j(d)\big)\Big)^{\sim} \\ &= \Big(\lambda_{ij}(\cdot, s) \cdot u_s^i\big(g_{ij}(e)\big)\Big) \otimes \Big(\lambda_{ij}(\cdot, s) \cdot v_s^i\big(h_{ij}(d)\big)\Big)^{\sim}, \end{aligned}$$

which is equal to $w_s^i\big(k_{ij}(e \otimes \tilde{d})\big)$ in $\mathfrak{E}_j^{F_{ij}} \otimes_{C_0(F_{ij})} \tilde{\mathfrak{D}}_j^{F_{ij}}$; in other words,

$$k_{ij}\Big(\big(w_s^j(y)\big)^{F_{ij}}\Big) = (w_s^i)^{F_{ij}}\big(k_{ij}(y^{F_{ij}})\big).$$

This last equation means there is an action w of G on $\mathfrak{Y}'$ such that $w_s(y) = \big(w_s^i(y_i)\big)$; because $\|a(t)\| = \|\alpha_s(a)(s \cdot t)\|$, $y \in \mathfrak{Y}$ implies $w_s(y) \in \mathfrak{Y}$, and w is also an action of G on $\mathfrak{Y}$. Routine calculations show that

$${}_A\langle w_s(y), w_s(y')\rangle = \alpha_s({}_A\langle y, y'\rangle) \quad \text{and} \quad \langle w_s(y), w_s(y')\rangle_B = \beta_s(\langle y, y'\rangle_B),$$

so $(\mathfrak{Y}, u)$ implements the required equivalence between (A, α) and (B, β). ∎

PROPOSITION 6.6. *If (T, G) is a locally compact transformation group with T paracompact, and $(\lambda, \nu) \in Z_G^2(T, \mathcal{S})$, there is a system (A, G, α) which is locally Morita equivalent to $\big(C_0(T), \tau\big)$ and has $\delta(A, \alpha) = [\lambda, \nu]$.*

PROOF. As in [17, Proposition 3.6], we use the construction of [14]. First, we set $F_i = \overline{p^{-1}(N_i)}$, refine the cover $\{N_i\}$ to ensure it is locally finite, and shrink the sets N_i

a little to ensure that λ_{ij}, ν_{ijk} are defined on F_{ij}, F_{ijk}. Next, we adjust by a coboundary to make $\{\nu_{ijk}\}$ alternating [17, Corollary 3.5]. Then we let

$$A = \Big\{ \sum_{j,k} \phi_{jk} e_{jk} : \phi_{jk} \in C_0(T), \phi_{jk} \equiv 0 \text{ outside } F_{jk} \Big\},$$

where $e_{ij}e_{jk} = \overline{\nu_{ijk}} e_{jk}$. (See [14, Theorem 1] for a more precise definition; the idea is that A is a continuous-trace algebra with spectrum T, and Dixmier-Douady class $\delta(A) = [\nu]$.) We define an action α of G on A by

$$\alpha_s\Big(\sum \phi_{jk} e_{jk}\Big) = \sum \overline{\lambda_{jk}(\cdot, s)} \tau_s(\phi_{jk}) e_{jk},$$

where $\lambda_{jk}(\cdot, s)\tau_s(\phi_{jk})$ is by definition 0 off F_{jk}. To see that A is locally Morita equivalent to $C_0(T)$, we let

$$\mathfrak{X}_i = \Big\{ \sum_k \phi_k e_k : \phi_k \in C_0(F_i), \phi_k \equiv 0 \text{ off } F_k \Big\},$$

and define

$$\Big\langle \sum \phi_j e_j, \sum \psi_k e_k \Big\rangle_{C_0(F_i)} = \sum \bar{\phi}_j \psi_j$$

$${}_{A^{F_i}}\Big\langle \sum \phi_j e_j, \sum \psi_k e_k \Big\rangle = \sum_{j,k} \nu_{ijk} \phi_j \bar{\psi}_k e_{jk}$$

$$\Big(\sum \phi_{jk} e_{jk}\Big) \cdot \Big(\sum \psi_l e_l\Big) = \sum_j \Big(\sum_k \overline{\nu_{ijk}} \phi_{jk} \psi_k\Big) e_j$$

$$\Big(\sum \psi_l e_l\Big) \cdot f = \sum f \psi_l e_l.$$

A lot of boring calculations confirm that the completion of $\mathfrak{X}_i$ is an $A^{F_i} - C_0(F_i)$-imprimitivity bimodule: the cocycle identity for $\{\nu_{ijk}\}$ is required to prove both that $a\langle x, y\rangle = \langle a \cdot x, y\rangle$, and that $(ab) \cdot x = a \cdot (b \cdot x)$, but the rest seems to be routine. We define an action u^i of G on $\mathfrak{X}_i$ by

$$u^i_s\Big(\sum \phi_j e_j\Big) = \sum \lambda_{ij}(\cdot, s) \phi_j e_j,$$

and verify equally tediously that $(\mathfrak{X}_i, u^i)$ is an $(A^{F_i}, \alpha^{F_i}) - \big(C_0(F_i), \tau\big)$-imprimitivity bimodule, so that (A, α) is locally Morita equivalent to $\big(C_0(T), \tau\big)$.

Before computing the Dixmier-Douady class of (A, α), we note that restricting the coefficients ϕ_k to F_{ij} induces an isomorphism of $\mathfrak{X}_i^{F_{ij}}$ onto

$$\mathfrak{X}_{ij} = \Big\{ \sum \phi_k e_k : \phi_k \in C_0(F_{ij}), \phi_k \equiv 0 \text{ off } F_k \Big\};$$

to see the surjectivity, note that if $\phi_k \in C_0(F_{ij})$ vanishes off F_k, setting $\phi_k = 0$ in $F_i \backslash (F_j \cup F_k)$ gives a continuous function on the closure of $\big(F_i \backslash (F_j \cup F_k)\big) \cup F_{ijk}$, which by Urysohn's Lemma extends to a continuous function on F_i which vanishes off F_k. We can now define isomorphisms g_{ij} of $\mathfrak{X}_{ij} \cong \mathfrak{X}_j^{F_{ij}}$ onto $\mathfrak{X}_{ij} \cong \mathfrak{X}_i^{F_{ij}}$ by

$$g_{ij}\Big(\sum \phi_k e_k\Big) = \sum \nu_{ijk} \phi_k e_k,$$

and verify that Equations (6.1) and (6.2) both hold, so that (λ, ν) represents $\delta(A, \alpha)$, as claimed. ∎

This proposition completes the proof of Theorem 6.3.

7. **The Dixmier-Douady classes of N-principal systems.** If (A, G, α) is an N-principal system (so that G is abelian), then the dual system $(A \times_\alpha G, \hat{G}, \hat{\alpha})$ is an $N^\perp$-principal system [13, §2]. In this section we shall show how to compute the Dixmier-Douady class $\delta(A \times_\alpha G, \hat{G}, \hat{\alpha}) \in \check{H}^2_{\hat{G}}\big((A \rtimes_\alpha G)^\wedge, \mathcal{S}\big)$ from $\delta(A, G, \alpha)$, and, in particular, how to determine the spectrum $(A \rtimes_\alpha G)^\wedge$ and the Dixmier-Douady class $\delta(A \rtimes_\alpha G)$. We saw in [17] that the principal bundles $p: A \longrightarrow \hat{A}/G$ and $q: (A \rtimes_\alpha G)^\wedge \longrightarrow \hat{A}/G$ determine $\delta(A \rtimes_\alpha G)$ modulo classes pulled back along q from $\check{H}^2(\hat{A}, \mathcal{S})$. After deriving our formula, we shall show that it is consistent with the restrictions imposed by [17], and that it generalizes the one obtained in [12, 3.5] for the special case in which $\hat{A} = Y/N$ for some principal G-bundle Y.

Just to be sure of our conventions, we recall how $\check{H}^1(Z, \mathcal{G})$ is identified with the collection of isomorphism classes of G-bundles. Given $\{s_{ij}\}$ in $Z^1(\mathfrak{U}, \mathcal{G})$, representing $\omega \in \check{H}^1(Z, \mathcal{G})$, we define a corresponding G-bundle by forming the quotient space

$$F_\omega = \coprod_{\iota \in I} U_\iota \times G \sim, \tag{7.1}$$

in which (i, z, r) is identified with $\big(j, z, rs_{ij}(z)\big)$, and G acts via $s \cdot (i, z, r) = (i, z, sr)$.

LEMMA 7.1. *Let G be an abelian group, let T be a locally compact paracompact space which is a G/N-bundle over Z, and let $\mathfrak{U} = \{N_\iota\}_{\iota \in I}$ be an open cover of Z. If (λ, ν) is in $Z^2_G\big(p^{-1}(\mathfrak{U}), \mathcal{S}\big)$, then for $n \in N$, $\lambda_{ij}(\cdot, n)$ is constant on G-orbits, and there is a cocycle $\{\gamma^\lambda_{ij}\}$ in $Z^1(\mathfrak{U}, \hat{\mathcal{N}})$ such that $\gamma^\lambda_{ij}\big(p(x)\big) = \lambda_{ij}(x, n)$. The map $(\lambda, \nu) \longrightarrow [\gamma^\lambda_{ij}]$ induces a homomorphism $b: \check{H}^2_G(X, \mathcal{S}) \longrightarrow \check{H}^1(Z, \hat{\mathcal{N}})$.*

PROOF. Because N acts trivially on T, and G is abelian, the cocycle identity Equation (5.1) implies that $\lambda_{ij}(\cdot, n)$ is constant on orbits (*cf.* Corollary 4.12). The last assertion follows similarly from Equations (5.3) and (5.4). ∎

LEMMA 7.2. *Suppose that $\hat{G} \longrightarrow \hat{N}$ has local sections, $p: T \longrightarrow G$ is a principal G/N-bundle, and (A, G, α) is locally Morita equivalent to $\big(C_0(T), G, \tau\big)$. Then (A, G, α) is an N-principal system, and the quasi-orbit map $q: (A \rtimes_\alpha G)^\wedge \longrightarrow T/G$ is a principal $\hat{N}$-bundle satisfying $[q] = b\big(\delta(A, G, \alpha)\big)$.*

PROOF. Corollary 4.5 says that (A, G, α) is N-principal, and hence it follows from [13, Theorem 2.2] that the quasi-orbit map q is a principal $\hat{N}$-bundle. To compute transition functions for q, we use [17, Proposition 2.6], and resume the notation of Lemmas 6.1 and 6.2. Let $v^\iota: N \longrightarrow UM(A^{F_\iota})$ be the local Green twisting maps obtained from u^ι by identifying $\mathcal{L}\big((\mathcal{X}_\iota)_{C_0(F_\iota)}\big)$ with $M(A^{F_\iota})$. Then for $x, y, z \in \mathcal{X}^{F_{ij}}_j$, Equations (4.2) and (6.1) imply

$$\begin{aligned}
(v^\iota_n {}_{A^{F_{ij}}}\langle x, y\rangle) \cdot g_{ij}(z) &= \big(v^\iota_n {}_{A^{F_{ij}}}\langle g_{ij}(x), g_{ij}(y)\rangle\big) \cdot g_{ij}(z) \\
&= u^\iota_n\big(g_{ij}(x)\big)\langle g_{ij}(y), g_{ij}(z)\rangle_{C_0(F_{ij})} \\
&= \lambda_{ij}(\cdot, n) \cdot g_{ij}(u^j_n(x)\, {}_{C_0(F_{ij})}\langle y, z\rangle) \\
&= \gamma_{ij}(\cdot)(n) g_{ij}(v^j_n {}_{A^{F_{ij}}}\langle x, y\rangle \cdot z) \\
&= \gamma_{ij}(\cdot)(n)(v^j_n {}_{A^{F_{ij}}}\langle x, y\rangle) \cdot g_{ij}(z).
\end{aligned}$$

Thus $v_n^\iota = \gamma_{\iota j}(\cdot)(n)v_n^j$, and [17, Proposition 2.6] implies that $\{\gamma_{\iota j}\}$ is a cocycle in $Z^1(\{N_\iota\}, \mathcal{S})$ representing $[q]$.

■

THEOREM 7.3. *Let (A, G, α) be a dynamical system, in which A has continuous trace, G is abelian, and the spectrum $p: T \longrightarrow T/G$ is a paracompact principal G/N-bundle. Suppose that (A, α) is locally Morita equivalent to $\big(C_0(T), \tau\big)$, and that both $G \longrightarrow G/N$ and $\hat{G} \longrightarrow \hat{N}$ have local sections. Then the quasi-orbit map $q: (A \rtimes_\alpha G)^\wedge \longrightarrow T/G$ is a principal $\hat{N}$-bundle, and $\big((A \rtimes_\alpha G)^\wedge, \hat{G}, \hat{\alpha}\big)$ is locally Morita equivalent to $\Big(C_0\big((A \rtimes_\alpha G)^\wedge\big), \hat{\tau}\Big)$. Let $\mathfrak{N} = \{N_\iota\}_{\iota \in I}$ be any open cover of T/G which is fine enough to ensure there are equivariant projections $w_\iota: p^{-1}(N_\iota) \longrightarrow G/N$, there are continuous functions $s_{\iota j}: N_{\iota j} \longrightarrow G$ such that $s_{\iota j}(z)N = w_\iota(z)^{-1}w_j(z)$, and there is a representative $(\lambda, \nu) \in Z_G^2\big(p^{-1}(\mathfrak{N}), \mathcal{S}\big)$ for $\delta(A, G, \alpha)$. Then there are equivariant projections $\sigma_\iota: q^{-1}(N_\iota) \longrightarrow \hat{N}$ such that $\gamma_{\iota j}^\lambda\big(q(\pi)\big) = \sigma_\iota(\pi)^{-1}\sigma_j(\pi)$, and $\delta(A \rtimes_\alpha G, \hat{G}, \hat{\alpha}) = [\hat{\lambda}, \hat{\nu}]$ where $(\hat{\lambda}, \hat{\nu}) \in Z_{\hat{G}}^2\big(q^{-1}(\mathfrak{N}), \mathcal{S}\big)$ is defined by*

$$\hat{\lambda}_{\iota j}(\pi, \chi) = \overline{\chi\big(s_{\iota j}(z)\big)}, \text{ and} \tag{7.2}$$

$$\hat{\nu}_{\iota jk}(\pi) = \lambda_{jk}\big(c_\iota(z), s_{\iota j}(z)\big)\nu_{\iota jk}\big(c_\iota(z)\big)\overline{\sigma_k(\pi)\big(n_{\iota jk}(z)\big)}, \tag{7.3}$$

where we have written $z = q(\pi)$, $n_{\iota jk}(z) = s_{\iota j}(z)s_{jk}(z)s_{\iota k}(z)^{-1}$, and $c_\iota(z)$ is defined by $w_\iota\big(c_\iota(z)\big) = N$.

PROOF. Lemma 7.2 implies that q is a principal $\hat{N}$-bundle with $[q] = b\big(\delta(A, G, \alpha)\big) = [\gamma^\lambda]$. Therefore open covers $\mathfrak{N}' = \{N'_\iota\}_{\iota \in I}$ of T/G satisfying the requirements of the proposition do exist. Since the class of $(\hat{\lambda}, \hat{\nu})$ is invariant under refinement, we may assume that $\mathfrak{N}'$ is locally finite and that there is an open cover $\mathfrak{N} = \{N_\iota\}_{\iota \in I}$ such that $\overline{N_\iota} \subseteq N'_\iota$. Refining $\mathfrak{N}$ as necessary, we may assume that there exist $\{F_\iota\}$, $(\mathfrak{X}_\iota, u^\iota)$, and $g_{\iota j}$, as in Lemma 6.1. We also define local trivializations for p by $h_\iota(x) = \big(p(x), w_\iota(x)\big)$. Similarly, we put $Q_\iota = \overline{q^{-1}(N_\iota)}$ and $\phi_\iota(\pi) = \big(q(\pi), \sigma_\iota(\pi)\big)$.

We begin by building the appropriate local modules. For each i we need an $A^{F_\iota} \rtimes_\alpha G -_{Q_\iota} C_0(Q_\iota)$ imprimitivity bimodule $\mathfrak{E}_\iota$, but we shall actually construct a $C_c(G, A^{F_\iota}) - C_c(\bar{N}_\iota \times N)$ pre-imprimitivity bimodule $\mathfrak{E}_\iota^0$ and complete. We view $C_c(\bar{N}_\iota \times N)$ as a dense subalgebra of $C_0(Q_\iota)$ using the Fourier transform: if $b \in C_c(\bar{N}_\iota \times N)$, then b represents the function $\hat{b} \in C_0(Q_\iota)$ defined by

$$\hat{b}\big(\phi_\iota^{-1}(z, \gamma)\big) = \int_N b(z, n)\gamma(n)\, dn.$$

Recall from [2; §6] that $\mathfrak{Y}_\iota^0 = C_c(G, \mathfrak{X}_\iota)$ is a pre-$A^{F_\iota} \rtimes_\alpha G - C_0(F_\iota) \rtimes_\tau G$-imprimitivity bimodule. To ease the notation, we shall write B_ι for $A^{F_\iota} \rtimes_\alpha G$ and D_ι for $C_0(F_\iota) \rtimes_\tau G$. Then the inner products are given by the formulas

$$_{B_\iota}\langle \xi, \eta\rangle(r) = \int_G {}_{A^{F_\iota}}\Big\langle \xi(s), u_r^\iota\big(\eta(r^{-1}s)\big)\Big\rangle\, ds \tag{7.4}$$

$$\langle \xi, \eta\rangle_{D_\iota}(r)\big(h_\iota^{-1}(z, tN)\big) = \int_G \langle \xi(s), \eta(sr)\rangle_{C_0(F_\iota)}\big(h_\iota^{-1}(z, stN)\big)\, ds.$$

The left action of B_t on $\mathfrak{Y}_t$ is the integrated form of the covariant pair (V^t, M_t) where

$$V_t^t\xi(s) = u_t^t\big(\xi(t^{-1}s)\big), \text{ and } M_t(a)\xi(s) = a\cdot\xi(s),$$

for $s,t\in G$, $\xi\in\mathfrak{Y}_t^0$, and $a\in A^{F_t}$. The right action of D_t is determined by

$$\eta\cdot\phi(r) = \int_G \eta(s)\cdot\phi\big(s^{-1}r, s^{-1}\cdot(\cdot)\big)\,ds,$$

for $\eta\in\mathfrak{Y}_t^0$ and $\phi\in C_c(G\times F_t)$.

Next we observe that $\mathfrak{Z}_t^0 = C_c(\bar{N}_t\times G)$ is a pre-D_t–$C_0(\bar{N}_t\times\hat{N})$-imprimitivity bimodule with inner products

$$\text{(7.5)}\qquad {}_{D_t}\langle f,g\rangle(r)\big(h_t^{-1}(z,sN)\big) = \int_N f(z,sn^{-1})\overline{g(z,r^{-1}sn^{-1})}\,dn$$

$$\langle f,g\rangle_{C_0(Q_t)}(z,n) = \int_G \overline{f(z,s^{-1})}g(z,s^{-1}n)\,ds,$$

and actions given by

$$\text{(7.6)}\qquad \phi\cdot f(z,s) = \int_G \phi\big(r, h_t^{-1}(z,sN)\big)f(z,r^{-1}s)\,dr$$

$$\text{(7.7)}\qquad f\cdot b(z,s) = \int_N f(z,sn^{-1})b(z,n)\,dn.$$

Note that the completion $\mathfrak{Z}_t$ of $\mathfrak{Z}_t^0$ is isomorphic to the imprimitivity bimodule tensor product $C_0(\bar{N}_t)\otimes\mathfrak{W}_t$ where $\mathfrak{W}_t$ denotes the usual $C_0(G/N)\rtimes_\tau G - C^*(N)$-imprimitivity bimodule [19; §7] and $C_0(\bar{N}_t)$ is viewed as a $C_0(\bar{N}_t) - C_0(\bar{N}_t)$-imprimitivity bimodule in the standard way.

Our modules will be given by the module tensor product $\mathfrak{E}_t = \mathfrak{Y}_t\otimes_{D_t}\mathfrak{Z}_t$. Notice that on elementary tensors,

(7.8)

$$\langle\!\langle \xi\otimes f, \eta\otimes g\rangle\!\rangle_{C_0(Q_t)}(z,n)$$
$$= \langle\langle\eta,\xi\rangle_{D_t}\cdot f, g\rangle_{C_0(Q_t)}(z,n),$$

which by Equation (7.5) is

$$= \int_G \overline{\langle\eta,\xi\rangle_{D_t}\cdot f(z,s^{-1})}g(z,s^{-1}n)\,ds,$$

which by Equation (7.6) is

$$= \int_G\int_G \overline{\langle\eta,\xi\rangle_{D_t}(r)\big(h_t^{-1}(z,s^{-1}N)\big)f(z,r^{-1}s^{-1})}\,dr\,g(z,s^{-1}n)\,ds,$$

which by Equation (7.4) is

$$= \int_G\int_G\int_G \overline{\langle\eta(t),\xi(tr)\rangle_{C_0(F_t)}\big(h_t^{-1}(z,ts^{-1}N)\big)}\cdot$$
$$\cdot\overline{f(z,r^{-1}s^{-1})}g(z,s^{-1}n)\,dt\,dr\,ds$$
$$= \int_G\int_G\int_G \langle\xi(tr),\eta(t)\rangle_{C_0(F_t)}\big(h_t^{-1}(z,ts^{-1}N)\big)\cdot$$
$$\cdot\overline{f(z,r^{-1}s^{-1})}g(z,s^{-1}n)\,dt\,dr\,ds$$
$$= \int_G\int_G\int_G \langle f(z,r^{-1}s^{-1})\xi(tr), g(z,s^{-1}n)\eta(t)\rangle_{C_0(F_t)}\cdot$$
$$\cdot\big(h_t^{-1}(z,ts^{-1}N)\big)\,dt\,dr\,ds$$

https://doi.org/10.4153/CJM-1993-057-8 Published online by Cambridge University Press

The module $\mathfrak{E}_\iota$ is the completion of the algebraic tensor product $\mathfrak{E}_\iota^{00} = C_c(G, \mathfrak{X}_\iota) \odot C_c(\bar{N}_\iota \times G)$ with respect to $\langle\langle \cdot, \cdot \rangle\rangle_{C_0(Q_\iota)}$. We can view $\mathfrak{E}_\iota^{00}$ as a subspace of $\mathfrak{E}_\iota^0 = C_c(G \times \bar{N}_\iota \times G, \mathfrak{X}_\iota)$, and using Equation (7.8), we may extend $\langle\langle \cdot, \cdot \rangle\rangle_{C_0(Q_\iota)}$ to a pre-inner product on $\mathfrak{E}_\iota^0$: if $\xi, \eta \in \mathfrak{E}_\iota^0$, then

(7.9)

$$\langle\langle \xi, \eta \rangle\rangle_{C_0(Q_\iota)}(z, n) = \int_G\int_G\int_G \langle \xi(tr, z, r^{-1}s^{-1}), \eta(t, z, s^{-1}n) \rangle_{C_0(F_\iota)}\big(h_\iota^{-1}(z, ts^{-1}N)\big)\, dt\, dr\, ds.$$

Notice that if $\xi \in C_c(G \times \bar{N}_\iota \times G, \mathfrak{X}_\iota)$ then there are compact sets $K \subseteq G$ and $K' \subseteq \bar{N}_\iota$ such that $e \in K$, $K = K^{-1}$, and $\operatorname{supp} \xi \subseteq K \times K' \times K$. Then $\operatorname{supp} \langle\langle \xi, \xi \rangle\rangle_{C_0(Q_\iota)} \subseteq K' \times K^4 \cap N$, and

$$|\langle\langle \xi, \xi \rangle\rangle_{C_0(Q_\iota)}(x, n)| \leq \|\xi\|_\infty^2 \int_{K^3}\int_{K^2}\int_K dt\, dr\, ds \leq \|\xi\|_\infty^2 \mu(K^3)^3.$$

It follows that convergence in the inductive limit topology in $\mathfrak{E}_\iota^0 = C_c(G \times \bar{N}_\iota \times G, \mathfrak{X}_\iota)$ implies convergence in the (semi-) norm on $\mathfrak{E}_\iota^0$ induced by $\langle\langle \cdot, \cdot \rangle\rangle_{C_0(Q_\iota)}$, and we can therefore view $\mathfrak{E}_\iota$ as the completion of $\mathfrak{E}_\iota^0$ with respect to the inner product determined by Equation (7.9).

Similarly, the B_ι-inner product extends to all of $\mathfrak{E}_\iota^0 = C_c(G \times \bar{N}_\iota \times G, \mathfrak{X}_\iota)$, and satisfies

(7.10)

$${}_{B_\iota}\langle\langle \xi, \eta \rangle\rangle(r)\big(h_\iota^{-1}(z, vN)\big) = \int_G\int_G\int_N {}_{A^{F_\iota}}\big\langle \xi(s, z, s^{-1}vn^{-1}), u_r^\iota\big(\eta(r^{-1}t, z, t^{-1}vn^{-1})\big)\big\rangle\, dt\, ds\, dn.$$

There is a subtlety in this calculation: ${}_{B_\iota}\langle\langle \xi, \eta \rangle\rangle(r)$ is an element of A^{F_ι}, which is determined by its images in the primitive quotients $A^{F_\iota}(t)$ for $t = h_\iota^{-1}(z, vN)$. To avoid the question of defining elements of $\mathfrak{X}_\iota$ by their images in the corresponding quotients, we observe that if $x, y \in \mathfrak{X}_\iota$ and $f \in C_0(F_\iota)$, then

$${}_{A^{F_\iota}}\langle x, y \cdot f \rangle(t) = {}_{A^{F_\iota}}\langle x, y \rangle(t) f(t).$$

Using this trick, we can compute the left-hand side of Equation (7.10) for ξ, η of the form $x \otimes f$, $y \otimes g$, much as before. (One might be surprised to notice that the right-hand side of Equation (7.10) depends on the functions

$$(z, vN) \longmapsto \int \xi(s, z, s^{-1}vN)\, ds;$$

this arises because vectors of the form $\eta \cdot \phi \otimes f - \eta \otimes \phi \cdot f$ have length zero with respect to ${}_{B_\iota}\langle\langle \cdot, \cdot \rangle\rangle$, and hence are modded out when completing $\mathfrak{E}_\iota^0$.)

For $\xi \in \mathfrak{E}_{J\iota}^0 = C_c(G \times \bar{N}_{\iota J} \times G, \mathfrak{X}_J^{F_{\iota J}})$, we define $\tilde{\kappa}_{\iota J}(\xi) \in \mathfrak{E}_{\iota J}^0$ by

$$\tilde{\kappa}_{\iota J}(\xi)(r, z, s) = \lambda_{\iota J}\big(h_\iota^{-1}(z, rsN), rs\big) g_{\iota J}\Big(\xi\big(r, z, ss_{\iota J}(z)\big)\Big).$$

We wish to show that $\tilde{\kappa}_{\iota J}$ induces an imprimitivity bimodule isomorphism $\kappa_{\iota J}$ of $\mathfrak{E}_J^{F_{\iota J}}$ onto $\mathfrak{E}_\iota^{F_{\iota J}}$. We claim that to do this, it is enough to check the following:

(1) $\tilde{\kappa}_{\iota J}\big(M_J(a) \cdot \xi\big) = M_\iota(a) \cdot \tilde{\kappa}_{\iota J}(\xi)$,

(2) $\tilde{\kappa}_{\iota J}\big(V_t^J(\xi)\big) = V_t^\iota\big(\tilde{\kappa}_{\iota J}(\xi)\big)$,

(3) $\langle\langle \tilde{\kappa}_{\iota J}(\xi), \tilde{\kappa}_{\iota J}(\eta) \rangle\rangle_{C_0(Q_\iota)}(z, n) = \hat{\alpha}_{\gamma_{\iota J}(z)}^{-1}(\langle\langle \xi, \eta \rangle\rangle_{C_0(Q_J)})(z, n)$, and

(4) $\tilde{\kappa}_{\iota J}(\xi \cdot b) = \tilde{\kappa}_{\iota J}(\xi) \cdot \hat{\alpha}_{\gamma_{\iota J}(z)}^{-1}(b)$,

where $a \in A^{F_{ij}}$, $\xi, \eta \in \mathfrak{X}_j^{F_{ij}}$, $b \in C_c(\bar{N}_{ij} \times N)$, and $\hat{\alpha}_\gamma(b)(z,n) = \overline{\gamma(n)}b(z,n)$.

To verify the claim, we first observe that property (3) will imply that $\tilde{\kappa}_{ij}$ preserves the $C_0\big(q^{-1}(\bar{N}_{ij})\big)$-inner products:

$$\begin{aligned}\langle\!\langle \tilde{\kappa}_{ij}(\xi), \tilde{\kappa}_{ij}(\eta)\rangle\!\rangle^{\wedge}_{C_0(Q_i)}\big(\phi_i^{-1}(z,\gamma)\big) &= \int_N \langle\!\langle \tilde{\kappa}_{ij}(\xi), \tilde{\kappa}_{ij}(\eta)\rangle\!\rangle_{C_0(Q_i)}(z,n)\gamma(n)\,dn \\ &= \int_N \langle\!\langle \xi, \eta\rangle\!\rangle_{C_0(Q_j)}(z,n)\gamma_{ij}(z)(n)\gamma(n)\,dn \\ &= \langle\!\langle \xi, \eta\rangle\!\rangle^{\wedge}_{C_0(Q_j)}\Big(\phi_j^{-1}\big(z, \gamma\gamma_{ij}(z)\big)\Big).\end{aligned}$$

It follows that $\tilde{\kappa}_{ij}$ extends to an inner product preserving map κ_{ij} from $\mathfrak{E}_j^{F_{ij}}$ onto $\mathfrak{E}_i^{F_{ij}}$. On the other hand, (4) implies that κ_{ij} is $C_0\big(q^{-1}(\bar{N}_{ij})\big)$-linear, and therefore that κ_{ij} is an isomorphism of Hilbert $C_0\big(q^{-1}(N_{ij})\big)$-modules. Since (1) and (2) imply that κ_{ij} is $A^{F_{ij}} \rtimes_\alpha G$-linear, it must also preserve the left inner product, and is actually an isomorphism of imprimitivity bimodules.

Fortunately, (1) follows readily from the fact that $g_{ij}(a \cdot x) = a \cdot g_{ij}(x)$ for $a \in A^{F_{ij}}$ and $x \in \mathfrak{X}_j^{F_{ij}}$.

Next, consider $\langle\!\langle \tilde{\kappa}_{ij}\big(V^j_\ell(\xi)\big), \eta\rangle\!\rangle_{C_0(Q_i)}(z,n)$. Writing $c_i(z)$ for $h_i^{-1}(z,N)$ and u^{ji} for $(u^j)^{F_{ij}}$, this expands to

$$\int_G\int_G\int_G \overline{\lambda_{ij}(ts^{-1}\cdot c_i(z), ts^{-1})} \cdot \Big\langle g_{ij}\Big(u^{ji}_\ell\Big(\xi\big(\ell^{-1}tr, z, r^{-1}s^{-1}s_{ij}(z)\big)\Big)\Big), \eta(t,z,s^{-1}n)\Big\rangle_{C_0(F_i)}\big(ts^{-1}\cdot c_i(z)\big)\,dt\,dr\,ds,$$

which, using Lemma 6.2(1), the identity $\langle f\cdot x, y\rangle(t) = f(t)\langle x,y\rangle(t)$ for $f \in C_0(F_i)$, $x, y \in \mathfrak{X}_i$, and the cocycle identity Equation (5.1), is equal to

$$\int_G\int_G\int_G \overline{\lambda_{ij}\big(\ell^{-1}ts^{-1}\cdot c_i(z), \ell^{-1}ts^{-1}\big)} \cdot \Big\langle u^{ij}_\ell\Big(g_{ij}\Big(\xi\big(\ell^{-1}tr, z, r^{-1}s^{-1}s_{ij}(z)\big)\Big)\Big), \eta(t,z,s^{-1}n)\Big\rangle_{C_0(F_i)}\big(ts^{-1}\cdot c_i(z)\big)\,dt\,dr\,ds,$$

which equals $\langle\!\langle V^i_\ell\big(\tilde{\kappa}_{ij}(\xi)\big), \eta\rangle\!\rangle_{C_0(Q_i)}(z,n)$. It follows that (2) holds.

Recall that $\lambda_{ij}(t, sn) = \gamma_{ij}(z)(n)\lambda_{ij}(t,s)$. Thus, if $\xi, \eta \in \mathfrak{E}^0_{ij}$, then

$$\begin{aligned}\langle\!\langle \tilde{\kappa}_{ij}(\xi), \tilde{\kappa}_{ij}(\eta)\rangle\!\rangle_{C_0(Q_i)}(z,n) &= \gamma_{ij}(z)(n)\int_G\int_G\int_G \Big\langle g_{ij}\Big(\xi\big(tr, z, r^{-1}s^{-1}s_{ij}(z)\big)\Big), \\ &\qquad g_{ij}\Big(\eta\big(t,z,s^{-1}ns_{ij}(z)\big)\Big)\Big\rangle_{C_0(F_i)}\big(ts^{-1}\cdot c_i(z)\big)\,dt\,dr\,ds \\ &= \gamma_{ij}(z)(n)\int_G\int_G\int_G\big\langle \xi\big(tr, z, r^{-1}s^{-1}s_{ij}(z)\big), \\ &\qquad \eta\big(t,z,s^{-1}ns_{ij}(z)\big)\big\rangle_{C_0(F_j)}\big(ts^{-1}\cdot c_i(z)\big)\,dt\,dr\,ds,\end{aligned}$$

which, replacing s by $s_{ij}(z)s$ and using $s_{ij}(z)\cdot c_j(z) = c_i(z)$, equals

$$\begin{aligned}\gamma_{ij}(z)(n)\int_G\int_G\int_G \langle \xi(tr,z,r^{-1}s^{-1}), \eta(t,z,s^{-1}n)\rangle_{C_0(F_j)}\big(ts^{-1}\cdot c_j(z)\big)\,dt\,dr\,ds \\ = \gamma_{ij}(z)(n)\langle\!\langle \xi,\eta\rangle\!\rangle_{C_0(Q_j)}(z,n) \\ = \hat{\alpha}^{-1}_{\gamma_{ij}(z)}(\langle\!\langle \xi,\eta\rangle\!\rangle_{C_0(Q_j)})(z,n).\end{aligned}$$

This verifies (3).

Now suppose that $b \in C_c(\bar{N}_\iota \times N)$ and $\xi \in \mathfrak{E}^0_{\iota\jmath}$. Then by Equation (7.7),

$$\begin{aligned}\tilde{\kappa}_{\iota\jmath}(\xi \cdot b)(r,z,s) &= \lambda_{\iota\jmath}(rs \cdot c_\iota(z), rs) g_{\iota\jmath}\Big(\xi \cdot b\big(r,z,ss_{\iota\jmath}(z)\big)\Big)\\ &= \int_N \lambda_{\iota\jmath}(rs \cdot c_\iota(z), rs) g_{\iota\jmath}\Big(\xi\big(r,z,ss_{\iota\jmath}(z)n^{-1}\big)\Big) b(z,n)\,dn,\end{aligned}$$

which, using $\lambda_{\iota\jmath}(t,sn) = \gamma_{\iota\jmath}(z)(n)\lambda_{\iota\jmath}(t,s)$, is

$$\begin{aligned}&= \int_N \lambda_{\iota\jmath}(rs \cdot c_\iota(z), rsn^{-1}) g_{\iota\jmath}\Big(\xi\big(r,z,sn^{-1}s_{\iota\jmath}(z)\big)\Big)\gamma_{\iota\jmath}(z)(n) b(z,n)\,dn\\ &= \int_N \tilde{\kappa}_{\iota\jmath}(\xi)(r,z,sn^{-1})\hat{\alpha}^{-1}_{\gamma_{\iota\jmath}(z)}(b)(z,n)\,dn\\ &= \tilde{\kappa}_{\iota\jmath}(\xi) \cdot \hat{\alpha}^{-1}_{\gamma_{\iota\jmath}(z)}(b)(r,z,s).\end{aligned}$$

This verifies (4), and proves the assertions about $\kappa_{\iota\jmath}$.

Next we define an action ν^ι of $\hat{G}$ on $\mathfrak{E}_\iota$: for $\xi \in \mathfrak{E}^0_\iota$, let $\nu^\iota_\chi(\xi)(r,z,s) = \overline{\chi(rs)}\xi(r,z,s)$. It is easy to verify that

$$\begin{aligned}{}_{B_\iota}\langle\langle \nu^\iota_\chi(\xi), \nu^\iota_\chi(\eta)\rangle\rangle(r) &= \overline{\chi(r)}\,{}_{B_\iota}\langle\langle \xi,\eta\rangle\rangle(r) = \hat{\alpha}_\chi({}_{B_\iota}\langle\langle \xi,\eta\rangle\rangle)(r),\\ \langle\langle \nu^\iota_\chi(\xi), \nu^\iota_\chi(\eta)\rangle\rangle^\wedge_{C_0(Q_\iota)}\big(\phi_\iota^{-1}(x,\gamma)\big) &= \langle\langle \xi,\eta\rangle\rangle^\wedge_{C_0(Q_\iota)}\big(\phi_\iota^{-1}(z,\gamma\bar{\chi}|_N)\big)\\ &= \hat{\tau}_\chi(\langle\langle \xi,\eta\rangle\rangle^\wedge_{C_0(Q_\iota)})\big(\phi_\iota^{-1}(z,\gamma)\big), \text{ and}\\ \tilde{\kappa}_{\iota\jmath}\big((\nu^\jmath_\chi)^{F_{\iota\jmath}}(\xi^{F_{\iota\jmath}})\big)(r,z,s) &= \chi\big(s_{\iota\jmath}(z)\big)(\nu^\iota_\chi)^{F_{\iota\jmath}}\big(\tilde{\kappa}_{\iota\jmath}(\xi^{F_{\iota\jmath}})\big)(r,z,s).\end{aligned}$$

It follows that ν^ι extends to an action on all of $\mathfrak{E}_\iota$. (Note that $\chi \longmapsto \nu^\iota_\chi$ is continuous in the inductive limit topology, and therefore strongly continuous.)

The $N^\perp$-principal system $(A \rtimes_\alpha G, \hat{G}, \hat{\alpha})$ is locally Morita equivalent to the system $C_0((A \rtimes_\alpha G)^\wedge, \hat{\tau})$ by Corollary 4.5, and at this point we have explicitly found a cover $\mathfrak{U}$, modules $\mathfrak{E}_\iota$, actions ν^ι, and isomorphisms $\kappa_{\iota\jmath}$ as in Lemma 6.1. The last equation above shows that the invariant $\hat{\lambda}$ associated to this data as in Lemma 6.2 is given by Equation (7.2).

Before we verify Equation (7.3), we need an observation. Suppose that $\phi \in C_0(F_{\iota\jmath})$ and that $\xi \in \mathfrak{E}^0_{\iota\jmath}$. Define $\xi_1, \xi_2 \in \mathfrak{E}^0_{\iota\jmath}$ by the formulas

$$\begin{aligned}\xi_1(r,z,s) &= \phi \cdot \xi(r,z,s), \text{ and}\\ \xi_2(r,z,s) &= \phi\big(h_\iota^{-1}(z,rsN)\big)\xi(r,z,s).\end{aligned}$$

It is immediate from Equation (7.9) that $\langle\langle \xi_1,\eta\rangle\rangle_{C_0(Q_\iota)} = \langle\langle \xi_2,\eta\rangle\rangle_{C_0(Q_\iota)}$ for all η in $\mathfrak{E}^0_{\iota\jmath}$, and therefore that $[\xi_1] = [\xi_2]$ in $\mathfrak{E}_\iota^{F_{\iota\jmath}}$.

The second component of our invariant is the function $\rho_{\iota\jmath k}\colon F_{\iota\jmath k} \longrightarrow \mathbb{T}$ defined by

$$\kappa_{\iota\jmath}^{F_{\iota\jmath k}} \circ \kappa_{\jmath k}^{F_{\iota\jmath k}} = \rho_{\iota\jmath k} \cdot \kappa_{\iota k}^{F_{\iota\jmath k}},$$

which we can compute from the relationship

(7.11)
$$\langle\langle \xi, \kappa_{\iota\jmath}^{F_{\iota\jmath k}}\big(\kappa_{\jmath k}^{F_{\iota\jmath k}}(\eta)\big)\rangle\rangle^\wedge_{C_0(Q_\iota)}\big(\phi_\iota^{-1}(z,\gamma)\big) = \rho_{\iota\jmath k}\big(\phi_\iota^{-1}(z,\gamma)\big)\langle\langle \xi, \kappa_{\iota k}^{F_{\iota\jmath k}}(\eta)\rangle\rangle^\wedge_{C_0(Q_\iota)}\big(\phi_\iota^{-1}(z,\gamma)\big).$$

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If $\eta \in \mathfrak{E}^0_{kij}$, then

$$\begin{aligned}\tilde{\kappa}_{ij}^{F_{ijk}}\Big(\tilde{\kappa}_{jk}^{F_{ijk}}(\eta)\Big)(r,z,s) &= \lambda_{ij}(rs\cdot c_i(z), rs) g_{ij}\Big(\tilde{\kappa}_{jk}(\eta)\big(r,z,ss_{ij}(z)\big)\Big) \\ &= \lambda_{ij}(rs\cdot c_i(z), rs)\lambda_{jk}\big(rss_{ij}(z)\cdot c_j(z), rss_{ij}(z)\big) \\ &\quad \cdot g_{ij}\Big(g_{jk}\Big(\eta\big(r,z,s_{ij}(z)s_{jk}(z)\big)\Big)\Big) \\ &= \lambda_{ij}\big(rs\cdot c_i(z), rs\big)\lambda_{jk}\big(rs\cdot c_i(z), rs\big)\lambda_{jk}\big(c_i(z), s_{ij}(z)\big) \\ &\quad \cdot \nu_{ijk}\cdot g_{ik}\Big(\eta\big(r,z,ss_{ik}(z)n_{ijk}(z)\big)\Big),\end{aligned}$$

which is equivalent in $\mathfrak{E}_{ijk}$ to

$$\lambda_{ij}\big(rs\cdot c_i(z), rs\big)\lambda_{jk}\big(rs\cdot c_i(z), rs\big)\lambda_{jk}\big(c_i(z), s_{ij}(z)\big)\cdot \nu_{ijk}(rs\cdot c_i(z)) g_{ik}\Big(\eta\big(r,z,ss_{ik}(z)n_{ijk}(z)\big)\Big),$$

which by Equation (5.2) equals

$$\begin{aligned}(7.12)\qquad &\lambda_{jk}\big(c_i(z), s_{ij}(z)\big)\nu_{ijk}\big(c_i(z)\big)\lambda_{ik}\big(rs\cdot c_i(z), rs\big)g_{ik}\big(\eta(r,z,ss_{ik}(z)n_{ijk}(z)\big) \\ &= \lambda_{jk}\big(c_i(z), s_{ij}(z)\big)\nu_{ijk}\big(c_i(z)\big)\overline{\gamma_{ik}\big(n_{ijk}(z)\big)}\tilde{\kappa}_{ik}(\eta)\big(r,z,sn_{ijk}(z)\big).\end{aligned}$$

Next, notice that functions of z pull right through the formula Equation (7.9) for the inner product, and that, if $\eta \in \mathfrak{E}^0_i$, $m \in N$, and η^m is defined by $\eta^m(r,z,s) = \eta(r,z,sm)$, then

$$\langle\!\langle \xi, \eta^m\rangle\!\rangle^{\wedge}_{C_0(Q_i)}\big(\phi_i^{-1}(z,\gamma)\big) = \overline{\gamma(m)}\langle\!\langle \xi,\eta\rangle\!\rangle^{\wedge}_{C_0(Q_i)}\big(\phi_j^{-1}(z,\gamma)\big).$$

Combining these two observations with Equations (7.11) and (7.12), we obtain

$$(7.13)\qquad \rho_{ijk}\big(\phi_i^{-1}(z,\gamma)\big) = \lambda_{jk}\big(c_i(z), s_{ij}(z)\big)\nu_{ijk}\big(c_i(z)\big)\overline{\gamma_{ik}\big(n_{ijk}(z)\big)\gamma\big(n_{ijk}(z)\big)}.$$

Since $\rho_{ijk}\big(\phi_i^{-1}(z,\gamma)\big) = \rho_{ijk}\Big(\phi_k^{-1}\big(z,\gamma\gamma_{ik}(z)\big)\Big)$, this gives Equation (7.3).

That the class of $(\hat{\lambda},\rho)$ in $Z^2_{\hat{G}}\big((A\rtimes_\alpha G)^\wedge,\mathcal{S}\big)$ is independent of our choices follows from part (4) of Lemma 6.2 (*i.e.*, $[\hat{\lambda},\rho] = \delta(A\rtimes_\alpha G, \hat{G},\hat{\alpha})$). This completes the proof of Theorem 7.3. ∎

Our next task is to check that Theorem 7.3 is consistent with [17, §4]: since the dual system $(A\rtimes_\alpha G,\hat{G},\hat{\alpha})$ is $N^\perp$-principal, and $(A\rtimes_\alpha G)\times_{\hat{\alpha}}\hat{G}$ is isomorphic as a G-bundle to $p\colon T\longrightarrow T/G$, [17, Theorem 4.2] says that $\delta(A\rtimes_\alpha G)$ should belong to the coset $d_q([p])$ in $\check{H}^2\big((A\rtimes_\alpha G)^\wedge,\mathcal{S}\big)/q^*\big(\check{H}^2(T/G,\mathcal{S})\big)$. Thus we need:

PROPOSITION 7.4. *The cocycle* $\{\hat{\nu}_{ijk}\}$ *constructed in Theorem 7.3 represents a class in* $d_q([p])$.

We retain the notation of Theorem 7.3, and in addition set $\chi_{ijk} = \tilde{\gamma}_{ij}\tilde{\gamma}_{jk}\tilde{\gamma}_{ik}^{-1}$. Then, following [17, Equations (4.1)–(4.4)], we choose a cochain $\mu_{ijk}\colon N_{ijk}\longrightarrow T$ such that $(\partial\mu)_{ijkl}(z) = \chi_{jkl}(z)\big(s_{ij}(z)\big)$, and $d_q([p])$ is represented by the cocycle

$$(7.14)\qquad \rho_{ijk}(\pi) = \overline{\sigma_k(\pi)\Big(n_{ijk}\big(q(\pi)\big)\Big)}\tilde{\gamma}_{jk}\big(q(\pi)\big)\Big(s_{ij}\big(q(\pi)\big)\Big)\mu_{ijk}\big(q(\pi)\big).$$

While this looks very like the formula for $\hat{\nu}$ with ν_{ijk} replaced by μ_{ijk}, there is a subtlety: although $\tilde{\gamma}_{jk}\big(p(t)\big)$ is by definition an extension of $\lambda_{jk}(t,\cdot)|_N$ to G, $\lambda_{jk}(\cdot,s)$ need not be constant on G/N-orbits, and therefore may not equal $\tilde{\gamma}_{jk}$. If we want to replace λ_{ij} by $\tilde{\gamma}_{ij}$, we have to multiply (λ,ν) by a coboundary to compensate:

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LEMMA 7.5. *Suppose $p\colon T \longrightarrow T/G$ is a principal G/N-bundle, $(\lambda,\nu) \in Z^2_G\big(p^{-1}(\mathfrak{U}),\mathcal{S}\big)$, and continue to use the same notation. Then*

$$\phi_{ij}(t) = \lambda_{ij}(t,s)\overline{\tilde\gamma_{ij}\big(p(t)\big)(s)} \text{ where } sN = w_i(t) \tag{7.15}$$

gives a well-defined continuous function $\phi_{ij}\colon p^{-1}(N_{ij}) \longrightarrow \mathbb{T}$ such that

$$\lambda_{ij}(t,s)\overline{\phi_{ij}(t)}\phi_{ij}(s^{-1}\cdot t) = \tilde\gamma_{ij}\big(p(t)\big)(s). \tag{7.16}$$

If we then set $\mu_{ijk} = (\nu_{ijk}\phi_{ij}\phi_{jk}\phi_{ik}^{-1})\circ c_i$, we have

$$(\partial\mu)_{ijkl}(z) = \chi_{jkl}(z)\big(s_{ij}(z)\big). \tag{7.17}$$

PROOF. If we replace s by sn, the right-hand side of Equation (7.15) becomes

$$\lambda_{ij}(t,sn)\overline{\tilde\gamma_{ij}\big(p(t)\big)(sn)} = \lambda_{ij}(t,n)\lambda_{ij}(t,s)\overline{\tilde\gamma_{ij}\big(p(t)\big)(s)\tilde\gamma_{ij}\big(p(t)\big)(n)}.$$

But on N we have $\tilde\gamma_{ij}\big(p(t)\big) = \gamma_{ij}\big(p(t)\big) = \lambda_{ij}(t,\cdot)$, and hence the right-hand side of Equation (7.15) is well-defined on $p^{-1}(N_{ij})\times G/N$. To verify Equation (7.16), note that $rN = w_i(t)$ implies $s^{-1}rN = w_i(s^{-1}\cdot t)$, so the right-hand side of Equation (7.16) is

$$\lambda_{ij}(t,s)\overline{\lambda_{ij}(t,r)}\lambda_{ij}(s^{-1}\cdot t, s^{-1}r)\tilde\gamma_{ij}\big(p(t)\big)(r)\overline{\tilde\gamma_{ij}\big(p(t)\big)(s^{-1}r)},$$

which reduces to $\tilde\gamma_{ij}\big(p(t)\big)(s)$ because $\tilde\gamma_{ij}\big(p(t)\big)(\cdot)$ is a homomorphism.

To verify Equation (7.17), we set $\rho_{ijk} = \nu_{ijk}\phi_{ij}\phi_{jk}\phi_{ik}^{-1}$, and observe that $\{\rho_{ijk}\}$ is a cocycle. Thus (suppressing the variables)

$$\begin{aligned}(\partial\{\rho_{ijk}\circ c_i\})_{ijkl} &= (\rho_{jkl}\circ c_i)^{-1}(\rho_{jkl}\circ c_j)\\ &= \Big[(\nu_{jkl}\circ c_i)^{-1}\big(\nu_{jkl}\circ(s_{ij}^{-1}\cdot c_i)\big)\Big][(\phi_{jk}\circ c_i)^{-1}(\phi_{jk}\circ c_j)]\\ &\quad\cdot[(\phi_{kl}\circ c_i)^{-1}(\phi_{kl}\circ c_j)][(\phi_{jl}\circ c_i)(\phi_{jl}\circ c_j)^{-1}]\end{aligned}$$

Now using Equation (5.2) and the observation that $s_{ij}(z)N = w_j\big(c_i(z)\big)$, we have

$$\begin{aligned}(\partial\{\rho_{ijk}\circ c_i\})_{ijkl} &= [\lambda_{jk}(c_i,s_{ij})\lambda_{kl}(c_i,s_{ij})\lambda_{jl}(c_i,s_{ij})^{-1}][\lambda_{jk}(c_i,s_{ij})^{-1}\tilde\gamma_{jk}(s_{ij})]\\ &\quad\cdot[\lambda_{kl}(c_i,s_{ik})^{-1}\tilde\gamma_{kl}(s_{ik})\lambda_{kl}(c_j,s_{jk})\tilde\gamma_{kl}(s_{jk})^{-1}][\lambda_{jl}(c_i,s_{ij})\tilde\gamma_{jl}(s_{ij})^{-1}]\\ &= \tilde\gamma_{kl}(s_{ik}s_{jk}^{-1})\tilde\gamma_{jk}(s_{ij})\tilde\gamma_{jl}(s_{ij})^{-1}\lambda_{kl}(c_i,s_{ij})\lambda_{kl}(s_{ij}^{-1}\cdot c_i,s_{jk})\lambda_{kl}(c_i,s_{ik})^{-1}\\ &= \tilde\gamma_{kl}(s_{ij}n_{ijk}^{-1})\tilde\gamma_{jk}(s_{ij})\tilde\gamma_{jl}(s_{ij})^{-1}\lambda_{kl}(c_i,s_{ij}s_{jk})\lambda_{kl}(c_i,s_{ik})^{-1}\\ &= \tilde\gamma_{kl}(n_{ijk})^{-1}\chi_{jkl}(s_{ij})\lambda_{kl}(c_i,n_{ijk})\lambda_{kl}(c_i,s_{ik})\lambda_{kl}(c_i,s_{ik})^{-1}\\ &= \chi_{jkl}(s_{ij})\end{aligned}$$

since $\tilde\gamma_{kl} = \gamma_{kl} = \lambda_{kl}$ on N. ∎

PROOF OF PROPOSITION 7.4. We first replace (λ,ν) by the cocycle (λ',ν'), where

$$\lambda'_{ij}(t,s) = \lambda_{ij}(t,s)\overline{\phi_{ij}(t)}\phi_{ij}(s^{-1}\cdot t) = \tilde\gamma_{ij}\big(p(t)\big)(s)$$
$$\nu'_{ijk}(t) = \nu_{ijk}(t)\phi_{ij}(t)\phi_{jk}(t)\overline{\phi_{ik}(t)}.$$

Since (λ, ν) and (λ', ν') have the same class in $\check{H}^2_G(T, \mathcal{S})$, we have $[\hat{\nu}_{ijk}] = [\hat{\nu}'_{ijk}]$. (One could presumably see this directly by writing down a coboundary, but it also follows from our earlier results: the dynamical systems (A, G, α), (A', G, α') corresponding to (λ, ν), (λ', ν') are Morita equivalent by Proposition 6.5, so that $A \rtimes_\alpha G \cong A' \times_{\alpha'} G$ by [2, 3], and $[\hat{\nu}] = \delta(A \rtimes_\alpha G)$ is the same as $[\hat{\nu}'] = \delta(A' \times_{\alpha'} G)$ by Theorem 7.3.) Equation (7.17) says we can take $\mu_{ijk} = \nu'_{ijk} \circ c_i$ when computing $d_q([p])$, and now formula (7.14) for a cocycle representing $d_q([p])$ is precisely the formula (7.3) describing $\hat{\nu}'$. ∎

EXAMPLE 7.6. We now want to work through our constructions in the situation of [12], where we discussed a family of N-principal systems (A, G, α) for which all the associated topological invariants could be simultaneously non-trivial. The algebras A are constructed from a system (D, N, θ) which is locally Morita equivalent to $\big(C_0(T), \mathrm{id}\big)$, first by inducing, and then taking a quotient. The formula for $\delta(A)$ given in [12, 3.2] was shown in [17, §5(c)] to be consistent with the results of [17]. However, it was also shown in [12, 3.5] that $\delta(A \rtimes_\alpha G) = q^*\big(\delta(D)\big)$, and we should check that this is consistent with our Theorem 7.3. The calculation is itself rather interesting: it shows that the complicated-looking invariant we have introduced is actually computable. We assume as usual that $G \longrightarrow G/N$ and $\hat{G} \longrightarrow \hat{N}$ have local sections.

As in [12, §3], we start with a principal G-bundle $r: Y \longrightarrow Z$ and a locally unitary action of the subgroup N on a continuous-trace algebra D with spectrum Z—that is, with a system (D, N, θ) which is locally Morita equivalent to $\big(C_0(Z), \mathrm{id}\big)$. The algebra A is the quotient of the induced algebra

$$\begin{aligned} \operatorname{Ind}\theta &= \operatorname{Ind}_N^G(D, N, \theta) \\ &= \big\{ g \in C_0(Y, D) : g(ny) = \theta_n\big(g(y)\big) \text{ for } n \in N,\ y \in Y, Ny \longrightarrow \|g(y)\| \in C_0(Y/N) \big\} \end{aligned}$$

by the ideal

$$I(\theta) = \big\{ g \in \operatorname{Ind}\theta : g(y)\big(r(y)\big) = 0 \text{ for all } y \in Y \big\};$$

here we have lapsed into bundle notation, so $d(z)$ makes sense for $d \in D$ and $z \in Z = \hat{D}$. The action of G by left translation on $\operatorname{Ind}\theta$ leaves $I(\theta)$ invariant, and hence descends to an action α of G on $A = \operatorname{Ind}\theta / I(\theta)$. For $y \in Y$, the formula $M(y)(g) = g(y)\big(r(y)\big)$ defines an irreducible representation of $\operatorname{Ind}\theta$ which vanishes on $I(\theta)$, and the map M induces a homeomorphism of $T = Y/N$ onto $\hat{A}$ [12, 3.1]. We need to compute a representative (λ, ν) for $\delta(A, G, \alpha)$, and for this we make some careful choices. First of all, we choose a cover $\{N_i\}$ of $Z = Y/G$ by closed sets, such that $\{\operatorname{Int} N_i\}$ is also a cover, and:

(1) $r: Y \longrightarrow Z$ is trivial over N_i, so that there are G-equivariant surjections q_i of $r^{-1}(N_i)$ onto G;

(2) there are $D^{N_i} -_{N_i} C_0(N_i)$-imprimitivity bimodules F_i, and imprimitivity bimodule isomorphisms $h_{ij}: F_j^{N_{ij}} \longrightarrow F_i^{N_{ij}}$, so that we can define $\mu_{ijk}: N_{ijk} \longrightarrow \mathbb{T}$ by $h_{ij}^{N_{ijk}} \circ h_{jk}^{N_{ijk}} = \mu_{ijk} \cdot h_{ik}^{N_{ijk}}$, and have $\delta(D) = [\mu_{ijk}] \in \check{H}^2(Z, \mathcal{S})$;

(3) there are strictly continuous homomorphisms $w^i: N \longrightarrow UM(D^{N_i})$ such that $\theta_n^{N_i} = \operatorname{Ad} w_n^i$.

We then define $\gamma_{ij}: N_{ij} \longrightarrow \hat{N}$ by

$$\gamma_{ij}(\cdot)(n)(w_n^i)^{N_{ij}} = (w_n^j)^{N_{ij}},$$

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and $[\gamma_{ij}]$ represents Res: $D \times_\theta N \longrightarrow Z$ [10, p. 224]. Since $\hat{G} \longrightarrow \hat{N}$ has local sections, we may further suppose that there exist $\tilde{\gamma}_{ij}: N_{ij} \longrightarrow \hat{G}$ satisfying $\gamma_{ij}(z) = \tilde{\gamma}_{ij}(z)|_N$.

For future reference, we let $p: Y/N \longrightarrow Z$ be the quotient map for the G/N-action on Y/N, and define G/N-equivariant projections $w_\iota: p^{-1}(N_\iota) \longrightarrow G/N$ by $w_\iota(N\cdot y) = q_\iota(y)N$. Note that if we define $s_{ij}: N_{ij} \longrightarrow G$ by $q_i s_{ij} = q_j$, then $\{s_{ij}\}$ is actually a cocycle, and hence the cocycle $\{n_{ijk}\}$ appearing in Theorem 7.3 is trivial.

Now the notation has been set up, we let

$$\mathfrak{Y}_\iota = \left\{ g \in C_b\big(r^{-1}(N_\iota), F_\iota\big) \,\middle|\, \begin{array}{l} g(ny) = w_n^\iota\big(g(y)\big) \text{ for } y \in Y, n \in N, \text{ and} \\ Ny \longmapsto \|g(y)\| \text{ vanishes at } \infty \text{ on } r^{-1}(N_\iota)/N \end{array} \right\}$$

With the natural pointwise actions, and the inner products defined pointwise by $\langle g,h\rangle(y) = \langle g(y),h(y)\rangle$, $\mathfrak{Y}_\iota$ is an $\mathrm{Ind}(\theta^{N_\iota}) -_{p^{-1}(N_\iota)\times N_\iota} C_0\big(p^{-1}(N_\iota), C_0(N_\iota)\big)$-imprimitivity bimodule (the argument of [15, 3.2(2)] carries over). Further, $v_s^\iota(g)(y) = g(s^{-1}\cdot y)$ defines an action of G on $\mathfrak{Y}_\iota$, which is compatible with the actions of G on $\mathrm{Ind}(\theta^{N_\iota})$ and $C_0\big(p^{-1}(N_\iota), C_0(N_\iota)\big)$, and leaves the submodule

$$J_\iota = \big\{ g \in \mathfrak{Y}_\iota : g(y)\big(r(y)\big) = 0 \text{ for all } y\big\}$$

invariant. If

$$K_\iota = \big\{ f \in C_0\big(r^{-1}(N_\iota), C_0(N_\iota)\big) : f(y)\big(r(y)\big) = 0 \text{ for all } y\big\},$$

then J_ι is an $I(\theta^{N_\iota}) - K_\iota$-imprimitivity bimodule, and the quotient $\mathfrak{X}_\iota = \mathfrak{Y}_\iota / J_\iota$ implements a Morita equivalence between $A^{p^{-1}(N_\iota)} \cong \mathrm{Ind}(\theta^{N_\iota})/I(\theta^{N_\iota})$ and $C_0\big(p^{-1}(N_\iota), C_0(N_\iota)\big)/K_\iota \cong C_0\big(p^{-1}(N_\iota)\big)$. This bimodule respects the left and right actions of $C_0\big(p^{-1}(N_\iota)\big)$, and the action of G on $\mathfrak{X}_\iota$ induced by v^ι gives us an $(A^{p^{-1}(N_\iota)}, \alpha^{p^{-1}(N_\iota)}) -_{p^{-1}(N_\iota)} \Big(C_0\big(p^{-1}(N_\iota)\big), \tau\Big)$-imprimitivity bimodule $(\mathfrak{X}_\iota, u^\iota)$.

Before defining our local isomorphisms, we note that if $M \subset N_\iota$, then

$$C\big(p^{-1}(N_\iota), C_0(N_\iota)\big)^{p^{-1}(M)\times M} = C\big(p^{-1}(M), C_0(M)\big),$$

and that we can similarly identify $\mathfrak{Y}_\iota^M$ with

$$\big\{ g: r^{-1}(M) \longrightarrow F_\iota^M : g(ny) = \big(w_n^\iota\big)^M\big(g(y)\big), \ etc. \big\}.$$

Now for $g \in \mathfrak{Y}_j^{N_{ij}}$, we set

$$v_{ij}(g)(y) = \tilde{\gamma}_{ij}\big(r(y)\big)\big(q_j(y)\big)^{-1} h_{ij}\big(g(y)\big),$$

and calculate

$$\begin{aligned} v_{ij}(g)(ny) &= \tilde{\gamma}_{ij}\big(r(y)\big)(n)^{-1}\tilde{\gamma}_{ij}\big(r(y)\big)\big(q_j(y)\big)^{-1}\Big(h_{ij}\big(w_n^j \cdot g(y)\big)\Big) \\ &= \tilde{\gamma}_{ij}\big(r(y)\big)(n)^{-1} w_n^j \cdot \Big(\tilde{\gamma}_{ij}\big(r(y)\big)\big(q_j(y)\big)^{-1} h_{ij}\big(g(y)\big)\Big) \\ &= w_n^\iota \cdot v_{ij}(g)(y); \end{aligned}$$

thus $v_{ij}:\mathfrak{Y}_j^{N_{ij}} \longrightarrow \mathfrak{Y}_i^{N_{ij}}$. We verify routinely that v_{ij} is an isomorphism of imprimitivity bimodules, which is compatible with the actions G by left translation, and maps the submodule $K_j^{N_{ij}}$ onto $K_i^{N_{ij}}$; we let $g_{ij}: \mathfrak{X}_j^{N_{ij}} \longrightarrow \mathfrak{X}_i^{N_{ij}}$ be the isomorphisms of $(A^{p^{-1}(N_{ij})}, \alpha^{p^{-1}(N_{ij})}) -_{p^{-1}(N_{ij})} \Big(C_0\big(p^{-1}(N_{ij})\big), \tau\Big)$-bimodules induced by v_{ij}.

To find λ_{ij}, we first calculate on $\mathfrak{Y}_j$ (or rather, on $\mathfrak{Y}_j^{N_{ij}}$):

$$\begin{aligned} v_s^l \circ v_{ij}(g)(y) &= \tilde{\gamma}_{ij}\big(r(y)\big)(s^{-1}\cdot y)^{-1} h_{ij}\big(g(s^{-1}\cdot y)\big) \\ &= \tilde{\gamma}_{ij}\big(r(y)\big)(s)\big(v_{ij}\circ v_s^l(g)(y)\big). \end{aligned}$$

Since $\tilde{\gamma}_{ij}$ is constant on G-orbits, this formula passes through the quotient map, giving

$$\lambda_{ij}(t,s) = \tilde{\gamma}_{ij}\big(p(t)\big)(s) \text{ for } t \in p^{-1}(N_i) \subset Y/N. \tag{7.18}$$

(Note that Lemma 7.2 says that $(A \rtimes_\alpha G)^\wedge$ has transition functions $z \longrightarrow \tilde{\gamma}_{ij}(z)/N = \gamma_{ij}(z)$, which is consistent with the statement in [12, 3.5] identifying $(A \rtimes_\alpha G)^\wedge$ with $(D \times_\theta N)^\wedge$, and with the description of $(D \times_\theta N)^\wedge$ in [10, p. 224].]) To compute $\{\nu_{ijk}\}$, we again start on $\mathfrak{Y}_k$:

$$\begin{aligned} v_{ij}\circ v_{jk}(g)(y) &= \overline{\tilde{\gamma}_{ij}\big(r(y)\big)\big(q_j(y)\big)\tilde{\gamma}_{jk}\big(r(y)\big)\big(q_k(y)\big)} h_{ij}\circ h_{jk}\big(g(y)\big) \\ &= \overline{\tilde{\gamma}_{ij}\big(r(y)\big)\Big(q_k(y)s_{jk}\big(r(y)\big)^{-1}\Big)\tilde{\gamma}_{jk}\big(r(y)\big)\big(q_k(y)\big)}\big[\mu_{ijk}\cdot h_{ik}\big(g(y)\big)\big] \\ &= \tilde{\gamma}_{ij}\big(r(y)\big)\Big(s_{jk}\big(r(y)\big)\Big)\overline{\chi_{ijk}\big(r(y)\big)\big(q_k(y)\big)\tilde{\gamma}_{ik}\big(r(y)\big)\big(q_k(y)\big)}\big[\mu_{ijk}\cdot h_{ik}\big(g(y)\big)\big] \\ &= \tilde{\gamma}_{ij}\big(r(y)\big)\Big(s_{jk}\big(r(y)\big)\Big)\overline{\chi_{ijk}\big(r(y)\big)\big(q_k(y)\big)}\mu_{ijk}\cdot[v_{ik}(g)(y)]. \end{aligned}$$

When we pass to the quotient $\mathfrak{X}_k^{N_{ijk}} = \big(\mathfrak{Y}_k/J_k\big)^{N_{ijk}} = \mathfrak{Y}_k^{N_{ijk}}/J_k^{N_{ijk}}$, the pointwise action of $\mu_{ijk} \in C(N_{ijk}, \mathbb{T})$ on the $F_k^{N_{ijk}}$-valued functions in $\mathfrak{Y}_k^{N_{ijk}}$ becomes multiplication by the function $\mu_{ijk}\circ r$. Since $\chi_{ijk} \in N^\perp$, we have

$$\chi_{ijk}\big(r(y)\big)\big(q_k(y)\big) = \chi_{ijk}\big(r(y)\big)\big(w_k(N\cdot y)\big),$$

and hence on $\mathfrak{X}_k$ we have $g_{ij}\circ g_{jk} = \nu_{ijk}\cdot g_{ik}$, where

$$\nu_{ijk}(t) = \tilde{\gamma}_{ij}\big(p(t)\big)\Big(s_{jk}\big(p(t)\big)\Big)\overline{\chi_{ijk}\big(p(t)\big)\big(w_k(t)\big)}\mu_{ijk}\big(p(t)\big). \tag{7.19}$$

(If $\mu_{ijk} = 1$, this formula for a cocycle representing $\delta(A)$ is the one appearing in [17, p. 31, line 6] as a representative for $d_p([q])$. Thus adding $[\mu_{ijk}] = \delta(D)$ to the proof of [17, 5.2] shows that $\delta(A) = [\nu_{ijk}]$ is given by the formula $\delta(D) + \langle p^*([q]), r\rangle$, as in [12, 3.5].])

At last we are ready to check that the cocycle in Equation (7.3) also represents the class $q^*\big(\delta(D)\big) = [\mu_{ijk}\circ q]$. If we retain the notation of Theorem 7.3, plug (7.18) and (7.19) into (7.3), and suppress the variable $z = q(\pi)$, then

$$\begin{aligned} \hat{\nu}_{ijk}(\pi) &= \lambda_{jk}\Big(c_i\big(q(\pi)\big), s_{ij}\big(q(\pi)\big)\Big)\nu_{ijk}\Big(c_i\big(q(\pi)\big)\Big) \\ &= \tilde{\gamma}_{jk}(s_{ij})\tilde{\gamma}_{ij}(s_{jk})\overline{\chi_{ijk}\big(w_k(c_i)\big)}\mu_{ijk} \\ &= \tilde{\gamma}_{jk}(s_{ij})\tilde{\gamma}_{ij}(s_{jk})\overline{(\tilde{\gamma}_{ij}\tilde{\gamma}_{jk}\tilde{\gamma}_{ik}^{-1})\Big(w_k\big(s_{ik}\cdot c_k\big)\Big)}\mu_{ijk}. \end{aligned}$$

Since by definition $w_k\big(c_k(z)\big) = N$, we have $w_k\big(s_{ik} \cdot c_k\big) = s_{ik}$, and

$$\begin{aligned}\hat{\nu}_{ijk}(\pi) &= \tilde{\gamma}_{jk}(s_{ij}s_{ik}^{-1})\tilde{\gamma}_{ij}(s_{jk}s_{ik}^{-1})\overline{\tilde{\gamma}_{ik}(s_{ik}^{-1})}\mu_{ijk}\\ &= \overline{\tilde{\gamma}_{jk}(s_{jk})\tilde{\gamma}_{ij}(s_{ij})}\tilde{\gamma}_{ik}(s_{ik})\mu_{ijk}.\end{aligned}$$

Thus $\{\hat{\nu}_{ijk}\}$ differs from $\{\mu_{ijk} \circ q\}$ by a coboundary, and we have verified the formula $\delta(A \rtimes_\alpha G) = q^*\big(\delta(D)\big)$ of [12, 3.5].

8. Concluding remarks.

8.1. When G acts trivially on T, Equations (5.1) and (5.2) reduce to the assertion that $t \longmapsto \lambda_{ij}(t, \cdot)$ is a cocycle with values in $\hat{G}$, and we have a natural isomorphism

$$[\lambda, \nu] \in \check{H}^2_G(T, \mathcal{S}) \longrightarrow \big([\nu], [\lambda]\big) \in \check{H}^2(T, \mathcal{S}) \oplus \check{H}^1(T, \hat{G}). \tag{8.1}$$

If $v^i: G \longrightarrow UM(A^{N_i})$ implement α over N_i, and $\gamma_{ij}: N_{ij} \longrightarrow \hat{G}$ satisfy $v^j = \gamma_{ij}u^i$, then Equation (6.1) implies that $\lambda_{ij}(t, \cdot) = \gamma_{ij}$. (Since $v^i_s \in UM(A^{N_i})$, the isomorphisms g_{ij} in Equation (6.1) commute with left multiplication by v^i_s.) Thus the isomorphism Equation (8.1) takes $\delta(A, G, \alpha)$ to the pair $\big(\delta(A), \zeta(\alpha)^{-1}\big)$, where $\zeta(\alpha)$ is the obstruction of [10]. The surjectivity in (8.1) is a reformulation of [10, Theorem 3.8], and our Theorem 6.3 says that the pair $\big(\delta(A), \zeta(\alpha)\big)$ determines (A, G, α) up to Morita equivalence.

8.2. Suppose $G = \mathbb{Z}/2\mathbb{Z}$ acts trivially on T, so that by Equation (8.1),

$$\check{H}^2_{\mathbb{Z}_2}(T, \mathcal{S}) \cong \check{H}^2(T, \mathcal{S}) \oplus \check{H}^1(T, \mathbb{Z}_2). \tag{8.2}$$

In this case every system $(A, \mathbb{Z}_2, \alpha)$ in which A is a continuous-trace algebra with spectrum T is locally Morita equivalent to $\big(C_0(T), \tau\big)$. (To see this, note that the non-trivial automorphism α_1 is locally inner, hence locally has the form $\operatorname{Ad} u$. Since $\operatorname{Ad} u^2 = 1$, $u^2 = f1$ for some complex function f, and we can locally replace u by $f^{-1/2}u$ to see that $(A, \mathbb{Z}_2, \alpha)$ is locally unitary—or, in view of Proposition 4.3, locally Morita equivalent to $\big(C_0(T), \tau\big)$.) Since actions of $\mathbb{Z}/2$ amount to gradings of A, Theorem 6.3 classifies graded continuous-trace algebras with spectrum T, and from the isomorphism Equation (8.2), we recover the main results of [9].

8.3. We have called $\check{H}^2_G(T, \mathcal{S})$ an equivariant cohomology group because it is a modification of the ordinary cohomology group $\check{H}^2(T, \mathcal{S})$ designed to accommodate an action of G. We have not, however, shown that it is part of a larger cohomology theory in which there is an algebraic apparatus to help us compute. As we have hinted earlier, there are similar groups in all positive dimensions, and a long exact sequence associated to a principal G/N-bundle $p: T \longrightarrow Z$, which reduces to the Gysin sequence of a principal circle bundle in the case $G = \mathbb{R}$, $N = \mathbb{Z}$. This sequence will resolve some of the obvious questions about $\check{H}^2_G$—for example, it will identify the range of the homomorphism $b: \check{H}^2_G \longrightarrow \check{H}^1(Z, \hat{\mathcal{N}})$ of Lemma 7.1 as the kernel of the homomorphism $[q] \longmapsto \langle [p], [q] \rangle_G$ of [17]—and will also provide non-trivial information about dynamical systems. For, example, by taking $N = \{e\}$, so that G acts freely, we shall obtain an isomorphism

$$\check{H}^2_G(T, \mathcal{S}) \cong p^*\check{H}^2(T/G, \mathcal{S})$$

This implies that every system (A, G, α) with spectral system $\big(C_0(T), \tau\big)$ is Morita equivalent to the pull-back of a system (B, G, id) with spectrum T/G, and we recover [13, Theorem 1.1].

There are other questions about $\check{H}^2_G(\cdot, \mathcal{S})$ which the Gysin sequence will not resolve. For example, what can we say about the homomorphism $[\lambda, \nu] \longmapsto [\nu]$ of $\check{H}^2_G(T, \mathcal{S})$ into $\check{H}^2(T, \mathcal{S})$? This is certainly a question of some interest: a class δ is in the range exactly when there is an algebra A with $\delta(A) = \delta$ carrying an action of G which is locally Morita equivalent to $\big(C_0(T), \tau\big)$. In [18], we shall prove that, when $G = \mathbb{R}$, $N = \mathbb{Z}$ and T is a principal $\mathbb{T} = \mathbb{R}/\mathbb{Z}$-bundle, this map is surjective, and indeed is an isomorphism. More generally, we show that when $G = \mathbb{R}^k$, $N = \mathbb{Z}^k$ and T is a principal $\mathbb{T}^k$-bundle, the range consists precisely of the classes in $\check{H}^2(T, \mathcal{S})$ which can be realized by cocycles defined on covers by invariant sets. We shall also consider in detail the case $G = \mathbb{T}$, $N = \{e^{2\pi i k/m}\}$, where the map $[\lambda, \nu] \longmapsto [\nu]$ is neither surjective or injective. We hope that our presentation of these results in [18] will inspire topologists to bombard us with useful information about $\check{H}^2_G$.

8.4. In Theorem 7.3, we have shown how to compute the Dixmier-Douady class of the system dual to an N-principal system (A, G, α) in terms of a cocycle (λ, ν) representing $\delta(A \rtimes_\alpha G, \hat{G}, \hat{\alpha})$. It is tempting to ask whether this construction is a special case of some duality for equivariant cohomomology, which maps $\check{H}^2_G$ into $\check{H}^2_{\hat{G}}$. However, while Lemma 7.1 shows how $(\lambda, \nu) \in Z^2_G(T, \mathcal{S})$ determines a cocycle in $Z^1(T/G, \hat{\mathcal{N}})$, we would have to *choose* a principal $\hat{N}$-bundle $q\colon Y \longrightarrow T/G$ realizing this cocycle before we could talk about a dual cohomology group $\check{H}^2_{\hat{G}}(Y, \mathcal{S})$. When we start with a system (A, G, α), the spectrum $q\colon (A \rtimes_\alpha G)^\wedge \longrightarrow T/G$ provides a canonical choice, but we have not been able to find a purely topological analogue of this construction which would allow us to discuss duality without reference to C^*-algebras.

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