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BEURLING'S ORDINARY VALUE

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Let $n(w, f)$ be the number of w-points of f meromorphic in $D = \{|z| < 1\}$. Beurling defined the quantity $\overline{n}(w, f)$ and called w an ordinary value of f if $\overline{n}(w, f) < \infty$. We shall consider the intermediate quantity $\underline{n}(w, f)$ in the sense that $n(w, f) \le \underline{n}(w, f) \le \overline{n}(w, f)$, and construct two bounded holomorphic functions f_1 and f_2 of finite Dirichlet integrals in D for which

$$0 = n(0, f_1) < \underline{n}(0, f_1) < \overline{n}(0, f_1) < \infty$$

and

$$0 = n(0, f_2) < \underline{n}(0, f_2) < \overline{n}(0, f_2) = \infty .$$

1. Introduction

Let W be the Riemann sphere of radius $\frac{1}{2}$ touching the complex plane $\mathbb{C}$ at 0. The sphere W is endowed with the chordal distance $X(\cdot, \cdot)$ and with the element of the spherical area $d\omega(w)$ at $w \in W$, being expressed as $d\omega(w) = \left(1+|w|^2\right)^{-2}dxdy$, if $w \ne \infty$ is identified with its projection $x + iy \in \mathbb{C}$. Then the area of the Riemannian image of $D = \{|z| < 1\}$ by f meromorphic in D over the spherical cap

$$C(a, r) = \{w \in W;\ X(w, a) < r\} \quad (a \in W,\ 0 < r \le 1)$$

is

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$$A(a,\ r,\ f) = \iint_{C(a,r)} n(w,\ f)\,d\omega(w)\ ,$$

where $n(w,\ f)$ is the number of the zeros of $f - w$ in D, the order being counted. We then set, for $a \in W$,

$$\overline{n}(a,\ f) = \limsup_{r\to 0}\ (\pi r^2)^{-1}A(a,\ r,\ f)\ ,$$

$$\underline{n}(a,\ f) = \liminf_{r\to 0}\ (\pi r^2)^{-1}A(a,\ r,\ f)\ ;$$

here

$$\pi r^2 = \iint_{C(a,r)} d\omega(w)\ .$$

It follows from the lower semicontinuity of $n(w,\ f)$ that

$$n(a,\ f) \le \underline{n}(a,\ f) \le \overline{n}(a,\ f)$$

at each $a \in W$. If $\overline{n}(a,\ f) < \infty$ $\big(\underline{n}(a,\ f) < \infty$, respectively$\big)$, then a is called an ordinary value (a lower ordinary value, respectively) of f; the definition of ordinary value is due to Beurling [1, p. 11]. Furthermore, if

$$\text{(1.1)}\qquad \iint_D \big(|f'(z)|/(1+|f(z)|^2)\big)^2 dxdy = A(0,\ 1,\ f) < \infty\ ,$$

where $z = x + iy$, then

$$n(a,\ f) = \underline{n}(a,\ f) = \overline{n}(a,\ f) < \infty$$

for $d\omega$-almost every $a \in W$; see [2, Theorem 6.3 on p. 118, and the inequality at line 11 from below on p. 149].

Now, if $n(a,\ f) = \infty$, then, apparently,

$$\underline{n}(a,\ f) = \overline{n}(a,\ f) = \infty$$

without the assumption (1.1). *Does the equaltiy*

$$\underline{n}(a,\ f) = \overline{n}(a,\ f)\quad \text{(possibly equal to } \infty\text{)}$$

hold if $n(a,\ f) < \infty$ *for* f *satisfying* (1.1)?

We shall construct two examples which answer this question in the negative.

REMARK. An obvious example erases the doubt that $n(a, f) < \infty$ for each $a \in W$ if (1.1) is satisfied.

THEOREM 1. *There exists a bounded univalent holomorphic function* f *in* D *, satisfying* ((1.1) *and*)

$$0 = n(0, f) < \underline{n}(0, f) < \overline{n}(0, f) < \infty . \tag{1.2}$$

We note that if f is bounded in D , then (1.1) is equivalent to

$$\iint_D |f'(z)|^2 dxdy < \infty \quad (z = x+iy) .$$

The proof of Theorem 1 is rather easy in contrast with that of

THEOREM 2. *There exists a bounded holomorphic function* f *in* D *, satisfying* (1.1) *and*

$$0 = n(0, f) < \underline{n}(0, f) < \overline{n}(0, f) = \infty . \tag{1.3}$$

We note that 0 is a lower ordinary value yet not an ordinary value of f in Theorem 2.

2. Proof of Theorem 1

First of all, $\arg w$ of $w \in W - \{0, \infty\}$ means that of the projection of w into $\mathbb{C}$. Letting $a_k = 2^{-k}$, $k = 1, 2, \ldots$, we consider the simply connected domain S over W defined by

$$S = \{w \in C(0, a_2) ; -\pi/2 < \arg w < 0\}$$

$$\cup \bigcup_{n=1}^{\infty} \{w \in C(0, a_{2n}) - \overline{C}(0, a_{2n+1}) ; 0 \le \arg w < \pi/2\} .$$

Let f be a one-to-one conformal mapping from D onto S , which may be considered as a bounded holomorphic function satisfying (1.1).

Let $r = 2^{-t}$, $2n \le t < 2n+1$, so that $a_{2n+1} < r \le a_{2n}$ $(n = 1, 2, \ldots)$. Then

$$A(0, r, f) = \tfrac{1}{4}\pi r^2 + I_n + \tfrac{1}{4}\pi \left(r^2 - a_{2n+1}^2\right) ,$$

where

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$$I_n = \sum_{k=n+1}^{\infty} \tfrac{1}{4}\pi\left(a_{2k}^2 - a_{2k+1}^2\right) = \frac{\pi}{5} \cdot \frac{1}{16^{n+1}} .$$

Therefore

$$(2.1) \qquad \left(\pi r^2\right)^{-1} A(0, r, f) = \tfrac{1}{2} - \frac{4}{5} \cdot \frac{4^t}{16^{n+1}}$$

and

$$3/10 < \left(\pi r^2\right)^{-1} A(0, r, f) \le 9/20 .$$

Let $r = 2^{-t}$, $2n+1 \le t < 2n+2$, so that $a_{2n+2} < r \le a_{2n+1}$ $(n = 1, 2, \ldots)$. Then

$$A(0, r, f) = \tfrac{1}{4}\pi r^2 + I_n ,$$

so that

$$(2.2) \qquad \left(\pi r^2\right)^{-1} A(0, r, f) = \frac{1}{4} + \frac{1}{5} \cdot \frac{4^t}{16^{n+1}} ,$$

together with

$$3/10 \le \left(\pi r^2\right)^{-1} A(0, r, f) < 9/20 .$$

It now follows from (2.1) and (2.2) that

$$3/10 = \underline{n}(0, f) < \overline{n}(0, f) = 9/20 ,$$

whence follows (1.2).

3. Proof of Theorem 2

We shall make use of the following

LEMMA. *Given* $A > 0$, $B > 0$, *and* $s > 0$, *there exist a natural number* N *and a positive number* $\lambda < s$ *such that*

$$\pi N\left[A^2 - (A-\lambda)^2\right] = B .$$

The proof is elementary and is omitted.

To prove Theorem 2 we choose a pair $\{p_m\}_{m=1}^{\infty}$ and $\{q_m\}_{m=1}^{\infty}$ of sequences of natural numbers inductively as follows. First, let $p_1 = 2$.

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Then, given p_m $(m \geq 1)$ we select $q_m > p_m$ such that $q_m^{-\frac{1}{2}} < 2^{-p_m}$. We choose then $p_{m+1} > q_m$ such that $2^{-p_{m+1}} < q_m^{-\frac{1}{2}}$.

We set $a_m = 2^{-p_m}$ and $b_m = q_m^{-\frac{1}{2}}$ $(m = 1, 2, \ldots)$, so that

$$1/4 = a_1 > b_1 > a_2 > \ldots > a_m > b_m > a_{m+1} > \ldots \downarrow 0 .$$

It then follows from the lemma that there exist a natural number ν_m and a positive number $\varepsilon_m < a_m - b_m$ such that

$$\pi\nu_m\left[a_m^2 - \left(a_m - \varepsilon_m\right)^2\right] = p_m^{-2} \quad (m = 1, 2, \ldots) .$$

We notice that $\nu_m^{-1}p_m^{-2}$ is the area of the spherical ring $R\left(a_m\right) = C\left(0, a_m\right) - \bar{C}\left(0, a_m - \varepsilon_m\right)$ $(m = 1, 2, \ldots)$. It also follows from the lemma that there exist a natural number μ_m and a positive number $\delta_m < b_m - a_{m+1}$ such that

$$\pi\mu_m\left[b_m^2 - \left(b_m - \delta_m\right)^2\right] = 2^{-q_m} \quad (m = 1, 2, \ldots) .$$

In the present case, $\mu_m^{-1}2^{-q_m}$ is the area of the spherical ring $R\left(b_m\right) = C\left(0, b_m\right) - \bar{C}\left(0, b_m - \delta_m\right)$ $(m = 1, 2, \ldots)$.

Let S_1 be the Riemann surface over W, in the form of a ribbon, which winds its way just ν_m times over $R\left(a_m\right)$, and just μ_m times over $R\left(b_m\right)$ $(m = 1, 2, \ldots)$, and which tends to the origin; see Figure 1 where the case $\nu_m = \mu_m = 1$ $(m = 1, 2, \ldots)$ is expressed. More precisely, S_1 covers

$$C\left(0, a_1\right) - \bigcup_{m=1}^{\infty} \left[R\left(a_m\right) \cup R\left(b_m\right)\right]$$

once by the parts which we shall call bridges, while S_1 covers $R\left(a_m\right)$

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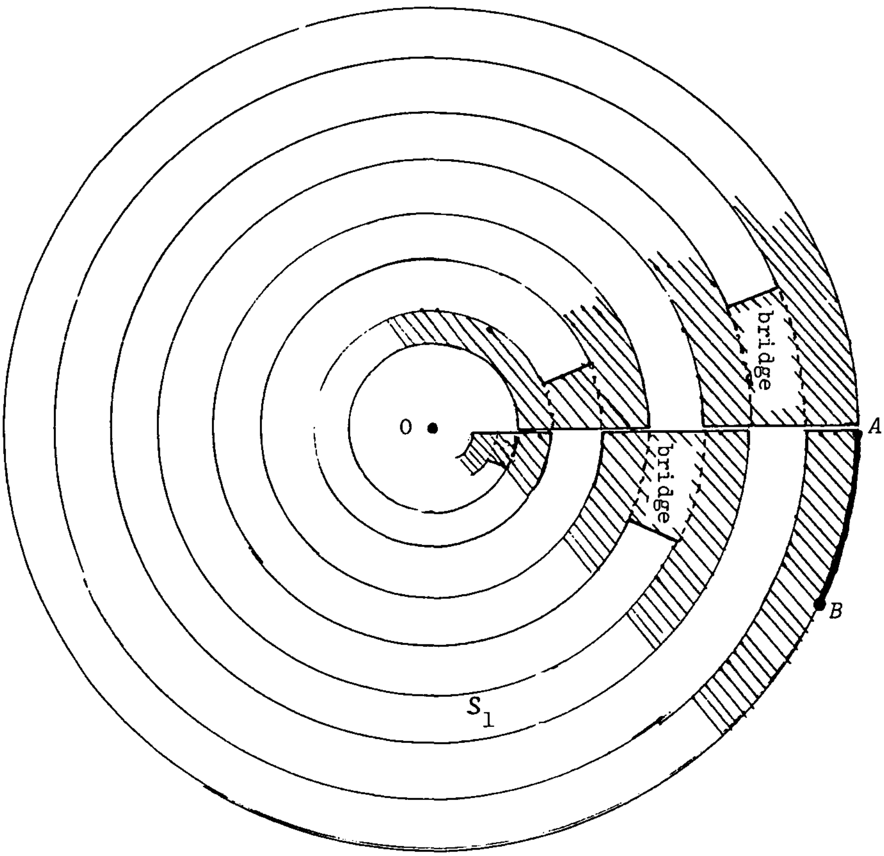

FIGURE 1

$(R(b_m)$ respectively) just ν_m $(\mu_m$ respectively) times except for a cross cut of $R(a_m)$ $(R(b_m)$ respectively) which S_1 covers just $\nu_m - 1$ $(\mu_m - 1$ respectively) times $(m = 1, 2, \ldots)$.

Let S_2 be a one-sheeted ribbon over $C(0, 1/3)$ such that S_2 ends at 0 in the form

$$S_2 \cap C(0, r_0) = \{w \in C(0, r_0) ; |\arg w| < \pi/2\}$$

for a certain $r_0 > 0$; see Figure 2. We then paste S_1 and S_2 along the circular arc AB to obtain the resulting simply connected Riemann surface S over W . Let f be a one-to-one conformal mapping from D onto S , which we may consider as a bounded holomorphic function in D .

We first consider the sequence $a_m \downarrow 0$. Then

$$A(0, a_m, f) \geq \text{the area of } S_1 \text{ over } R(a_m) = p_m^{-2} ,$$

so that

$$\left(\pi a_m^2\right)^{-1} A(0, a_m, f) \geq \pi^{-1} 2^{2p_m} p_m^{-2} \to \infty ,$$

whence $\overline{n}(0, f) = \infty$.

We next observe that, for each sequence $r_n \downarrow 0$ $(n \geq 1)$, the following holds for $r_n < r_0$;

$$A(0, r_n, f) \geq \text{the area of the part of } S_2 \text{ over } C(0, r_n) = \tfrac{1}{2}\pi r_n^2 ,$$

so that $\underline{n}(0, f) \geq \frac{1}{2}$.

To prove $\underline{n}(0, f) < \infty$ we consider the sequence $b_m \downarrow 0$. Then

(3.1) $A(0, b_m, f)$ = the area of the bridges over $C(0, b_m)$ + the area of

$$\text{the part of } S_2 \text{ over } C(0, b_m) + \sum_{k=m+1}^{\infty} p_k^{-2} + \sum_{k=m}^{\infty} 2^{-q_k} ,$$

so that

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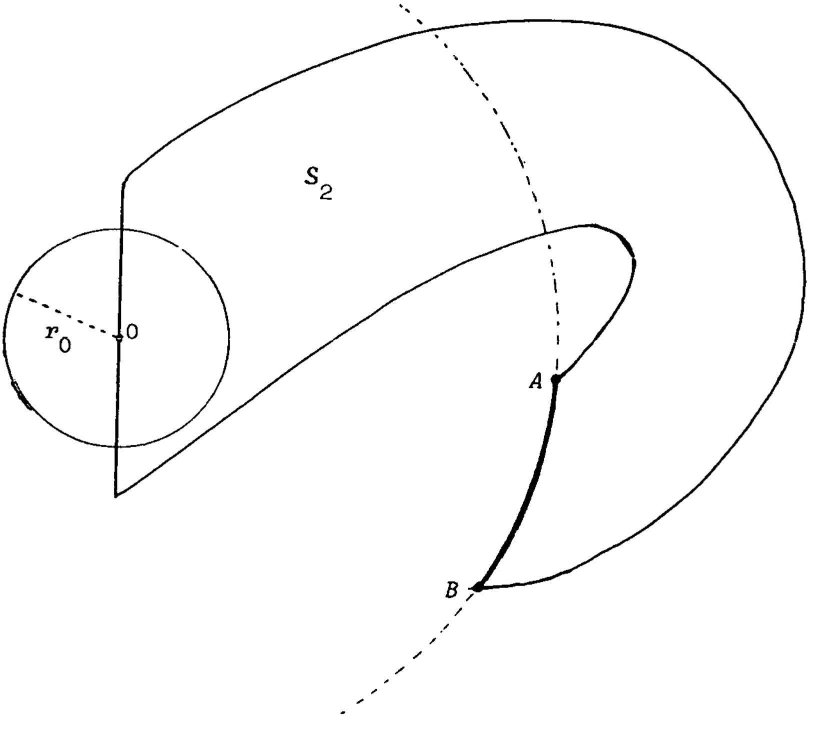

FIGURE 2

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the third term + the fourth term of (3.1)

$$\leq \sum_{k=p_{m+1}}^{\infty} k^{-2} + \sum_{k=q_m}^{\infty} 2^{-k} \leq \sum_{k=p_{m+1}}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k}\right) + 2^{1-q_m}$$

$$= \frac{1}{p_{m+1}-1} + 2^{1-q_m} .$$

Since the first term of (3.1) is less than πb_m^2 , it follows that

$$\left(\pi b_m^2\right)^{-1} A\left(0, b_m, f\right) \leq 1 + \tfrac{1}{2} + \pi^{-1} + \pi^{-1} q_m 2^{1-q_m}$$

for $b_m < r_0$, because of $q_m \leq p_{m+1}-1$. Letting $m \to \infty$ one observes that $\underline{n}(0, f) \leq 3/2 + \pi^{-1}$.

Since

$$\sum_{m=1}^{\infty} p_m^{-2} + \sum_{m=1}^{\infty} 2^{-q_m} \leq \sum_{k=p_1}^{\infty} k^{-2} + \sum_{k=q_1}^{\infty} 2^{-k} \leq \frac{1}{p_1-1} + 2^{1-q_1} ,$$

it is easy to observe that f satisfies (1.1).

References

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