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Construction of Fourier multipliers

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The classical Wiener-Pitt phenomenon for measures may be formulated as an existence theorem for Fourier multipliers with irregular spectral properties and the result has been refined in various ways over the years. The most recent development is due to Zafran, who exhibits abnormal spectral behaviour in multipliers whose transforms vanish at infinity and which operate on L_p, where $1 < p < \infty$, $p \neq 2$. Zafran's methods use Littlewood-Paley theory, interpolation, and the construction of a certain class of measures. We show here how the constructive element of his proof may be considerably simplified and sharpened.

1. Introduction and notation

The Banach algebra approach to the spectral analysis of L_1-multipliers, in the guise of the study of convolution measure algebras, has attained a level of development where the Wiener-Pitt phenomenon appears as a relatively crude first approximation. See for example the survey of work up to 1972 in [3]. There exists very little in the nature of a parallel development for L_p-multipliers, where $1 < p < 2$, and this is due in large measure to the difficulty of constructing examples. Zafran's announcement [4] and subsequent exposition [5] are therefore of considerable interest and the full consequences of his techniques remain to be assessed. Our purpose here is limited. It will be shown that the formidable technicalities of Section 3 of [5] can be replaced by some simple lemmas which follow the same general lines but which achieve

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stronger results with economy of effort.

So that results can be stated quickly, we introduce the minimum of notation. Attention will be restricted to the one dimensional torus, T, and its dual group, Z, regarded as the discrete additive group of integers. Given an integrable function f on T, we write $f^\wedge$ for the Fourier transform of f. Given a bounded measure on T, we write $\mu^\wedge$ for the Fourier-Stieltjes transform of μ. The symbol $\| \|_p$ denotes the L_p-norm as usual. Given a regular bounded measure μ on T, we write $\|\mu\|_{M_p}$ for the norm of that operator on $L_p(T)$ which maps the function f to the function $\mu * f$. In particular, $\|\mu\|_{M_1}$ denotes the usual total variation norm. Theorem 3.1 of [5] can be restated as follows:

THEOREM A. *Suppose that* $1 < p < 2$, *and let* $1/p + 1/p' = 1$. *Then there exist measures* $\{\mu_j\}_{j=1}^{\infty}$ *and a positive measure* μ *on* T, *a sequence* $\{n(j)\}$ *of positive integers tending to infinity, and a positive constant* c *(depending only on* p *) such that*

(i) $(\|\mu_j^\wedge\|_\infty)^2 \le 2^{1-n(j)}$,

(ii) $\left\|(\mu_j)^j\right\|_{M_p} \ge c(j!/j^j)2^{-jn(j)/p'}$,

(iii) $|\mu_j| \le \mu$, $j = 1, 2, \ldots$.

The result which we regard as the most natural sharpening of Theorem A and which is given a simple proof in this note is the following:

THEOREM B. *Suppose that* $1 < p < 2$, *and let* $1/p + 1/p' = 1$. *Then there is a sequence of measures* $\{\mu_n\}_{n=1}^{\infty}$ *and a probability measure* μ *such that*

(i) $(\|\mu_n^\wedge\|_\infty)^2 \le 2^{1-n}$,

(ii) $\left\|(\mu_n)^n\right\|_{M_p} \ge (n!/n^n)2^{-n^2/p'}$,

(iii) $|\mu_n| \le \mu$, $n = 1, 2, \ldots$.

The simple methods used here make it possible to sharpen Theorem A still further. In fact one is at liberty to specify the sequence $\{n(j)\}_{j=1}^{\infty}$ in advance. Moreover the estimates in the inequalities labelled *(ii)* can be improved. To make this precise we introduce the quantity, $B_p(j, n)$, defined by the formula

$$B_p(j, n) = 2^{-nj}j^{-j}\left(\sum_{i=1}^{N} |m_i|^p\right)^{1/p} ,$$

where the vector $(m_i)_{i=1}^{N}$ is a list of the multiplicities of the distinct words with j letters on $2^n j$ commuting symbols. [For example, in the case where $n = 1$, $j = 3$, one sees that $N = 56$ and that the vector $(m_i)_{i=1}^{N}$ has the entry 1 six times, the entry 3 thirty times, and the entry 6 twenty times.]

Because $B_p(j, n(j))$ is (much) larger than $(j!/j^j)2^{-jn(j)/p'}$, the following result improves both Theorem A and Theorem B.

THEOREM C. *Suppose that* $1 < p < 2$, *and let* $1/p + 1/p' = 1$. *Suppose further that* $\{n(j)\}_{j=1}^{\infty}$ *is an arbitrary sequence of positive integers and that* $\{c_j\}_{j=1}^{\infty}$ *is an arbitrary sequence of positive real numbers such that* $c_j < B_p(j, n(j))$ *for each* $j = 1, 2, \ldots$. *Then there is a sequence* $\{\mu_j\}_{j=1}^{\infty}$ *of measures and a probability measure* μ *such that*

(i) $\left(\|\hat{\mu}_j\|_\infty\right)^2 \le 2^{1-n(j)}$,

(ii) $\left\|(\mu_j)^j\right\|_{M_p} \ge c_j$,

(iii) $|\mu_j| \le \mu$, $j = 1, 2, \ldots$.

Before passing to the proof of Theorem C, we indicate briefly how the result relates to the Wiener-Pitt phenomenon. Of course this point is covered in both [4], [5], and Theorem A suffices. The object is to define a Fourier multiplier S on $L_p(T)$ whose Fourier transform $S^\wedge$ vanishes at infinity but whose spectrum is strictly larger than the closure of the range of $S^\wedge$. This is achieved by matching $S^\wedge$ with translates of $2^{-n/p'}\mu_n^\wedge$ (where μ_n is given by Theorem B) on successive dyadic blocks. It is immediately clear that property *(i)* of Theorem B ensures that $S^\wedge$ vanishes at infinity. Meanwhile property *(ii)* of that result leads readily to the appropriate spectral radius estimate. The problem is to show that the recipe does indeed define a Fourier multiplier on $L_p(T)$. Of course this depends on the classical Littlewood-Paley Theorem, but the crucial estimate depends on a simple but ingenious interpolation result ([5], Theorem 2.1) which makes essential use of property *(iii)* in Theorems A, B, or C. Accordingly one function of our simplification of Theorem A and its proof is to underline the significance of Zafran's interpolation theorem. In that sense the intent of this note is largely expository.

2. Proofs of Theorems A, B, C

The proofs are given in a sequence of lemmas. We suppose throughout that p is a fixed real number lying strictly between 1 and 2 and that p' is the conjugate index defined by $1/p + 1/p' = 1$. A major freeing manoeuvre is that we do not allow property *(iii)* of the theorems to worry us until the last moment. The first lemma is almost banal. $\delta(x)$ denotes the probability atom at x .

LEMMA 1. *Let* $\omega = \sum_{i=1}^{N} m_i\delta(x_i)$ *, where the points* x_i *are distinct. Then*

$$\|\omega\|_{M_p} \geq \left(\sum_{i=1}^{N} |m_i|^p\right)^{1/p} .$$

Proof. For any ϕ in $L_p(T)$ such that $\|\phi\|_p = 1$, we have

$$\|\omega\|_{M_p} \geq \|\omega * \phi\|_p = \left\|\sum_{i=1}^{N} m_i\phi_{x_i}\right\|_p ,$$

where $\phi_{x_i}(t) = \phi(t-x_i)$. Now choose $\phi = K\chi_{[0,\delta]}$, where K is a normalizing constant, $\chi_{[0,\delta]}$ denotes the indicator function of the interval $[0, \delta]$, and the positive number δ is less than half the

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minimum distance between the x_i's . Then the functions ϕ_{x_i} have disjoint supports, so that

$$\int \left|\sum m_i \phi_{x_i}(t)\right|^p dt = \sum \int |m_i \phi_{x_i}(t)|^p dt = \sum |m_i|^p .$$

The required result follows immediately.

The next lemma is merely the standard application of the Rudin-Shapiro construction ([1], pp. 34-36). The statement is slightly adapted to our needs.

LEMMA 2. *Suppose that* $\mu = \mathop{*}\limits_{i=1}^{n} \frac{1}{2}\left(\delta(s_i)+\delta(t_i)\right)$ *, and that the support of* μ *has* 2^n *distinct points. Then there exists a measure* τ *absolutely continuous with respect to* μ *such that* $\frac{d\tau}{d\mu} = \pm 1$ *everywhere and*

$$\left(\|\tau^\wedge\|_\infty\right)^2 \le 2^{1-n} .$$

Proof. For the sake of completeness we note that one defines

$$\rho_0 = \sigma_0 = \delta(0) ,$$

$$\rho_{i+1} = \delta(s_{i+1}) * \rho_i + \delta(t_{i+1}) * \sigma_i ,$$

$$\sigma_{i+1} = \delta(s_{i+1}) * \rho_i - \delta(t_{i+1}) * \sigma_i ,$$

for $i = 0, 1, \ldots, n-1$. Now one observes that

$$|\rho_{i+1}^\wedge|^2 + |\sigma_{i+1}^\wedge|^2 = 2|\rho_i^\wedge|^2 + 2|\sigma_i^\wedge|^2 = 2^{i+2} ,$$

and defines

$$\tau = 2^{-n}\rho_n .$$

Lemmas 1, 2 have been simple versions of Lemmas 3.3, 3.4 of [5]. Now we avoid the effort of 3.5-3.8 of that paper.

LEMMA 3. *Let* j, n *be positive integers and suppose that the subset* $\{s_{ik}, t_{ik} : i = 1, \ldots, n, k = 1, \ldots, j\}$ *of* T *is independent (over the* *rationals). Let*

$$\mu_k = \mathop{*}_{i=1}^{n} \tfrac{1}{2}\left(\delta\left(s_{ik}\right)+\delta\left(t_{ik}\right)\right) , \quad k = 1, \ldots, j .$$

Then there exist measures $\{\tau_k\}_{k=1}^{j}$ *such that* $\frac{d\tau_k}{d\mu_k} = \pm 1$ *everywhere and such that the measure* ν *, defined by* $\nu = \nu_{j,n} = j^{-1} \sum_{k=1}^{j} \tau_k$ *, satisfies*

(i) $(\|\nu^\wedge\|_\infty)^2 \le 2^{1-n}$,

(ii) $\|\nu^j\|_{M_p} \ge B_p(j, n) > (j!/j^j)2^{-nj/p'}$,

(iii) $\|\nu\|_{M_1} \le 1$.

Proof. We choose τ_k corresponding to μ_k as in Lemma 2. The hypothesis concerning the support is satisfied because of independence. It follows immediately that *(i)* and *(iii)* hold. Moreover the measure ν^j is of the form

$$\left(2^n j\right)^{-j} \sum_{i=1}^{N} m_i \delta\left(x_i\right) ,$$

where the x_i are the distinct words written (additively) using j letters chosen from the points of support of ν . Once more we have used independence. It now follows from Lemma 1 that

$$\|\nu^j\|_{M_p} \ge \left(2^n j\right)^{-j}\left\{\sum_{i=1}^{N} |m_i|^p\right\}^{1/p} = B_p(j, n) .$$

The crude lower estimate for $B_p(j, n)$ is obtained by considering only those x_i which use points of support from *every* τ_k . [There are 2^{nj} words of this type, and each occurs with multiplicity $j!$. Accordingly these terms contribute $j!2^{nj/p}$ to the p-norm of the multiplicity vector.]

The proof of the next lemma follows closely the Riesz product construction used on pp. 367-369 of [5]. The flexible use of related techniques is well known to harmonic analysts (*cf.* [2], p. 100). However the simple observation that the result may be isolated in some generality is what led us to the broad simplification of Theorem A. Accordingly we go through the details of the proof.

LEMMA 4. *Let* $\{\nu_j\}_{j=1}^{\infty}$ *be a sequence of measures on* T *, and* $\{a_j\}_{j=1}^{\infty}$, $\{b_j\}_{j=1}^{\infty}$, $\{c_j\}_{j=1}^{\infty}$ *sequences of positive constants such that* $c_j < b_j$ *, and*

(i)' $\|\hat{\nu}_j\|_\infty \le a_j$,

(ii)' $\left\|(\nu_j)^j\right\|_{M_p} \ge b_j$,

(iii)' $\|\nu_j\|_{M_1} \le 1$, $j = 1, 2, \ldots$.

Then there is a sequence $\{\mu_j\}_{j=1}^{\infty}$ *of measures and a probability measure* μ *such that*

(i) $\|\hat{\mu}_j\|_\infty \le a_j$,

(ii) $\left\|(\mu_j)^j\right\|_{M_p} \ge c_j$,

(iii) $|\mu_j| \le \mu$, $j = 1, 2, \ldots$.

Proof. Let $\{\sigma_j\}_{j=1}^{\infty}$ denote the Fejêr kernel (regarded as a sequence of measures). Using the fact that $\{\sigma_j\}_{j=1}^{\infty}$ is an approximate identity for $L_p(T)$ together with hypothesis *(ii)'*, choose a sequence $\{\phi_j\}_{j=1}^{\infty}$ of trigonometric polynomials in the unit ball of $L_p(T)$ and an increasing sequence $\{r(j)\}_{j=1}^{\infty}$ of positive integers such that

$$(1) \qquad \left\|(\sigma_{r(j)})^j * (\nu_j)^j * \phi_j\right\|_p \ge c_j , \quad j = 1, 2, \ldots .$$

Now let m denote Haar measure on T and, for each j , choose some probability measure ω_j with $|\nu_j| \le \omega_j$. (The choice is guaranteed by *(iii)'*.) Set

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(2) $$\sigma_{r(j)} * \nu_j = f_j.m \ , \quad \sigma_{r(j)} * \omega_j = g_j.m \ ,$$

where the Radon-Nikodym derivatives f_j, g_j can be taken as trigonometric polynomials in view of the fact that $\{\sigma_j\}_{j=1}^{\infty}$ is the Fejér kernel. Observe that

(3) $$g_j \geq 0 \ , \quad \int g_j(t)dm(t) = 1 \ , \quad |f_j| \leq g_j \ , \quad j = 1, 2, \ldots .$$

Now we choose positive integers $\{N_j\}_{j=1}^{\infty}$ such that

(4) $$N_{j+1}/N_j > 2js(j) \ , \quad j = 1, 2, \ldots ,$$

where $s(j)$ is the maximum of the degrees of the polynomials f_j, g_j. For all positive integers n, j, let us define trigonometric polynomials h_n, $h_{n,j}$ by the formulae

(5) $$h_n(t) = \prod_{k=1}^{n} g_k(N_k t) \ ,$$

(6) $$h_{n,j}(t) = f_j(N_j t) \prod_{k=1, k \neq j}^{n} g_k(N_k t) \ .$$

For each j, the measure μ_j is chosen as the vague (weak*) limit of the sequence $\{h_{n,j}.m\}$. The measure μ is chosen as the vague limit of the sequence $\{h_n.m\}$.

It is necessary to check that these limits *do* exist but this follows readily from the lacunarity condition (4) together with the norm condition embodied in (3). The point is that the constant term in each g_j is one, and the orthogonality relations resulting from (4) show that $\int h_n(t)dm(t) = 1$ for each n. Taking account of the remaining statements in (3), we see first that each measure $h_n.m$ is a probability measure, and then that each measure $h_{n,j}.m$ lies in the unit ball. We are now guaranteed at least one vague limit point for each of the sequences under discussion but, in fact, we can use the lacunary orthogonality relations to write down explicitly the unique possible Fourier-Stieltjes transforms

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$\mu_j^\wedge$, $\mu^\wedge$, for limit points μ_j, μ . Each of $\mu_j^\wedge$, $\mu^\wedge$ vanishes on any integer which fails to be a "word" of the form

$$\sum_{k=1}^{M} \varepsilon_k N_k \text{ , with } |\varepsilon_k| \le s(k) \text{ ,}$$

and, on such a word, $\mu_j^\wedge$ takes the value

$$g_1^\wedge(\varepsilon_1) g_2^\wedge(\varepsilon_2) \cdots f_j^\wedge(\varepsilon_j) \cdots g_M^\wedge(\varepsilon_M) \text{ ,}$$

while $\mu^\wedge$ takes the value

$$g_1^\wedge(\varepsilon_1) g_2^\wedge(\varepsilon_2) \cdots g_j^\wedge(\varepsilon_j) \cdots g_M^\wedge(\varepsilon_M) \text{ .}$$

Of course we are now able to verify property *(i)*, because

$$\|\mu_j^\wedge\|_\infty \le \|f_j^\wedge\|_\infty = \|(\sigma_{r(j)} * \nu_j)^\wedge\|_\infty = \|\nu_j^\wedge\|_\infty \le a_j \text{ .}$$

It is also straightforward to check property *(ii)*. To this end it seems easiest to fix j and introduce some more notation. We let

(7) $\left(\sigma_{r(j)} * \nu_j\right)^j * \phi_j = p_j . m$, $q_j(t) = p_j\left(N_j t\right)$, $\psi_j(t) = \phi_j\left(N_j t\right)$,

and note that

(8) $$(\mu_j)^j * \psi_j = q_j . m \text{ .}$$

One verifies (8) by comparing Fourier-Stieltjes transforms. The definitions of ψ_j, q_j given in (7) show that both $\psi_j^\wedge$, $q_j^\wedge$ vanish off the multiples of N_j . On the other hand, on an integer of the form mN_j , $\psi_j^\wedge$ takes the value $\phi_j^\wedge(m)$, and $q_j^\wedge$ takes the value $p_j^\wedge(m)$ which in turn equals $f_j(m)^j \phi_j^\wedge(m)$. Equality (8) is now obvious. From (1), (7), (8), we obtain

$$\left\|(\mu_j)^j * \psi_j\right\|_p = \|q_j\|_p = \|p_j\|_p \ge c_j \text{ ,}$$

while

$$\|\psi_j\|_p = \|\phi_j\|_p \le 1 \text{ .}$$

This establishes property *(ii)* and the proof is complete.

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It remains to remark that the combination of Lemmas 3 and 4 yields Theorem C. Suppose that we have been given the sequences $\{c_j\}_{j=1}^{\infty}$, $\{n(j)\}_{j=1}^{\infty}$ as in the statement of the theorem. The existence of independent sets presents no problem so we apply Lemma 3 for each pair $j, n(j)$ and relabel the resulting measure $\nu_{j,n(j)}$ as ν_j. Now we simply apply Lemma 4 with $(a_j)^2 = 2^{1-n(j)}$, $b_j = B_p(j, n(j))$. We showed in the proof of Lemma 3 that

$$\left(j!/j^j\right)2^{-jn/p'} < B_p(j, n) ,$$

so it is clear that Theorems A and B are consequences of Theorem C. All assertions have now been verified.

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