INTEGRALS INVOLVING PRODUCTS OF MODIFIED BESSEL FUNCTIONS OF THE SECOND KIND

by F. M. RAGAB

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1. Introductory. It is proposed to establish the two following integrals.

If $R(k \pm \nu + \frac{1}{2}n) > 0$,

$$\int_0^\infty \lambda^{k-1} K_\nu(\lambda) K_\mu(x\lambda^{-n})\, d\lambda = 2^{k-n-3}\pi^{-n} n^{k-1}$$

$$\times \sum_{i,\,-i} \frac{1}{i} E\{\Delta(n;\, \tfrac{1}{2}k + \tfrac{1}{2}\nu),\, \Delta(n;\, \tfrac{1}{2}k - \tfrac{1}{2}\nu),\, \tfrac{1}{2}\mu,\, -\tfrac{1}{2}\mu,\, 1 :: \tfrac{1}{4}e^{i\pi}(2n)^{-2n}x^2\}, \tag{1}$$

where n is a positive integer, x is real and positive, μ and ν are complex, and $\Delta(n;\,\alpha)$ represents the set of parameters

$$\frac{\alpha}{n},\ \frac{\alpha+1}{n},\ \ldots,\ \frac{\alpha+n-1}{n}.$$

If $R(k \pm \nu \pm n\mu) > 0$,

$$\int_0^\infty \lambda^{k-1} K_\nu(\lambda) K_\mu(x\lambda^{n})\, d\lambda = -2^{k-n-2}\pi^{2-n} n^{k-1} \sum_{\mu,\,-\mu} \operatorname{cosec} \mu\pi \{\tfrac{1}{4}(2n)^{2n}x^2\}^{\frac{1}{2}\mu}$$

$$\times E\left\{\Delta\left(n;\, \frac{k+n\mu+\nu}{2}\right)\, \Delta\left(n;\, \frac{k+n\mu-\nu}{2}\right):\, 1+\mu:\, \tfrac{1}{4}e^{\pm i\pi}(2n)^{-2n}x^{-2}\right\}, \tag{2}$$

where n is a positive integer and x is real and positive.

The following formulae will be required in the proofs.

If $p \geqq q+1$, then [**1**, p. 409]

$$E(p;\, \alpha_r :\, q;\, \rho_s :\, z) = \pi^{p-q-1} \sum_{r=1}^{p} \prod_{s=1}^{q} \sin(\rho_s - \alpha_r)\pi \prod_{t=1}^{p}{}' \operatorname{cosec}(\alpha_t - \alpha_r)\pi$$

$$\times z^{\alpha_r} E\left\{\begin{matrix} \alpha_r,\, \alpha_r - \rho_1 + 1\,,\, \ldots, \alpha_r - \rho_q + 1 : \\ \alpha_r - \alpha_1 + 1,\, \ldots * \ldots,\, \alpha_r - \alpha_p + 1 \end{matrix}\ \frac{e^{\pm i\pi(p-q-1)}}{z}\right\}. \tag{3}$$

$$\sin\left(\frac{k}{n}\pi\right) \sin\left(\frac{k+1}{n}\pi\right) \ldots \sin\left(\frac{k+n-1}{n}\pi\right) = 2^{1-n} \sin(k\pi). \tag{4}$$

$$\sin\frac{\pi}{n} \sin\frac{2\pi}{n} \ldots \sin\frac{(n-1)\pi}{n} = 2^{1-n} n. \tag{5}$$

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If $R(k\pm\nu)>0$, [2, p. 119]

$$\int_0^\infty \lambda^{k-1}K_\nu(\lambda)E(p;\,\alpha_r:\,q;\,\rho_s:\,z\lambda^{-2n})\,d\lambda$$

$$=(2\pi)^{1-n}2^{k-2}n^{k-1}E\left\{\alpha_1,\,\dots,\,\alpha_p,\,\Delta\left(n;\,\frac{k+\nu}{2}\right),\,\Delta\left(n;\,\frac{k-\nu}{2}\right):\,q;\,\rho_s:\,(2n)^{-2n}z\right\}. \tag{6}$$

If $p\geqq q+1$, [1, p. 353]

$$E(p;\,\alpha_r:\,q;\,\rho_s:\,z)=\sum_{r=1}^{p}\prod_{t=1}^{p}{}'\,\Gamma(\alpha_t-\alpha_r)\left\{\prod_{s=1}^{q}\Gamma(\rho_s-\alpha_r)\right\}^{-1}$$

$$\times\Gamma(\alpha_r)z^{\alpha_r}F\left\{\begin{matrix}\alpha_r,\,\alpha_r-\rho_1+1,\,\dots,\,\alpha_r-\rho_q+1;\,(-1)^{p-q}z\\ \alpha_r-\alpha_1+1,\,\dots\,*\,\dots,\,\alpha_r-\alpha_p+1\end{matrix}\right\}. \tag{7}$$

The formula

$$K_\mu(z)=\frac{1}{4\pi}\sum_{i,\,-i}\frac{1}{i}E(1,\,\tfrac{1}{2}\mu,\,-\tfrac{1}{2}\mu::\,\tfrac{1}{4}e^{i\pi}z^2) \tag{8}$$

may be established by expanding the functions on the right by means of (7).

If $p\geqq q+1$, $|\operatorname{amp} z|<\pi$ and if $R(k\pm\nu+2n\alpha_r)>0$, where $r=1,2,\dots,p$, n being a positive integer,

$$\int_0^\infty \lambda^{k-1}K_\nu(\lambda)E(p;\,\alpha_r:\,q;\,\rho_s:\,z\lambda^{2n})\,d\lambda$$

$$=\frac{2^{k+n-3}\pi^{n+1}n^{k-1}}{\sin\left(\dfrac{k+\nu}{2}\pi\right)\sin\left(\dfrac{k-\nu}{2}\pi\right)}\,E\left[\begin{matrix}p;\,\alpha_r & :\,(2n)^{2n}z\\ \Delta\left(n;\,1-\dfrac{k+\nu}{2}\right),\,\Delta\left(n;\,1-\dfrac{k-\nu}{2}\right),\,\rho_1,\,\dots,\,\rho_q & \end{matrix}\right]$$

$$+2^{k+n-3}\pi^{n+1}n^{k-2}$$

$$\times\sum_{\nu,\,-\nu}\left[\sum_{l=0}^{n-1}(-1)^l\frac{\{(2n)^{2n}z\}^{-(k+\nu+2l)/2n}}{\sin\left(\dfrac{k+\nu+2l}{2n}\pi\right)\sin(\nu+l)\pi}\right.$$

$$\left.\times E\left\{\begin{matrix}p;\,\alpha_r+\dfrac{k+\nu+2l}{2n} & :\,(2n)^{2x}z\\ \dfrac{l+1}{n},\,\dfrac{l+2}{n},\,\dots\,*\,\dots,\,\dfrac{l+n}{n},\,1+\dfrac{k+\nu+2l}{2n}, & \\ \Delta(n;\,\nu+l+1),\,\rho_1+\dfrac{k+\nu+2l}{2n},\,\dots,\,\rho_q+\dfrac{k+\nu+2l}{2n} & \end{matrix}\right\}\right]. \tag{9}$$

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To prove this, start with the special case $p = 1, q = 0$; then, from (3), it follows that

$$\int_0^\infty \lambda^{k-1}K_\nu(\lambda)E(\alpha :: z\lambda^{2n})\, d\lambda = z^\alpha \int_0^\infty \lambda^{k+2n\alpha-1}K_\nu(\lambda)E(\alpha :: z^{-1}\lambda^{-2n})\, d\lambda;$$

and from (6) this is equal to

$$(2\pi)^{1-n}2^{k+2n\alpha-2}n^{k+2n\alpha-1}z^\alpha E\left\{\alpha, \Delta\left(n; n\alpha+\frac{k+\nu}{2}\right), \Delta\left(n; n\alpha+\frac{k-\nu}{2}\right) :: (2n)^{-2n}z^{-1}\right\}.$$

Here apply formula (3), using (4) and (5), and so obtain (9) with $p = 1, q = 0$. Formula (9) is then obtained by generalising.

If n is a positive integer

$$\Gamma(z)\Gamma\left(z+\frac{1}{n}\right) \dots \Gamma\left(z+\frac{n-1}{n}\right) = (2\pi)^{\frac{1}{2}n-\frac{1}{2}}n^{\frac{1}{2}-nz}\Gamma(nz). \tag{10}$$

If $p \leqq q$, then [**1**, p. 352]

$$E(p; \alpha_r : q; \rho_s : z) = \frac{\Gamma(\alpha_1) \dots \Gamma(\alpha_p)}{\Gamma(\rho_1) \dots \Gamma(\rho_q)} F\left(\begin{matrix}\alpha_1, \dots, \alpha_p; z\\ \rho_1, \dots, \rho_q\end{matrix}\right). \tag{11}$$

2. Proofs of the integrals. To prove (1) apply (6) twice with $p = 3, q = 0, \alpha_1 = 1, \alpha_2 = \frac{1}{2}\mu$ and $\alpha_3 = -\frac{1}{2}\mu$. In the first case take $z = \frac{1}{4}x^2e^{i\pi}$, in the second take $z = \frac{1}{4}x^2e^{-i\pi}$, divide by i and subtract, and then apply (8).

In order to obtain (2) take $p = 3, q = 0$ in (9) with $\alpha_1 = \frac{1}{2}\mu, \alpha_2 = -\frac{1}{2}\mu, \alpha_3 = 1$; then combine the two cases in which $z = \frac{1}{4}x^2e^{i\pi}$ and $z = \frac{1}{4}x^2e^{-i\pi}$, using (8). As the terms arising from the first E-function on the right of (9) cancel, it follows that

$$\int_0^\infty \lambda^{k-1}K_\nu(\lambda)K_\mu(x\lambda^n)\, d\lambda = -2^{k+n-4}\pi^n n^{k-2}$$

$$\times \sum_{\nu, -\nu}\left[\sum_{l=0}^{n-1} \frac{\{(2n)^{2n}x^2/4\}^{-(k+\nu+2l)/2n}}{\sin \nu\pi} E\left\{\begin{matrix}\frac{\mu}{2}+\frac{k+\nu+2l}{2n}, -\frac{\mu}{2}+\frac{k+\nu+2l}{2n} : -(2n)^{2n}x^2/4\\ \frac{l+1}{n}, \dots * \dots, \frac{l+n}{n}, \Delta(n; \nu+l+1)\end{matrix}\right\}\right].$$

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Here apply (11) and (10), and the expression becomes

$$-2^{k-3}\pi n^{k+\nu-1}$$

$$\times\sum_{\nu,-\nu}\left[\sum_{l=0}^{n-1}\frac{\{(2n)^{2n}x^2/4\}^{-(k+\nu+2l)/2n}n^{2l}\,\Gamma\left(\frac{\mu}{2}+\frac{k+\nu+2l}{2n}\right)\Gamma\left(-\frac{\mu}{2}+\frac{k+\nu+2l}{2n}\right)}{\sin\nu\pi\,\Gamma(l+1)\Gamma(\nu+l+1)}\times F\left(\begin{matrix}\frac{\mu}{2}+\frac{k+\nu+2l}{2n},\ -\frac{\mu}{2}+\frac{k+\nu+2l}{2n} & ;\ \frac{4}{(2n)^{2n}x^2}\\ \frac{l+1}{n},\ldots *\ldots,\frac{l+n}{n},\ \Delta(n;\ \nu+l+1) & \end{matrix}\right)\right]. \tag{A}$$

Again, apply (7) to the expression on the right of (2), so getting

$$-2^{k-n-2}\pi^{2-n}n^{k-1}\sum_{\mu,-\mu}\operatorname{cosec}\mu\pi\{\tfrac{1}{4}(2n)^{2n}x^2\}^{\frac{1}{2}\mu}$$

$$\times\sum_{\nu,-\nu}\sum_{l=0}^{n-1}\frac{\Gamma\left(-\frac{l}{n}\right)\ldots\Gamma\left(-\frac{1}{n}\right)\Gamma\left(\frac{1}{n}\right)\ldots\Gamma\left(\frac{n-l-1}{n}\right)\Gamma\left(\frac{-\nu-l}{n}\right)\ldots\Gamma\left(\frac{-\nu-l+n-1}{n}\right)}{\Gamma\left(1+\mu-\frac{\mu n+k+\nu+2l}{2n}\right)}$$

$$\times\Gamma\left(\frac{\mu n+k+\nu+2l}{2n}\right)\left\{\frac{4e^{\pm i\pi}}{(2n)^{2n}x^2}\right\}^{(\mu n+k+\nu+2l)/2n}$$

$$\times F\left\{\begin{matrix}\frac{\mu n+k+\nu+2l}{2n},\ \frac{-\mu n+k+\nu+2l}{2n} & :\ 4(2n)^{-2n}x^{-2}\\ 1+\frac{l}{n},\ 1+\frac{l-1}{n},\ldots,1+\frac{1}{n},\ 1-\frac{1}{n},\ldots,1-\frac{n-l-1}{n},\ \Delta(n;\ \nu+l+1) & \end{matrix}\right\}. \tag{B}$$

Here replace $$1\Big/\Gamma\left(1+\mu-\frac{\mu n+k+\nu+2l}{2n}\right)$$

by $$\Gamma\left(\frac{-\mu n+k+\nu+2l}{2n}\right)\sin\left(\frac{-\mu n+k+\nu+2l}{2n}\pi\right)\Big/\pi.$$

Then note that

$$\sum_{\mu,-\mu}\sin\left(\frac{-\mu n+k+\nu+2l}{2n}\pi\right)e^{\pm i\pi(\mu n+k+\nu+2l)/2}\Big/\sin\mu\pi=-1.$$

Also, from (10) it follows that

$$\Gamma\left(\frac{-\nu-l}{n}\right)\ldots\Gamma\left(\frac{-\nu-l+n-1}{n}\right)=(-1)^{l-1}\frac{(2\pi)^{\frac{1}{2}n-\frac{1}{2}}n^{\frac{1}{2}+\nu+l}\pi}{\Gamma(\nu+l+1)\sin\nu\pi},$$

and that

$$\Gamma\left(-\frac{l}{n}\right)\ldots\Gamma\left(-\frac{1}{n}\right)\Gamma\left(\frac{1}{n}\right)\ldots\Gamma\left(\frac{n-l-1}{n}\right)=(-1)^{l}\frac{(2\pi)^{\frac{1}{2}n-\frac{1}{2}}n^{l-\frac{1}{2}}}{\Gamma(l+1)}.$$

Hence the expression (B) is equal to the expression (A), so that formula (2) has been established.

REFERENCES

1. T. M. MacRobert, *Functions of a complex variable*, 5th edn. (London, 1962).

2. F. M. Ragab, An integral involving a modified Bessel function of the second kind and an *E*-function, *Proc. Glasgow Math. Assoc.* **1** (1953), 119–120.

Faculty of Science
Cairo University
Cairo, U.A.R.