CORRESPONDENCES OF CHARACTERS FOR RELATIVELY PRIME OPERATOR GROUPS

GEORGE GLAUBERMAN

1. Introduction and notation. Let G be a finite group and let A be a finite solvable operator group on G. Suppose that A and G have relatively prime orders. Let T be the fixed-point subgroup of G with respect to A. We say that A *fixes* a complex character ζ of G if $\zeta(g^\alpha) = \zeta(g)$ for all $g \in G$ and $\alpha \in A$. Our aim in this paper is to define a one-to-one correspondence between the irreducible characters of T and those irreducible characters of G that are fixed by A, and to prove some properties of this correspondence that were mentioned in (**8**). For example, if the character λ of T corresponds to the character ζ of G, then $\zeta(1)$ divides $[G:T]\lambda(1)$ (Theorem 5).

In § 2 we observe that we may extend the fixed characters of G to characters of the semi-direct product GA. (This result was first proved by Gallagher and does not require that A be solvable.) We also derive a bound on the order of a p-subgroup of a p-solvable linear group (Corollary 3). In § 3 we determine certain values of characters of GA extended from fixed characters of G; some of our methods and results were suggested in (**6**, § 13), in which T is assumed to be cyclic. For example, in Corollary 6 we consider the situation in which A is cyclic and every non-identity element of A has T as its fixed-point subgroup. This includes a case encountered in the proof of Theorem B of the Hall-Higman paper (**11**), where A is a p-group and B is an extra-special group. We obtain the following result:

Suppose that η is an irreducible character of GA and $\eta|_G$ is irreducible. Let $\zeta = \eta|_G$, and let λ be the character of T corresponding to ζ. Then $\eta|_A = \epsilon\lambda(1)\theta + b\rho$, where $\epsilon = \pm 1$, θ is an irreducible character of A, b is a non-negative integer, and ρ is the character of the regular representation of A.

In the Hall-Higman case, it can be shown that $\epsilon = -1$ and that $\lambda(1) = 1$. Thus $\eta|_A = b\rho - \theta$. This is analogous to the situation in (**11**), where one considers a faithful irreducible GA-module M over an algebraically closed field of characteristic p. There, it is proved that M is the direct sum of a number (possibly zero) of free A-modules and of one indecomposable A-module of dimension $|A| - 1$.

We establish the correspondence between characters of T and fixed characters of G in § 4.

Suppose that A and G are solvable and $|A| = p_1 p_2 \ldots p_n$ for some primes $p_1, p_2, \ldots, p_n$ (not necessarily distinct). The *Fitting height* $h(S)$ of a solvable

Received June 9, 1967 and in revised form, January 18, 1968.

group S is the least integer h for which S has a series of normal subgroups $1 = S_0 \subseteq S_1 \subseteq \ldots \subseteq S_h = S$, with S_i/S_{i-1} nilpotent, $1 \leqq i \leqq h$. In (**14**), Thompson proved that $h(G) \leqq 5^n h(T)$. We consider in Theorem 6 a situation that arises in (**14**, p. 261).

My thanks are due to the National Science Foundation for its support by a Graduate Fellowship during the preparation of most of this paper.

Most of our notation is standard. Let G be a finite group. Denote the order of G by $|G|$. A *class function* on G is a complex-valued function that is constant on the conjugate classes of G. The *inner product* of two class functions θ and η is given by

$$(\theta, \eta) = |G|^{-1} \sum_{g \in G} \theta(g)\overline{\eta(g)},$$

where $\overline{\eta(g)}$ denotes the complex conjugate of $\eta(g)$. We say that a class function is a *character* of G if it is the character of a complex representation of G; it is a *generalized character* of G if it is a linear combination of characters of G with integral coefficients. For every linear transformation T on a finite-dimensional complex vector space, let det T be the determinant of T. If $g \in G$ and ζ is a character of G, let $(\det \zeta)(g)$ be the determinant of $R(g)$ for any representation R that affords ζ. Denote the kernel of a character ζ by Ker ζ.

We denote the field of rational numbers by $\mathbf{Q}$. For every positive integer m, let $\mathbf{Q}_m$ be the cyclotomic field obtained from $\mathbf{Q}$ by adjoining the complex mth roots of unity. Thus $\mathbf{Q} = \mathbf{Q}_1$. For every character ζ of G, let $\mathbf{Q}(\zeta)$ be the field obtained by adjoining the values of ζ to $\mathbf{Q}$. For any automorphism σ of any field containing $\mathbf{Q}(\zeta)$, define ζ^σ by $\zeta^\sigma(g) = (\zeta(g))^\sigma$ for all $g \in G$. It is well known that ζ^σ is a character of G. We say that σ *fixes* ζ if $\zeta = \zeta^\sigma$.

Given elements $g, h, \ldots$ in G, let $\langle g, h, \ldots \rangle$ be the subgroup of G that they generate. Suppose that H is a subgroup of G. Let $[G:H]$ be the index of H in G. For every generalized character λ of H, let λ^G be the character of G induced by λ. For every generalized character η of G, let $\eta|_H$ be the restriction of η to H. If H is a normal subgroup of G, we shall sometimes identify characters of G/H with the corresponding characters of G that contain H in their kernels.

We call G an *elementary Abelian* group if G is the direct product of (any number of) groups of equal prime order. We say that A is an *operator group* on G if to every element α of A there is associated an automorphism $g \rightarrow g^\alpha$ of G and if $(g^\alpha)^\beta = g^{\alpha\beta}$ and $g^1 = g$ for all $g \in G$ and $\alpha, \beta \in A$. Assume that this is the case. Let B and H be arbitrary non-empty subsets of A and G. For $\alpha \in A$, let $H^\alpha = \{h^\alpha | \, h \in H\}$; then α *fixes* H if $H^\alpha = H$. Furthermore, B fixes H if every element of B fixes H. Let

$$C_H(B) = \{h \in H \mid B \text{ fixes } \{h\}\}.$$

Suppose that H is a subgroup of G. We call $C_H(B)$ the *fixed-point subgroup* of H with respect to B. If B is also a subgroup of A, we say that B acts *faithfully* on H if every non-identity element of B is associated with a non-identity automorphism of H.

Suppose that A is an operator group on G. We shall often assume that A and G are embedded in their semi-direct product GA. If $\alpha \in A$ and η is a generalized character of G, define η^α by $\eta^\alpha(g) = \eta(g^{\alpha^{-1}})$. Then α *fixes* η if $\eta^\alpha = \eta$, and A *fixes* η if every element of A fixes η. We say that A is a *relatively prime* operator group on G if A is finite and if $|A|$ and $|G|$ are relatively prime.

All groups considered in this paper are finite.

2. Existence of extensions. The following result is a theorem of Gallagher, who proved a slightly different version in (**7**).

THEOREM 1. *Let R be an absolutely irreducible representation of a group G on a vector space V over a field K. Let A be an operator group on G such that the order of A is relatively prime to the degree of R. Assume that for each $\alpha \in A$, R is equivalent to the representation R_α given by*

$$R_\alpha(g) = R(g^\alpha), \qquad g \in G.$$

Then there exists a unique representation R^* of GA on V such that $R^*(g) = R(g)$ for all $g \in G$ and such that $\det R^*(\alpha) = 1$ *for all $\alpha \in A$.*

Proof. For each $\alpha \in A$, there exists a linear transformation $S(\alpha)$ of V such that

$$R_\alpha(g) = R(g^\alpha) = S(\alpha)^{-1}R(g)S(\alpha) \quad \text{for all } g \in G.$$

Take $\alpha, \beta \in A$ and $g \in G$. Then

$$S(\alpha\beta)^{-1}R(g)S(\alpha\beta) = R(g^{\alpha\beta}) = R((g^\alpha)^\beta) = S(\beta)^{-1}R(g^\alpha)S(\beta) = S(\beta)^{-1}S(\alpha)^{-1}R(g)S(\alpha)S(\beta).$$

Thus, $S(\alpha)S(\beta)S(\alpha\beta)^{-1}$ centralizes $R(g)$ for every $g \in G$. Since R is an absolutely irreducible representation of G, $S(\alpha)S(\beta)S(\alpha\beta)^{-1}$ is a scalar multiple of the identity transformation. Take $c(\alpha, \beta) \in K$ such that

$$S(\alpha)S(\beta) = c(\alpha, \beta)S(\alpha\beta). \tag{1}$$

Now let $d(\alpha) = \det S(\alpha)$ for every $\alpha \in A$. Let r be the degree of R. From (1) we have that

$$d(\alpha)d(\beta) = c(\alpha, \beta)^r d(\alpha\beta) \tag{2}$$

and

$$\begin{aligned} c(\alpha, \beta)c(\alpha\beta, \gamma) &= S(\alpha)S(\beta)S(\alpha\beta)^{-1}S(\alpha\beta)S(\gamma)S(\alpha\beta\gamma)^{-1} \\ &= S(\alpha)S(\beta)S(\gamma)S(\alpha\beta\gamma)^{-1} \\ &= S(\alpha)S(\beta)S(\gamma)S(\beta\gamma)^{-1}S(\beta\gamma)S(\alpha\beta\gamma)^{-1} \\ &= S(\alpha)c(\beta, \gamma)S(\beta\gamma)S(\alpha\beta\gamma)^{-1} \\ &= c(\alpha, \beta\gamma)c(\beta, \gamma). \end{aligned} \tag{3}$$

https://doi.org/10.4153/CJM-1968-148-x Published online by Cambridge University Press

Let $e(\beta) = \prod_{\alpha \in A} c(\alpha, \beta)$ for each $\beta \in A$. Let $n = |A|$. Multiplying each side of (3) over all γ in A, we obtain

$$c(\alpha, \beta)^n e(\alpha\beta) = e(\alpha)e(\beta). \tag{4}$$

Since n and r are relatively prime, there exist integers i and j such that $in + jr = 1$. Let $f(\alpha) = d(\alpha)^j e(\alpha)^i$ for each $\alpha \in A$. From (2) and (4), we obtain

$$c(\alpha, \beta) = c(\alpha, \beta)^{in+jr} = f(\alpha)f(\beta)f(\alpha\beta)^{-1}. \tag{5}$$

Define $S'(\alpha) = f(\alpha)^{-1}S(\alpha)$, $\alpha \in A$. From (1) and (5), $S'(\alpha\beta) = S'(\alpha)S'(\beta)$ for all $\alpha, \beta \in A$. For each $\alpha \in A$, let $d'(\alpha) = \det S'(\alpha)$ and let $S''(\alpha) = d'(\alpha)^{-j}S'(\alpha)$. For $g \in G$ and $\alpha, \beta \in A$,

$$S''(\alpha)^{-1}R(g)S''(\alpha) = S(\alpha)^{-1}R(g)S(\alpha) = R(g^\alpha), \qquad S''(\alpha)S''(\beta) = S''(\alpha\beta),$$

and

$$\det S''(\alpha) = (d'(\alpha)^{-j})^r d'(\alpha) = d'(\alpha)^{in} = d'(\alpha^n)^i = d'(1)^i = 1.$$

Hence, we may define R^* by

$$R^*(\alpha g) = S''(\alpha)R(g), \qquad \alpha \in A, g \in G.$$

We claim that R^* is unique. Let R^{**} be a representation of GA on V. Suppose that $R^{**}(g) = R(g)$ for all $g \in G$ and the determinant of $R^{**}(\alpha)$ is 1 for all $\alpha \in A$. Take $\alpha \in A$. For each $g \in G$,

$$R(g^\alpha) = R^*(\alpha)^{-1}R(g)R^*(\alpha) = R^{**}(\alpha)^{-1}R(g)R^{**}(\alpha);$$

thus, $R^*(\alpha)R^{**}(\alpha)^{-1}$ centralizes $R(g)$ for every $g \in G$. Since R is absolutely irreducible, there exists a scalar $h(\alpha)$ in K such that $R^{**}(\alpha) = h(\alpha)R^*(\alpha)$. By comparing determinants, we obtain $h(\alpha)^r = 1$. Since $R(1) = R^{**}(\alpha)^n = h(\alpha)^n R^*(\alpha)^n = h(\alpha)^n R(1)$, $h(\alpha)^n = 1$. Hence $h(\alpha) = h(\alpha)^{in+jr} = 1^{i+j} = 1$. Now take any $\alpha \in A$ and $g \in G$; then $R^{**}(\alpha g) = R^{**}(\alpha)R(g) = R^*(\alpha)R(g) = R^*(\alpha g)$. This completes the proof of Theorem 1.

COROLLARY 1. *Let ζ be a character of a group G. Let $K = \mathbf{Q}_{|G|}$. Suppose that A is a relatively prime operator group on G that fixes ζ. Then there exists a representation of GA over K whose restriction to G affords ζ.*

Proof. We use induction on the degree of ζ. Let χ be an irreducible constituent of ζ and B the subgroup of A consisting of all those elements that fix χ. Let $B\alpha_1, \ldots, B\alpha_s$ be the distinct left cosets of B in A. Define

$$\zeta_1 = \sum_{i=1}^{s} \chi^{\alpha_i}.$$

Then A fixes ζ_1, and therefore fixes $\zeta - \zeta_1$. If $\zeta_1 \neq \zeta$, we may apply the induction hypothesis to ζ_1 and $\zeta - \zeta_1$ and take the direct sum of the corresponding representations.

Assume that $\zeta_1 = \zeta$. By a theorem of Brauer (**2**, p. 292), K is a splitting field for G. Therefore, some representation R of G over K affords χ. The degree of R divides $|G|$ and is therefore relatively prime to $|A|$. By Theorem 1, R can be extended to a representation R^* of GB over K. Let S be the representation of GA induced by R^*. Then the restriction of S to G affords ζ.

Remark. Suppose that in the proof of Corollary 1 we choose R^* such that $(\det R^*)(\alpha) = 1$ for all $\alpha \in B$. It is easy to see that $(\det S)(\alpha) = \pm 1$ for all $\alpha \in A$. Examples with G cyclic and $|A| = 2$ show that the value -1 can occur.

LEMMA 1. *Let m and n be relatively prime positive integers. Then*:

(a) $\mathbf{Q}_m \cap \mathbf{Q}_n = \mathbf{Q}$;

(b) *Let σ be any field automorphism of $\mathbf{Q}_m$ and let τ be any field automorphism of $\mathbf{Q}_n$. Then there exists a field automorphism ρ of $\mathbf{Q}_{mn}$ such that $x^\rho = x^\sigma$ for all $x \in \mathbf{Q}_m$ and $y^\rho = y^\tau$ for all $y \in \mathbf{Q}_n$.*

Proof. This is well known (see **15**, p. 162).

COROLLARY 2. *Let ζ be a character of degree r of a group G. Suppose that A is a relatively prime operator group on G that fixes ζ and acts faithfully on $G/\mathrm{Ker}\,\zeta$. Let $K = \mathbf{Q}_{|G|}$. Then*:

(a) *A possesses a faithful representation of degree r over K that has a rational valued character*;

(b) *$|A|$ divides $(2r)! = 2r(2r-1)(2r-2)\ldots 2\cdot 1$; and*

(c) *If A is an elementary Abelian p-group of order p^e, then $(p-1)e \leqq r$.*

Proof. (a) By the previous corollary, there exists a representation of GA over K whose restriction to G affords ζ. Let η be the character of this representation. Since A acts faithfully on $G/\mathrm{Ker}\,\zeta$, $\eta|_A$ is faithful. Also, η has degree r and $\mathbf{Q}(\eta) \subseteq K$. However, $\mathbf{Q}(\eta|_A) \subseteq \mathbf{Q}_{|A|}$. By Lemma 1,

$$\mathbf{Q} = K \cap \mathbf{Q}_{|A|} \supseteq \mathbf{Q}(\eta|_A).$$

(b) This follows from (a) by a theorem of Schur (**12**).

(c) By (a), A has a faithful rational character χ of degree r. Assume that $e > 0$. Then χ has at least one non-trivial irreducible constituent. Let S be the set of all irreducible characters of A that occur as constituents of χ. Define two elements θ and η of S to be *equivalent* if $\theta = \eta^\sigma$ for some automorphism σ of $\mathbf{Q}_{|A|}$ over $\mathbf{Q}$. Clearly, this yields an equivalence relation. Let $\theta_1, \ldots, \theta_f$ be representatives of the distinct equivalence classes of S. For $i = 1, \ldots, f$, let $K_i = \mathrm{Ker}\,\theta_i$. Since equivalent characters have the same kernel, $\bigcap_i K_i = 1$. Hence

$$p^e = |A/\textstyle\bigcap_i K_i| \leqq \pi_i |A/K_i| = p^f.$$

Thus, $e \leqq f$.

For $i = 1, \ldots, f$, $\mathbf{Q}(\theta_i) = \mathbf{Q}_p$. Thus, every equivalence class has exactly $(p-1)$ elements. Therefore,

$$r \geqq (p-1)f \geqq (p-1)e.$$

This completes the proof of Corollary 2.

For the following result, we say that a group G is *p-solvable* if each of its composition factors has order p or order relatively prime to p.

COROLLARY 3. *Let p be a prime. Suppose that G is a p-solvable group of linear transformations of a vector space V of finite dimension r over a field F. Assume that F has characteristic* 0 *or p and that G has no normal p-subgroup except the identity group. Then every Sylow p-subgroup of G has order dividing* $(2r)!$. *Moreover, if G contains an elementary Abelian subgroup of order p^e, then* $(p-1)e \leqq r$.

Proof. Let N be the largest normal subgroup of G that has order relatively prime to p. Suppose that P is a p-subgroup of G. By (**11**, Lemma 1.2.3), no non-identity element of P centralizes N.

Suppose that F has characteristic zero. Let ζ be the character of N on V. Then Ker $\zeta = 1$ and P fixes ζ. The result follows from parts (b) and (c) of Corollary 2.

Suppose that F has characteristic p. By (**13**, *Satz* 206, p. 223), there exists (up to equivalence) a one-to-one correspondence of representations of N over F with representations of N over the complex field. Moreover, we may assume that this correspondence is preserved under direct sums. Let R be the representation of N over the complex field that corresponds to its representation on V. Then N has a non-trivial constituent on V and, therefore, one in R. From the further properties of this correspondence, R has degree r and the character of R is fixed by P. Now we may apply parts (b) and (c) of Corollary 2. This completes the proof of Corollary 3.

Remark. Some results similar to Corollaries 2 and 3 have recently been obtained by J. D. Dixon (see **3**; **4**).

THEOREM 2. *Let ζ be an irreducible character of a group G. Let A be a relatively prime operator group on G such that A fixes ζ. Then*:

(a) *There exists a unique irreducible character η of GA such that $\eta|_G = \zeta$ and* $(\det \eta)(\alpha) = 1$ *for all $\alpha \in A$*;

(b) *If η satisfies* (a), *then* $\mathbf{Q}(\eta) = \mathbf{Q}(\zeta)$, *and $\eta(\alpha)$ is a rational integer for every $\alpha \in A$*;

(c) *Assume that η satisfies* (a). *If η' is an irreducible character of GA and ζ is a constituent of $\eta'|_G$, then there exists a unique irreducible character β of GA/G such that $\eta' = \eta\beta$. Conversely, for every irreducible character β of GA/G, $\eta\beta$ is an irreducible character of GA and ζ is a constituent of $\eta\beta|_G$.*

Proof. Let $m = |G|$ and $n = |A|$.

(a) Let K be the complex field and let R be a representation of G over K that affords ζ. The degree of R divides m and is, therefore, relatively prime to n. Take R^* as in Theorem 1, and let η be the character of R^*.

(b) Assume that η satisfies (a). Since $\eta|_G = \zeta$, $\mathbf{Q}(\zeta) \subseteq \mathbf{Q}(\eta)$. Conversely, for every automorphism ρ of $\mathbf{Q}_{mn}$ that fixes the elements of $\mathbf{Q}(\zeta)$, η^ρ is an irreducible character of GA that satisfies

$$\eta^\rho|_G = \zeta^\rho = \zeta \quad \text{and} \quad (\det \eta^\rho)(\alpha) = 1^\rho = 1 \quad \text{for all } \alpha \in A.$$

Hence $\eta^\rho = \eta$. Thus $\mathbf{Q}(\eta) \subseteq \mathbf{Q}(\zeta)$.

Take $\alpha \in A$; then $\eta(\alpha)$ is an algebraic integer and

$$\eta(\alpha) \in \mathbf{Q}(\eta) \cap \mathbf{Q}_n = \mathbf{Q}(\zeta) \cap \mathbf{Q}_n \subseteq \mathbf{Q}_m \cap \mathbf{Q}_n = \mathbf{Q}.$$

Thus $\eta(\alpha)$ is a rational integer.

(c) By hypothesis, GA fixes ζ. By the Frobenius Reciprocity Theorem, η' is a constituent of ζ^{GA}. Hence, by (**7**, Theorem 2), $\eta' = \eta\beta$ for some unique irreducible character β of GA/G. The converse also follows from (**7**, Theorem 2).

Note. Henceforth, the character η that satisfies part (a) of Theorem 2 will be called the *canonical extension* of ζ to GA.

3. Cyclic operator groups.

LEMMA 2. *Let A be a cyclic relatively prime operator group on a group G. Suppose that α is a generator of A and $T = C_G(A)$. Then:*

(a) *Every element of GA of the form αg, $g \in G$, is conjugate to an element of the form αt, $t \in T$;*

(b) *If $t_1, t_2 \in T$, then αt_1 and αt_2 are conjugate in GA if and only if t_1 and t_2 are conjugate in T;*

(c) *If $t \in T$, then $C_{GA}(\alpha t) = C_{AT}(t) = AC_T(t)$.*

Proof. Let $m = |G|$ and $n = |A|$.

(a) Let $g \in G$. Since αg lies in a coset of G that generates GA/G, n divides the order of αg. Hence, $\alpha g = \beta h = h\beta$ for some powers β and h of αg having the property that the order of β is n and the order of h divides m. Now, β generates a complement of G in GA that we shall call B. Since A is cyclic and m and n are relatively prime, B is conjugate to A in GA by (**16**, Theorem 27, pp. 162–163). We may assume that $B = A$. Then $\beta \in A$ and $\beta^{-1}\alpha \in G$; therefore $\beta = \alpha$. Thus $h \in C_{GA}(\alpha)$. Since the order of h divides $|G|$, $h \in G$. Hence $h \in T$.

(b) Suppose that $t_1, t_2 \in T$. If $t \in T$ and $t^{-1}t_1 t = t_2$, then $t^{-1}(\alpha t_1)t = \alpha t_2$. Conversely, suppose that $g \in G$ and $g^{-1}(\alpha t_1)g = \alpha t_2$. Let k be an integer such that $km \equiv 1 (\bmod\ n)$. Then $\alpha = (\alpha t_1)^{km} = (\alpha t_2)^{km}$. Hence $g^{-1}\alpha g = \alpha$, and $g \in T$. But

$$t_2 = \alpha^{-1}(\alpha t_2) = \alpha^{-1}g^{-1}(\alpha t_1)g = \alpha^{-1}(g^{-1}\alpha g)(g^{-1}t_1 g) = g^{-1}t_1 g.$$

(c) Let $t_1 = t_2$ in the proof of (b).

LEMMA 3. *Let A be a relatively prime operator group on a group G. Let $T = C_G(A)$. Then:*

(a) *Two elements of T are conjugate in G if and only if they are conjugate in T;*

(b) *A conjugate class of G is fixed by A if and only if it contains an element of T;*

(c) *If A fixes a subgroup H of G and a coset of H in G, the coset contains an element of T.*

Proof. These results follow from Corollary 1 of Theorem 3, Corollary 1 of Theorem 4, and (**9**, Theorem 1).

THEOREM 3. *Suppose that A is a cyclic relatively prime operator group on a group G. Let T be the fixed-point subgroup of G with respect to A.*

(a) *Suppose that ζ is an irreducible character of G that is fixed by A. Let η be the canonical extension of ζ to GA. Then there exists a unique sign $\epsilon = \pm 1$ and a unique irreducible character λ of T with the property that*

$$\eta(\alpha t) = \epsilon\lambda(t), \qquad t \in T, \tag{6}$$

for every element α that generates A.

(b) *For each irreducible character λ of T there exists a unique irreducible character ζ of G to which λ corresponds as in* (a).

Proof. Let $m = |G|$ and $n = |A|$, and let α be a generator of A.

(a) Suppose that β generates A. Then $\beta = \alpha^i$ for some integer i that is relatively prime to n. Take r and s such that $rm + ns = 1$. Let $j = i + ns(1 - i)$. Then $j \equiv 1 \pmod{m}$ and $j \equiv i \pmod{n}$. Therefore, j is relatively prime to mn. Let ω be a primitive mnth root of unity, and let ρ be the field automorphism of $\mathbf{Q}_{mn}$ determined by $\omega^\rho = \omega^j$. Then ρ fixes every element of $\mathbf{Q}_m$. By (**1**, p. 313), $\eta(x)^\rho = \eta(x^j)$ for all $x \in GA$. Since $\mathbf{Q}(\eta) \subseteq \mathbf{Q}_m$, $\eta^\rho = \eta$. In particular, for $t \in T$,

$$\eta(\alpha t) = \eta(\alpha t)^\rho = \eta((\alpha t)^j) = \eta(\alpha^j t^j) = \eta(\alpha^i t) = \eta(\beta t). \tag{7}$$

Let χ be the class function on GA defined by $\chi = \sum_\theta \theta(\alpha^{-1})\eta\theta$, where θ ranges over all the irreducible characters of GA/G. For $\beta \in A$ and $g \in G$, $\chi(\beta g) = \eta(\beta g)\sum_\theta \theta(\beta)\theta(\alpha^{-1})$. Hence

$$\chi(\beta g) = 0 \quad \text{if } \beta \neq \alpha \qquad \text{and} \qquad \chi(\alpha g) = n\eta(\alpha g). \tag{8}$$

By Theorem 2, every character $\eta\theta$ is irreducible. Therefore,

$$n = \sum_\theta |\theta(\alpha^{-1})|^2 = (\chi, \chi) = (1/mn)\sum_{g\in G} |\eta(\alpha g)|^2 n^2.$$

Consequently,

$$\sum_{g\in G} |\eta(\alpha g)|^2 = m. \tag{9}$$

Consider $\eta|_{AT}$. This is a character of AT, and is therefore a sum of irreducible characters of AT. Note that $AT = A \times T$. By (**2**, Corollary 51.13, p. 353), every irreducible character of T has the form $\theta\lambda$, where θ is an irreducible character of AT/T and λ is an irreducible character of AT/A. Therefore, there exist non-negative integers $c(\theta\lambda)$ such that $\eta|_{AT} = \sum_{\theta,\lambda} c(\theta\lambda)\theta\lambda$. Let $c(\lambda) = \sum_\theta c(\theta\lambda)\theta(\alpha)$. Then $\eta(\alpha t) = \sum_\lambda c(\lambda)\lambda(t)$ for all $t \in T$.

Choose a particular irreducible character λ_0 of T. Then $c(\lambda_0)$ is an algebraic integer in $\mathbf{Q}_n$. By the orthogonality relations, $c(\lambda_0) = (1/|T|) \sum_{t\in T} \eta(\alpha t)\lambda_0(t^{-1})$. Since $\mathbf{Q}(\eta) \subseteq \mathbf{Q}_m$ and $\mathbf{Q}(\lambda_0) \subseteq \mathbf{Q}_m$, $c(\lambda_0) \in \mathbf{Q}_m$. By Lemma 1 (a), $c(\lambda_0)$ is a rational integer.

Let $t_1, \ldots, t_w$ be a sequence of representatives of the distinct conjugate classes of T. By Lemma 2 and (9),

$$m = \sum_{g\in G} |\eta(\alpha g)|^2 = \sum_{1\leqq i\leqq w} |\eta(\alpha t_i)|^2[GA:C_{AT}(t_i)] = \sum_{1\leqq i\leqq w} |\eta(\alpha t_i)|^2[G:C_T(t_i)] = [G:T]\sum |\eta(\alpha t_i)|^2[T:C_T(t_i)] = [G:T]\sum_{t\in T} |\eta(\alpha t)|^2 = [G:T]\sum_{t\in T} \left|\sum_\lambda c(\lambda)\lambda(t)\right|^2 = [G:T]|T|\left(\sum_\lambda c(\lambda)\lambda, \sum_\lambda c(\lambda)\lambda\right) = |G|\sum_\lambda c(\lambda)^2 = m\sum_\lambda c(\lambda)^2.$$

Hence, there exists a unique irreducible character λ_0 and a unique sign $\epsilon = \pm 1$ such that $c(\lambda_0) = \epsilon$ and $c(\lambda) = 0$ for $\lambda \neq \lambda_0$. Clearly,

$$\eta(\alpha t) = \epsilon\lambda_0(t) \quad \text{for all } t \in T.$$

(b) Suppose that ζ' is any irreducible character of G that is fixed by A. Let η' be the canonical extension of ζ' to GA. Assume, for ζ, η, λ, and ϵ as in (a) and for some $\epsilon' = \pm 1$, that $\eta'(\alpha t) = \epsilon'\lambda(t)$ for all $t \in T$. Consider the class functions

$$\chi = \sum_\theta \theta(\alpha^{-1})\eta\theta, \qquad \chi' = \sum_\theta \theta(\alpha^{-1})\eta'\theta.$$

By (8), $\chi' = \epsilon\epsilon'\chi$. Hence, η' is a constituent of χ. By Theorem 2 (c), $\eta' = \eta\theta$ for some θ. Hence $\zeta' = \eta'|_G = (\eta|_G)(\theta|_G) = \zeta$.

Thus, different characters ζ determine different characters of T. Since A is cyclic and the character table of G is a non-singular matrix, we may apply a theorem of Brauer (**5**, p. 69). By this theorem, the number of irreducible characters of G fixed by A is equal to the number of conjugate classes of G fixed by A. By Lemma 3, this equals the number of conjugate classes of T, which, in turn, equals the number of irreducible characters of T. Hence, every irreducible character of T is determined by some (unique) irreducible character of G in the above manner. This completes the proof of Theorem 3.

Notation. From this point on, we will write $\zeta = \pi(A, G)(\lambda)$ and $\lambda = \pi^{-1}(A, G)(\zeta)$ if λ and ζ are related as in Theorem 3.

COROLLARY 4. *Suppose that ζ is an irreducible character of a group G and A is a relatively prime operator group on G that fixes ζ. Let η be the canonical extension of ζ to GA.*

https://doi.org/10.4153/CJM-1968-148-x Published online by Cambridge University Press

Take $\alpha \in A$. Let T_α be the fixed-point subgroup of G with respect to α. Then there exist a sign $\epsilon = \pm 1$ and an irreducible character λ of T_α such that

$$\eta(\alpha t) = \epsilon\lambda(t) \quad \textit{for all } t \in T_\alpha. \tag{10}$$

Proof. Let $B = \langle \alpha \rangle$ and $\eta' = \eta|_{GB}$. Then η' is the canonical extension of ζ to GB. Apply Theorem 3.

COROLLARY 5. *Suppose that A is a cyclic relatively prime operator group on a group G and B is a subgroup of A. Let T be the fixed-point subgroup of G with respect to A. Assume that $B \neq A$ and that, for every element α of A that lies outside B, T is the fixed-point subgroup of G with respect to α.*

Take an irreducible character ζ of G that is fixed by A. Let η be the canonical extension of ζ to GA and let $\lambda = \pi^{-1}(A, G)(\zeta)$. Let $\tilde{\lambda}$ be the character of AT that contains A in its kernel and coincides with λ on T. Then there exists $\epsilon = \pm 1$ and an irreducible character θ_0 of AT/T with the following properties:

(a) *For $t \in T$, $\alpha \in A$, and $\alpha \notin B$,*

$$\eta(\alpha t) = \epsilon\theta_0(\alpha)\lambda(t). \tag{11}$$

Moreover, $\theta_0(\alpha)^2 = 1$ for all $\alpha \in A$;

(b) *If $\zeta|_T = \epsilon\lambda$, then $G = T(\text{Ker } \zeta) = (\text{Ker } \zeta)T$. If $\zeta|_T \neq \epsilon\lambda$, then $|A/B|^{-1}(\zeta|_T - \epsilon\lambda)$ is a character of T;*

(c) *For every irreducible character ψ of GA/GB, let $\tilde{\psi} = \psi|_{AT}$. Then*

$$(\eta - \eta\psi)|_{AT} = \epsilon(\theta_0\tilde{\lambda} - \theta_0\tilde{\psi}\tilde{\lambda}) \textit{ and } \eta - \eta\psi = \epsilon(\theta_0\tilde{\psi} - \theta_0\tilde{\psi}\tilde{\lambda})^{GA}.$$

Proof. (a) Let θ be a faithful irreducible character of A/B. To simplify notation, we will also regard θ as a character of GA and as a character of AT. Let $\mu = \eta - \eta\theta$ and $\nu = \mu|_{AT}$. Then $\mu(x) = 0$ for $x \in GB$, and $(\mu, \mu) = (\eta - \eta\theta, \eta - \eta\theta) = 2$.

Let $t_1, \ldots, t_r$ be a set of representatives of the distinct conjugate classes of T. By Lemma 2,

$$\begin{aligned}
2 = (\mu, \mu) = |GA|^{-1}\sum_{x \in GA} |\mu(x)|^2 = |GA|^{-1}\sum_{x \notin GB} |\mu(x)|^2 &= \\
|GA|^{-1} \sum_{\alpha \in A, \alpha \notin B, 1 \leq i \leq r} |\mu(\alpha t_i)|^2 [GA : C_{TA}(t_i)] &= \\
|TA|^{-1} \sum_{\alpha \in A, \alpha \notin B, 1 \leq i \leq r} |\nu(\alpha t_i)|^2 [TA : C_{TA}(t_i)] &= \\
|TA|^{-1} \sum_{\alpha \in A, 1 \leq i \leq r} |\nu(\alpha t_i)|^2 [TA : C_{TA}(\alpha t_i)] &= (\nu, \nu).
\end{aligned}$$

Since ν is a sum of (possibly negative) integer multiples of irreducible characters of AT and since $\nu(1) = \mu(1) = 0$ and $(\nu, \nu) = 2$, there exist distinct characters ν_1 and ν_2 of AT such that $\nu = \nu_1 - \nu_2$. Since $AT = A \times T$, by (**2**, Corollary 51.13, p. 353), there exist irreducible characters θ_1 and θ_2 of AT/T and λ_1 and λ_2 of AT/A such that $\nu_i = \theta_i\lambda_i$, $i = 1, 2$. For all $t \in T$, $0 = \mu(t) = \nu(t) = \lambda_1(t) - \lambda_2(t)$. Hence, $\lambda_1 = \lambda_2$, and

$$\mu|_{AT} = \nu = \theta_1\lambda_1 - \theta_2\lambda_1. \tag{12}$$

Now take $\alpha \in A$ such that $\alpha \notin B$. Take ϵ and λ to satisfy (10). For all $t \in T$,

$$(\theta_1(\alpha) - \theta_2(\alpha))\lambda_1(t) = \mu(\alpha t) = \eta(\alpha t) - \eta\theta(\alpha t) = \epsilon(1 - \theta(\alpha))\lambda(t).$$

By the linear independence of the irreducible characters of T, $\lambda_1 = \tilde{\lambda}$ and $\theta_1(\alpha) - \theta_2(\alpha) = \epsilon(1 - \theta(\alpha))$. It is possible that ϵ depends upon α; however,

$$\theta_1(\alpha)^2 - 2\theta_1(\alpha)\theta_2(\alpha) + \theta_2(\alpha)^2 = 1 - 2\theta(\alpha) + \theta^2(\alpha). \tag{13}$$

Now, (13) holds for all $\alpha \in A$, since both sides are zero for $\alpha \in B$. Thus

$$(\theta_1)^2 + (\theta_2)^2 + 2\theta = 1_{AT} + \theta^2 + 2\theta_1\theta_2,$$

where 1_{AT} is the trivial character of AT. Since $\theta \neq 1_{AT}$, $\theta \neq \theta^2$. Hence, $\theta = \theta_1\theta_2$. Furthermore, $(\theta_1)^2 = 1_{AT}$ or $(\theta_2)^2 = 1_{AT}$.

Suppose that $(\theta_1)^2 = 1_{AT}$. Then $\theta_2 = \theta(\theta_1)^{-1} = \theta\theta_1$. If $\alpha \in A$, $t \in T$, and $\alpha \notin B$, then by (12),

$$(1 - \theta(\alpha))\eta(\alpha t) = \mu(t) = \nu(t) = \theta_1(\alpha)\lambda_1(t) - \theta_1(\alpha)\theta(\alpha)\lambda_1(t) = \theta_1(\alpha)(1 - \theta(\alpha))\tilde{\lambda}(t).$$

Since $1 - \theta(\alpha) \neq 0$, $\eta(\alpha t) = \theta_1(\alpha)\tilde{\lambda}(t) = \theta_1(\alpha)\lambda(t)$. Similarly, if $(\theta_2)^2 = 1_{AT}$, then $\eta(\alpha t) = -\theta_2(\alpha)\lambda_1(t)$.

(b) Suppose that $\zeta|_T = \epsilon\lambda$. Then $\epsilon = 1$. Let α be a generator of A; by (a), $\eta(\alpha) = \theta_0(\alpha)\lambda(1) = \theta_0(\alpha)\eta(1)$. By (**2**, Corollary (30.11), p. 212), α is contained in the centre of $GA/\text{Ker}\ \eta$. Since $\text{Ker}\ \zeta = (\text{Ker}\ \eta) \cap G$, α fixes each coset of $\text{Ker}\ \zeta$ in G. Now Lemma 3 (c) yields $G = T(\text{Ker}\ \zeta) = (\text{Ker}\ \zeta)T$.

Suppose that $\zeta|_T \neq \epsilon\lambda$. Then $A \neq 1$; thus $n > 1$. Let $\mu = \eta|_{AT} - \epsilon\theta_0\tilde{\lambda}$. Since $AT = A \times T = T \times A$, μ may be represented uniquely as a sum

$$\mu = \sum c(\psi)\psi,$$

where ψ runs over all the irreducible characters of AT/T and where, for each ψ, $c(\psi)$ is a generalized character of AT/A. As in (a), let θ be a faithful irreducible character of TA/TB. Let $n = |A/B|$. Since $\mu(x) = 0$ for $x \notin TB$, $\mu\theta = \mu$. Hence, for each ψ,

$$c(\psi) = c(\theta\psi) = \ldots = c(\theta^{n-1}\psi).$$

Therefore, $\mu|_T = \sum c(\psi)|_T = n\Delta$ for some generalized character Δ of T. By hypothesis, $\Delta \neq 0$, and Δ is a character of T unless $(\Delta, \lambda) \leqq -1$. But if $(\Delta, \lambda) \leqq -1$, then

$$(\eta|_T, \lambda) = \epsilon + n(\Delta, \lambda) \leqq 1 - n < 0,$$

which is impossible.

(c) Let $\mu = \epsilon(\theta_0\tilde{\lambda} - \theta_0\tilde{\psi}\tilde{\lambda})$. Suppose that $\alpha \in A$ and $t \in T$. If $\alpha \in B$, then $\psi(\alpha) = 1$; thus,

$$(\eta - \eta\psi)(\alpha t) = 0 = \epsilon\theta_0(\alpha)\lambda(t)(1 - \psi(\alpha)) = \mu(\alpha t).$$

https://doi.org/10.4153/CJM-1968-148-x Published online by Cambridge University Press

If $\alpha \notin B$, then

$$(\eta - \eta\psi)(\alpha t) = (1 - \psi(\alpha))\eta(\alpha t) = \epsilon(1 - \psi(\alpha))\theta_0(\alpha)\lambda(t) = \mu(\alpha t).$$

Thus $(\eta - \eta\psi)|_{AT} = \mu = \epsilon(\theta_0\tilde{\lambda} - \theta_0\tilde{\psi}\tilde{\lambda})$.

Let $\nu = \mu^{GA}$. By Lemma 2 and the hypothesis of this corollary, $\nu(x) = 0$ for all $x \in GB$ and $\nu(x) = \mu(x) = (\eta - \eta\psi)(x)$ if $x \in AT$ but $x \notin BT$. Hence, by Lemma 2, $\nu(x) = (\eta - \eta\psi)(x)$ for all $x \in GA$. This completes the proof of Corollary 5.

COROLLARY 6. *Suppose that A is a cyclic relatively prime operator group on a group G. Let T be the fixed-point subgroup of G with respect to A. Assume that for every non-identity element α of A, T is the fixed-point subgroup of G with respect to α.*

Suppose that η is an irreducible character of GA such that $\eta|_G$ is irreducible. Let $\zeta = \eta|_G$ and $\lambda = \pi^{-1}(A, G)(\zeta)$. Let $\tilde{\zeta}$ and $\tilde{\lambda}$ be characters of AT/A such that $\tilde{\zeta}|_T = \zeta|_T$ and $\tilde{\lambda}|_T = \lambda$. Denote the characters of the regular representations of AT/T and A by $\rho_{AT/T}$ and ρ_A. Then there exist $\epsilon = \pm 1$ and an irreducible character θ of AT/A with the following properties:

(a) $\eta|_{AT} = \epsilon\theta\tilde{\lambda} + |A|^{-1}(\tilde{\zeta} - \epsilon\tilde{\lambda})\rho_{AT/T}$;

(b) $\eta|_A = \epsilon\lambda(1)\theta + |A|^{-1}(\zeta(1) - \epsilon\lambda(1))\rho_A$;

(c) *if* $\zeta|_T = \epsilon\lambda$, *then* $G = T(\text{Ker } \zeta) = (\text{Ker } \zeta)T$, *and if* $\zeta|_T \neq \epsilon\lambda$, *then* $|A|^{-1}(\zeta|_T - \epsilon\lambda)$ *is a character of* T.

Proof. Let η_0 be the canonical extension of ζ to GA. By Theorem 2, there exists an irreducible character θ_1 of GA/G such that $\eta = \eta_0\theta_1$. Let $\lambda = \pi^{-1}(G, A)(\zeta)$. By Corollary 5, there exist $\epsilon = \pm 1$ and an irreducible character θ_0 of GA/G such that $\eta_0(\alpha t) = \epsilon\theta_0(\alpha)\lambda(t)$ whenever $\alpha \in A$, $t \in T$, and $\alpha \neq 1$. Let $\theta = \theta_0\theta_1$ and $\eta' = \epsilon\theta\tilde{\lambda} = |A|^{-1}(\tilde{\zeta} - \epsilon\tilde{\lambda})\rho_{AT/T}$.

Suppose that $\alpha \in A$ and $t \in T$. If $\alpha = 1$, then $\rho_{AT/T}(\alpha t) = |A|$ and

$$\eta'(\alpha t) = \eta'(t) = \epsilon\tilde{\lambda}(t) = |A|^{-1}(\tilde{\zeta}(t) - \epsilon\tilde{\lambda}(t))|A| = \epsilon\lambda(t) = (\zeta(t) - \epsilon\lambda(t)) = \zeta(t) = \eta(\alpha t).$$

If $\alpha \neq 1$, then $\rho_{AT/T}(\alpha t) = 0$ and

$$\eta'(\alpha t) = \epsilon\theta(\alpha)\tilde{\lambda}(t) = \epsilon\theta(\alpha)\lambda(t) = \epsilon\theta_0(\alpha)\theta_1(\alpha)\lambda(t) = \eta_0(\alpha t)\theta_1(\alpha) = (\eta_0\theta_1)(\alpha t) = \eta(\alpha t).$$

Thus $\eta' = \eta|_{AT}$. This proves (a). Clearly, (b) follows from (a). We obtain (c) from part (b) of Corollary 5.

4. Solvable operator groups.

LEMMA 4. *Let A be a cyclic relatively prime operator group on a group G. Suppose that A is a normal subgroup of an operator group B on G. Let T be the fixed-point subgroup of G with respect to A. Suppose that λ is an irreducible character of T and $\beta \in B$. Let $\zeta = \pi(A, G)(\lambda)$. Then λ^β is an irreducible character of T and $\zeta^\beta = \pi(A, G)(\lambda^\beta)$.*

https://doi.org/10.4153/CJM-1968-148-x Published online by Cambridge University Press

Proof. Since β normalizes A, β fixes $C_G(A)$, which equals T. Hence, λ^β is an irreducible character of T. Let η be the canonical extension of ζ to GA. Since we may consider GA as a normal subgroup of GB, η^β is an irreducible character of GA.

Let α be a generator of A. Take $\epsilon = \pm 1$ such that $\eta(\alpha t) = \epsilon\lambda(t)$ for all $t \in T$. Let $\alpha' = \beta\alpha\beta^{-1}$. Then α' is a generator of A; thus by Theorem 3, $\eta(\alpha' t) = \eta(\alpha t)$ for all $t \in T$. Now $(\eta^\beta)|_G = \zeta^\beta$, which is an irreducible character of G. Furthermore, $(\det \eta^\beta)(\alpha) = (\det \eta)(\beta\alpha\beta^{-1}) = 1$. Hence, η^β is the canonical extension of ζ^β to GA. For all $t \in T$,

$$\eta^\beta(\alpha t) = \eta((\alpha t)^{\beta^{-1}}) = \eta((\beta\alpha\beta^{-1})t^{\beta^{-1}}) = \epsilon\lambda(t^{\beta^{-1}}) = \epsilon\lambda^\beta(t).$$

Thus $\zeta^\beta = (\eta^\beta)|_G = \pi(A, G)(\lambda^\beta)$. This completes the proof of Lemma 4.

Let A be a relatively prime operator group on a group G. In § 3 we defined a one-to-one correspondence between the irreducible characters of $C_G(A)$ and the irreducible characters of G fixed by A in the case that A is cyclic. In this section, we define a similar correspondence whenever A is a solvable group.

Definition. Let A be a solvable relatively prime operator group on a group G. Let C be a composition series for A, given by

$$A = A_0 \supset A_1 \supset \ldots \supset A_n = 1. \tag{14}$$

Let $T_i = C_G(A_i)$, $i = 0, 1, \ldots, n$, and let $T = T_0 = C_G(A)$. Suppose that $1 \leqq i \leqq n$. Since A_{i-1} normalizes A_i, A_{i-1} fixes T_i. Consider A_{i-1}/A_i as an operator group on T_i, whose corresponding fixed-point subgroup is T_{i-1}. We shall define two sequences of characters.

(a) Let λ be an irreducible character of T. Define λ_i for $i = 0, 1, \ldots, n$ as follows:

(i) λ_i is an irreducible character of T_i;
(ii) $\lambda_0 = \lambda$;
(iii) if $i > 0$, then $\lambda_i = \pi(A_{i-1}/A_i, T_i)(\lambda_{i-1})$.

Define $\pi_C(\lambda) = \lambda_n$. Thus $\pi_C(\lambda)$ is an irreducible character of G.

(b) Assume that, in (14), each subgroup A_i is a normal subgroup of A. Let ζ be an irreducible character of G that is fixed by A. Define ζ_i for $i = n, n-1, \ldots, 1, 0$, as follows:

(i) ζ_i is an irreducible character of T_i that is fixed by A;
(ii) $\zeta_n = \zeta$;
(iii) if $i < n$, $\zeta_i = \pi^{-1}(A_i/A_{i+1}, T_{i+1})(\zeta_{i+1})$.

We define $(\pi_C)^{-1}(\zeta) = \zeta_0$. Thus, $(\pi_C)^{-1}(\zeta)$ is an irreducible character of T.

It is fairly clear that $\pi_C(\lambda)$ is well-defined.

LEMMA 5. *Assume the hypothesis of part* (b) *of the previous definition. Then* $(\pi_C)^{-1}(\zeta)$ *is well-defined and* $\zeta = \pi_C((\pi_C)^{-1}(\zeta))$. *Moreover, for every irreducible character* λ *of* T, $\pi_C(\lambda)$ *is fixed by* A *and* $\lambda = (\pi_C)^{-1}(\pi_C(\lambda))$.

Proof. Clearly, $(\pi_C)^{-1}(\zeta)$ is well-defined if ζ_i is fixed by A_i/A_{i+1} for $i = n, n-1, \ldots, 1, 0$. By hypothesis, A fixes ζ_n. Suppose that $i < n$ and A fixes

https://doi.org/10.4153/CJM-1968-148-x Published online by Cambridge University Press

ζ_{i+1}. We may regard A_i/A_{i+1} as a normal subgroup of A/A_{i+1}. By Lemma 4, A fixes ζ_i.

In a similar manner, we see that A fixes $\pi_C(\lambda)$ for every irreducible character λ of T. Let $\chi = \pi_C(\lambda)$; by induction, $\chi_i = \lambda_i$ for $i = n, n-1, \ldots, 1, 0$. Thus $(\pi_C)^{-1}(\pi_C(\lambda)) = \lambda$. Likewise, $\pi_C((\pi_C)^{-1}(\zeta)) = \zeta$.

THEOREM 4. *Let A be a solvable relatively prime operator group on a group G. Suppose that T is the fixed-point subgroup of G with respect to A, λ is an irreducible character of T, and C is a composition series for A. Let $\zeta = \pi_C(\lambda)$. Then*:

(a) *If A is a p-group for some prime p, there exists $\epsilon = \pm 1$ such that $(\zeta|_T - \epsilon\lambda)/p$ is either identically zero or is a character of T;*

(b) *If A is cyclic, then $\zeta = \pi(A, G)(\lambda)$;*

(c) *If D is any other composition series for A, then $\pi_D(\lambda) = \zeta$.*

Proof. Assume that C has the same form as in (14).

(a) We use induction on $|A|$. The assertion is obvious if $|A| = 1$. Assume that $|A| > 1$. We use the same notation as in the definition of π_C. By induction, $((\lambda_{n-1})|_T - \epsilon'\lambda)/p$ is a generalized character of T for some $\epsilon' = \pm 1$. Let $S = T_{n-1}$. Since $|A_{n-1}| = p$, by Corollary 6 (c) there exists $\epsilon'' = \pm 1$ such that $(\zeta|_S - \epsilon''\lambda_{n-1})/p$ is a generalized character of S. Since $T \subseteq S$, $(\zeta|_T - \epsilon''\lambda_{n-1}|_T)/p$ is a generalized character of T. Since

$$(\zeta|_T - \epsilon''\lambda_{n-1}|_T) - \epsilon''(\lambda_{n-1}|_T - \epsilon'\lambda) = \zeta|_T + \epsilon'\epsilon''\lambda,$$

$(\zeta|_T + \epsilon'\epsilon''\lambda)/p$ is a generalized character of T. Let $\epsilon = -\epsilon'\epsilon''$. Then $(\zeta|_T, \lambda) \equiv \epsilon \not\equiv 0 \pmod p$. Hence $(\zeta|_T, \lambda) \geqq 1$. Therefore, $\zeta|_T - \epsilon\lambda$ is either zero or a character of T. Thus, $(\zeta|_T - \epsilon\lambda)/p$ is either zero or a character of T.

(b) We use induction on $|A|$. Assume that $|A| > 1$. By Lemma 5, A fixes ζ. Let $\lambda_0 = \pi^{-1}(A, G)(\zeta)$ and let η be the canonical extension of ζ to GA. Take $\epsilon_0 = \pm 1$ such that $\epsilon_0\lambda_0(t) = \eta(\alpha t)$ for every $t \in T$ and for every generator α of A. Suppose that $|A|$ is a power of a prime p. Let B be the unique subgroup of index p in A. By Corollary 5 (b), $(\zeta|_T - \epsilon_0\lambda_0)/p$ is identically zero or is a character of T. Thus p does not divide $(\zeta|_T, \lambda_0)$. By part (a), $\lambda_0 = \lambda$. Thus $\zeta = \pi(A, G)(\lambda_0) = \pi(A, G)(\lambda)$.

Suppose that $|A|$ is not a prime power. Let $p = |A/A_1|$. Then $A = B \times E$ for a p-group B and a group E whose order is not divisible by p. Since $p = |A/A_1|$, $E \subseteq A_1$. Let $\lambda_1 = \pi(A/A_1, T_1)(\lambda_0)$, and let C^* be the composition series of A_1 obtained by deleting A from the series C. Then $\zeta = \pi_{C^*}(\lambda_1)$. By the induction hypothesis, $\zeta = \pi(A_1, G)(\lambda_1)$; thus ζ does not depend on C^*. Since $1 \subseteq E \subseteq A_1$, we may assume that E is one of the terms in C^*. Let $U = C_G(E)$ and $\lambda' = \pi(A/E, U)(\lambda)$. Similarly we obtain the result that $\zeta = \pi(E, G)(\lambda')$.

Let β and γ be generators of B and E, respectively. Consider E as a relatively prime operator group on GB. Then $C_{GB}(E) = BU$. Since $\eta|_G = \zeta$, $\eta|_{GB}$ is irreducible. By Theorems 2 and 3, there exist $\epsilon' = \pm 1$ and irreducible characters θ' of GA/GB and η' of BU such that

$$\eta(\gamma x) = \epsilon'\theta'(\gamma)\eta'(x) \quad \text{for all } x \in BU. \tag{15}$$

Consider (15) for $x \in U$. Since $\zeta = \pi(E, G)(\lambda')$, $\eta'|_U = \lambda'$. Now, $\lambda' = \pi(A/E, U)(\lambda) = \pi(B, U)(\lambda)$. Therefore, there exist $\epsilon = \pm 1$ and an irreducible character θ of BU/U such that

$$(16) \qquad \eta'(\beta t) = \epsilon\theta(\beta)\lambda(t) \quad \text{for all } t \in T.$$

By (15) and (16),

$$\eta(\beta\gamma t) = \epsilon\epsilon'\theta(\beta)\theta'(\gamma)\lambda(t) \quad \text{for all } t \in T.$$

Since $\beta\gamma$ generates A, $\zeta = \pi(A, G)(\lambda)$.

(c) Let D have the form

$$A = B_0 \supset B_1 \supset \ldots \supset B_n = 1.$$

We use induction on n. If $n \leqq 1$, $\pi_D(\lambda) = \zeta$. Suppose that $n \geqq 2$. If $A_1 = B_1$, then

$$\pi(A_0/A_1, C_G(A_1))(\lambda) = \pi(B_0/B_1, C_G(B_1))(\lambda);$$

thus $\pi_D(\lambda) = \pi_C(\lambda)$ by the induction hypothesis. Assume that $A_1 \neq B_1$. Let $J = A_1 \cap B_1$. Then J is a normal subgroup of A. By the induction hypothesis, $\pi_C(\lambda)$ and $\pi_D(\lambda)$ are unchanged if we assume that $J = A_2 = B_2$. Consider A/J as an operator group on $C_G(J)$ and J as an operator group on G. By the induction hypothesis, $\pi_C(\lambda) = \pi_D(\lambda)$ if $n > 2$.

Assume that $n = 2$. Since $A_1 \neq B_1$, $A = A_1 \times B_1$. Now A_1 and B_1 both have prime order. Either $|A_1| = |B_1|$ or A is cyclic. Suppose that $|A_1| = |B_1| = p$, say. Let $\zeta' = \pi_D(\lambda)$ and $\lambda' = (\pi_C)^{-1}(\zeta')$. By Lemma 5, $\zeta' = \pi_C(\lambda')$. By part (a) of the present theorem, (ζ', λ') is not divisible by p. Similarly, (ζ', λ) is not divisible by p, since $\zeta' = \pi_D(\lambda)$. By (a), $\lambda' = \lambda$. Hence $\zeta' = \pi_C(\lambda) = \zeta$.

Assume that $n = 2$ and that A is cyclic. By (b), $\pi_C(\lambda) = \pi(A, G)(\lambda) = \pi_D(\lambda)$. This completes the proof of Theorem 4.

COROLLARY 7. *Assume the hypothesis of Theorem 4. Suppose that A is a normal subgroup of an operator group B on G. If $\beta \in B$, then $\zeta^\beta = \pi_C(\lambda^\beta)$.*

Proof. Assume that C has the form (14). Let D be the composition series

$$A = (A_0)^\beta \supset (A_1)^\beta \supset \ldots \supset (A_n)^\beta = 1.$$

Since B normalizes A, B fixes T, which is equal to $C_G(A)$. An induction argument shows that $\zeta^\beta = \pi_D(\lambda^\beta)$. By Theorem 4, $\zeta^\beta = \pi_C(\lambda^\beta)$.

COROLLARY 8. *Assume the hypothesis of Theorem 4. Then A fixes ζ. Conversely, for every irreducible character ζ' of G that is fixed by A, there exists a unique irreducible character λ' of T such that $\zeta' = \pi_C(\lambda')$.*

Proof. Take $\alpha \in A$. Since $T \subseteq C_G(\alpha)$, $\lambda^\alpha = \lambda$. By Corollary 7, $\zeta^\alpha = \pi_C(\lambda^\alpha) = \pi_C(\lambda) = \zeta$.

We prove the converse by induction on the length of the series C. Suppose that C has the form (14). We may assume that $|A| > 1$. Let D be the series

$$A_1 \supset A_2 \supset \ldots \supset A_n = 1.$$

https://doi.org/10.4153/CJM-1968-148-x Published online by Cambridge University Press

Now, A_1 fixes ζ'. By the induction hypothesis, there exists a unique irreducible character μ of $C_G(A_1)$ such that $\zeta' = \pi_D(\mu)$.

Take $\alpha \in A$. Since A_1 is a normal subgroup of A, $\pi_D(\mu^\alpha) = \zeta'^\alpha = \zeta'$, by Corollary 7. Thus A fixes μ. Let $\lambda' = \pi^{-1}(A/A_1, C_G(A_1))(\mu)$. Then $\zeta' = \pi_C(\lambda')$. Moreover, μ uniquely determines λ'.

COROLLARY 9. *Let A be a solvable relatively prime operator group on a group G. Let T be the fixed-point subgroup of G with respect to A. The number of irreducible characters of G that are fixed by A is equal to the number of irreducible characters of T and is also equal to the number of conjugate classes of G that are fixed by A. In particular, if $T \neq 1$, A fixes some non-identity irreducible character of G.*

Proof. This follows directly from Corollary 8 and Lemma 3.

Notation. Assume the hypothesis of Theorem 4. We write

$$\zeta = \pi(A, G)(\lambda) \quad \text{and} \quad \lambda = \pi^{-1}(A, G)(\zeta).$$

By Theorem 4, this notation is independent of the composition series C, and it agrees with our previous notation when A is cyclic. By Corollary 9, $\pi(A, G)$ is a one-to-one correspondence between the irreducible characters of T and the irreducible characters of G that are fixed by A.

THEOREM 5. *Let A be a solvable relatively prime operator group on a group G. Let T be the fixed-point subgroup of G with respect to A. Suppose that λ is an irreducible character of T and $\zeta = \pi(A, G)(\lambda)$. Then:*

(a) *If β is an element of an operator group on G that contains A as a normal subgroup, then $\zeta^\beta = \pi(A, G)(\lambda^\beta)$;*

(b) *If σ is a field automorphism of $\mathbf{Q}_{|GA|}$, then $\zeta^\sigma = \pi(A, G)(\lambda^\sigma)$;*

(c) *The field $\mathbf{Q}(\lambda)$ is equal to $\mathbf{Q}(\zeta)$;*

(d) *The character λ is a constituent of $\zeta|_T$;*

(e) *The degree $\zeta(1)$ divides $[G:T]\lambda(1)$.*

Proof. Let C be a composition series for A. Then $\zeta = \pi_C(\lambda)$. Let η be the canonical extension of ζ to GA. Clearly, we may assume that $|A| > 1$.

(a) This is Corollary 7.

(b) By the definition of π_C and by induction, it is sufficient to prove this result when A has prime order. By Theorem 3, there exists a unique sign $\epsilon = \pm 1$ such that $\eta(\alpha t) = \epsilon\lambda(t)$ whenever $\alpha \in A$, $t \in T$, and $\alpha \neq 1$. Clearly, A fixes ζ^σ, and η^σ is the canonical extension of ζ^σ to GA. Since $\eta^\sigma(\alpha t) = \epsilon\lambda^\sigma(t)$ whenever $\alpha \in A$, $t \in T$, and $\alpha \neq 1$, we have that $\lambda^\sigma = \pi^{-1}(A, G)(\zeta^\sigma)$.

(c) This follows from (b); the field automorphisms of $\mathbf{Q}_{|GA|}$ that fix λ coincide with those that fix ζ.

(d) As in the proof of (b), we assume that A has prime order. Let $p = |A|$. By Theorem 4 (a), $(\zeta|_T, \lambda) \equiv \pm 1 \not\equiv 0 \pmod{p}$.

(e) We use induction on the length, n, of the composition series C. We may assume that $n > 0$. Take T_{n-1} and λ_{n-1} as in the definition of π_C. By the induction hypothesis, $\lambda_{n-1}(1)$ divides $[T_{n-1}:T]\lambda(1)$.

Suppose that $n > 1$. By the induction hypothesis, $\zeta(1)$ divides

$$[G:T_{n-1}]\lambda_{n-1}(1).$$

Hence, $\zeta(1)$ divides $[G:T_{n-1}][T_{n-1}:T]\lambda(1)$, which equals $[G:T]\lambda(1)$. Thus we may assume that $n = 1$. Let α be a generator of A. By (**10**, p. 287), $[GA:C_{GA}(\alpha)]\eta(\alpha)/\eta(1)$ is an algebraic integer, that is, $\pm[G:T]\lambda(1)/\zeta(1)$ is an algebraic integer. This completes the proof of Theorem 5.

5. An application. We require some additional notation. If A is an operator group on a group G, then $C_A(G)$ is the set of those elements of A that fix every element of G. If $H \subseteq C_G(A)$, we say that A *centralizes* H. We shall often consider a group of transformations of a vector space as an operator group on the additive group of the vector space.

Suppose that G is a group. For $x, y \in G$, let (x, y) be the commutator $x^{-1}y^{-1}xy$. If H and K are subgroups of G, let $C_H(K)$ be the centralizer of K in H and let (H, K) be the subgroup of G generated by the commutators (x, y) for $x \in H$ and $y \in K$. Note that this agrees with our previous notation if K normalizes H, and K is considered as an operator group on H. Let $G' = (G, G)$, and let $Z(G)$ be the centre of G. Denote by $F(G)$ the *Fitting subgroup* of G, that is, the maximal normal nilpotent subgroup of G. (By **10**, Theorem 10.5.2, p. 153, $F(G)$ must exist.) Let $F_2(G)$ be the subgroup of G that contains $F(G)$ and satisfies $F_2(G)/F(G) = F(G/F(G))$. We denote the characters of the regular representation and of the trivial representation of G by ρ_G and 1_G.

Suppose that π is a set of primes. An integer is a *π-number* if each of its prime divisors lies in π. For every positive integer n, let n_π be the largest π-number that divides n. A subgroup H of G is called a *Hall π-subgroup* of G if $|H| = |G|_\pi$.

Suppose that G is a p-group for some prime p. The *Frattini subgroup* of G, denoted by $D(G)$, is the subgroup of G generated by the elements x^p and (x, y) for $x, y \in G$. Since $D(G) \supseteq G'$, $D(G)$ is a normal subgroup of G. We say that G is a *special* p-group if $D(G) = G' = Z(G)$ and if $D(G)$ is an elementary Abelian group.

The direct sum of subspaces V and W of a given vector space will be denoted by $V \oplus W$.

In this section we apply our previous results to prove the following theorem.

THEOREM 6. *Let G be a finite solvable group. Let r and s be two primes that do not divide $|G|$. Suppose that $A \times B$ is an operator group on G such that A is a cyclic r-group, B is a group of order s, and $C_G(B) \subseteq C_G(A)$. Let $H = (G, A)$. Then H is a normal subgroup of G. Furthermore, if H is not contained in $F(G)$, then the following conditions hold*:

(a) $r = 2$;

(b) $2s - 1$ *is a power of some prime* q;

(c) $H/(H \cap F(G))$ *is a non-Abelian special q-group of exponent q which is centralized by α^2 for every $\alpha \in A$*;

https://doi.org/10.4153/CJM-1968-148-x Published online by Cambridge University Press

(d) *AB centralizes* $Z(H/(H \cap F(G)))$; *and*
(e) $H \subseteq F_2(G)$.

Remark. This theorem was originally announced (see **8**) *for the special case that* $|A| = r$. In (**14**, pp. 261–262), Thompson pointed out that A need only be a cyclic r-group.

LEMMA 6. *Let* A *be a relatively prime operator group on a group* G, *and let* $T = C_G(A)$. *Then* (G, A) *is a normal subgroup of* G *and is fixed by* A. *Moreover,*
(a) $((G, A)A) = (G, A)$;
(b) *if* G *is an Abelian group, then* $G = (G, A) \times T$; *and*
(c) *if* G *is a solvable group and* π *is a set of primes, then* A *fixes a Hall* π-*subgroup of* G.

Proof. By (**9**, proof of Corollary 3 of Theorem 1), (G, A) is normal in G and is fixed by A, and $((G, A), A) = (G, A)$. By (**9**, Corollary 2 (ii) of Theorem 4), (c) holds. Finally, consider G as a normal subgroup of the semi-direct product GA. Then (b) follows from (**16**, Lemma, p. 172).

LEMMA 7. *Let* A *be an operator group on a finite group* G. *Let* S *be a normal subgroup of* G *that is contained in* $C_G(A)$. *Then* (G, A) *centralizes* S.

Proof. Clearly, S is a normal subgroup of GA. Hence, $C_{GA}(S)$ is a normal subgroup of GA that contains A. Let $g \in G$ and $\alpha \in A$. Then $\alpha \in C_{GA}(S)$ and $g^{-1}\alpha^{-1}g \in C_{GA}(S)$. Thus $g^{-1}g^{\alpha} = g^{-1}\alpha^{-1}g\alpha \in C_{GA}(S)$.

LEMMA 8. *Let* A *be a relatively prime operator group on a finite solvable group* G. *Let* $T = C_G(A)$, *and let* π *be the set of all prime divisors of* $[G:T]$. *Then* (G, A) *is a* π-*group.*

Proof. By Lemma 6 (c), A fixes some Hall π-subgroup H of G. Moreover, $G = TH$, since

$$|TH| = |T|\,|H|/|T \cap H| \geqq |T|\,|H|/|T|_{\pi} = |T|\,|G|_{\pi}/|T|_{\pi} = |T|[G:T]_{\pi} = |T|[G:T] = |G|.$$

Now let $g \in G$ and $\alpha \in A$. Take $t \in T$ and $h \in H$ such that $th = g$. Then $g^{-1}g^{\alpha} = h^{-1}t^{-1}t^{\alpha}h^{\alpha} = h^{-1}h^{\alpha}$. Thus $(G, A) \subseteq H$.

LEMMA 9. *Let* χ *be a faithful irreducible complex character of a finite nilpotent group* G *of nilpotence class two. If* $x \in G$ *and* $x \notin Z(G)$, *then* $\chi(x) = 0$.

Proof. Let R be a representation of G that affords χ. Take $y \in G$ such that $xy \neq yx$. Let $z = x^{-1}y^{-1}xy$. Then $z \in Z(G)$ and $z \neq 1$. Hence $R(z)$ is a scalar multiple of the identity transformation I, say, $R(z) = aI$. Since $y^{-1}xy = xz$, $R(y^{-1}xy) = aR(x)$. Therefore, $\chi(y^{-1}xy) = a\chi(x)$. Since characters are class functions, $\chi(y^{-1}xy) = \chi(x)$. Since $a \neq 1$, $\chi(x) = 0$.

The main step in the proof of Theorem 6 is the following lemma.

LEMMA 10. *Let* G *be a finite group, and let* q, r, *and* s *be distinct primes. Suppose that* G *contains subgroups* Q, A, *and* B *with the following properties*:

https://doi.org/10.4153/CJM-1968-148-x Published online by Cambridge University Press

(i) *Q is a normal non-identity q-subgroup of G;*
(ii) *A is a cyclic r-group;*
(iii) *B is a group of order s;*
(iv) *B centralizes A and $C_Q(B) \subseteq C_Q(A)$;*
(v) *$G = QAB$; and*
(vi) *$(Q, A) = Q$.*

Suppose that G is represented by a group of linear transformations on a vector space V of finite dimension over a field F. Assume that Q is faithfully represented, that the characteristic of F does not divide $|G|$, and that $C_V(B) \subseteq C_V(A)$. Then:
(a) *$r = 2$ and $2s - 1$ is a power of q;*
(b) *Q is a non-Abelian special q-group of exponent q;*
(c) *AB centralizes $Z(Q)$; and*
(d) *α^2 centralizes Q for every $\alpha \in A$.*

Proof. Let d be the dimension of V over F. We use induction on $|G| + d$. Let S be a basis for V over F, and let E be an algebraically closed field that contains F. Clearly, we may consider V as a subset of a vector space U over E that has S as a basis. Then G is represented by a group of linear transformations on U over E. An easy calculation shows that a basis for $C_V(B)$ over F is also a basis for $C_U(B)$ over E. Thus $C_U(B) \subseteq C_U(A)$. Since G, U, and E satisfy the hypothesis of the lemma, and since U has dimension d over E, we may assume that $U = V$ and $E = F$.

Suppose that G is not represented irreducibly on V. Let W be a non-trivial proper G-invariant subspace of V. By Maschke's Theorem (**10**, p. 253), V contains a G-invariant subspace X such that $V = W \oplus X$. We may assume that Q is represented non-trivially on W. Since $Q/C_Q(W)$ is non-trivial and is faithfully represented on W, $G/C_Q(W)$ and W satisfy the hypothesis of Lemma 10. (We use Lemma 3 (c) to obtain condition (iv).) By induction, we obtain (a) and observe that $C_Q(W)$ contains $(AB, D(Q))$ and $(AB, Z(Q))$ and also contains (α^2, Q) and g^q for all $\alpha \in A$ and $g \in Q$. Similarly, $C_Q(x)$ contains the same groups and elements if Q is represented non-trivially on X. It obviously contains them if Q is represented trivially on X. Hence

$$(AB, D(Q)) \subseteq C_Q(W) \cap C_Q(X) = 1.$$

Furthermore, (c) and (d) hold, and Q has exponent q. Since $Q = (Q, A)$, by Lemma 7 we have that $D(Q) \subseteq Z(Q)$. Regarding A as an operator group on Q/Q', we obtain $Q/Q' = (Q/Q', A)$. Thus, by Lemma 6 (b), A has no fixed points on Q/Q'. Since A centralizes $Z(Q)$, we have that $Z(Q) \subseteq Q'$. Therefore, $D(Q) \subseteq Z(Q) \subseteq Q' \subseteq D(Q)$. Hence, Q is a non-Abelian special q-group of exponent q.

Thus, it suffices to prove the lemma when F is algebraically closed and when G is represented irreducibly on V. Suppose that the characteristic of F is not zero. As in the proof of Corollary 3, there exists a representation of G on a complex vector space W that corresponds to the representation of G on V.

Since the multiplicity of the trivial representation of subgroups is preserved by the correspondence in (**13**, *Satz* 206), Q is faithful on V. Similarly, the dimensions of $C_W(AB)$ and $C_W(B)$ coincide, since $C_V(AB) = C_V(B)$. But $C_W(AB) \subseteq C_W(B)$. Therefore, $C_W(B) = C_W(AB) \subseteq C_W(A)$.

Thus, we may assume that F is the complex field and that G is represented irreducibly on V. Let W be a homogeneous submodule of V under the action of Q, and let I be the largest subgroup of G which fixes W. Let $Ig_1, \ldots, Ig_n$ be the distinct cosets of I in G. By Clifford's Theorem (**2**, pp. 343–345),

$$V = W^{g_1} \oplus \ldots \oplus W^{g_n}. \tag{17}$$

Suppose that $B \not\subseteq I$. Then $I/Q \subseteq AQ/Q$; thus $I \subseteq QA$. Let $X = \sum_{\alpha \in A} W^{\alpha}$. By (17), $V = \oplus \sum_{\beta \in B} X^{\beta}$. Now, let ν be an arbitrary element of X. Then B fixes $\sum_{\beta \in B} \nu^{\beta}$. Hence A fixes $\sum_{\beta \in B} \nu^{\beta}$. Since A fixes X^{β} for each $\beta \in B$, A fixes ν^{β} for each β. Thus A centralizes X^{β} for each $\beta \in B$. But then, A centralizes V, which contradicts the hypothesis that $(Q, A) = Q \neq 1$ and that Q acts faithfully on V. Therefore, $B \subseteq I$.

Since W is a homogeneous Q-module, each element of $Z(Q)$ is represented on W by a scalar multiple of the identity transformation. Therefore, $(Z(Q), B)$ centralizes W. Since G normalizes $(Z(Q), B)$, $(Z(Q), B)$ is contained in $C_G(W^g)$ for every $g \in G$. By (17), $(Z(Q), B)$ centralizes V. Hence $(Z(Q), B) = 1$, $Z(Q) \subseteq C_Q(B) \subseteq C_Q(A)$. Since $Q = (Q, A)$, $Q \neq Z(Q)$. Thus, Q is a non-Abelian group and AB centralizes $Z(Q)$. Since $G = QAB$, $Z(Q) \subseteq Z(G)$.

Suppose that Q has nilpotence class $c \geqq 3$. Let $Q = Q_1, Q_2, \ldots,$ be the lower central series of Q. Since $(Q_{c-1}, Q) = Q_c \neq 1$, we have that $Q_{c-1} \not\subseteq Z(Q)$. Therefore $(Q_{c-1}, A) \neq 1$ by Lemma 7. However, Q_{c-1} is Abelian since $(Q_{c-1}, Q_{c-1}) \subseteq Q_{2c-2} \subseteq Q_{c+1} = 1$ (**10**, Corollary 10.3.5, p. 156). Let $R = (Q_{c-1}, A)$. By Lemma 6 (a), $R = (R, A)$. Since A and B normalize R, RAB and V satisfy the hypothesis of Lemma 10. By the induction hypothesis, R is not Abelian. This is impossible, since R is contained in the Abelian group Q_{c-1}. Thus Q has nilpotence class two.

Take W as above, and let $K = C_Q(W)$. Since G centralizes $Z(Q)$, $K \cap Z(Q) \subseteq C_Q(W^g)$ for every $g \in G$. By (17), $K \cap Z(Q)$ centralizes V. Hence $K \cap Z(Q) = 1$. Since

$$(K, Q) \subseteq K \cap Q' \subseteq K \cap Z(Q) = 1,$$

we have that $K \subseteq Z(Q)$. Thus $K = 1$. Consequently, Q is represented faithfully on W. Let χ be the character of Q on an irreducible constituent of W with respect to Q. By Lemma 9, $\chi(x) = 0$ whenever $x \in Q$ and $x \notin Z(Q)$. Moreover, $Z(Q) \subseteq Z(G)$. Hence, $\chi(g^{-1}xg) = \chi(x)$ for all $x \in Q$ and $g \in G$. By Clifford's Theorem, Q is homogeneous on V, that is, $V = W$. Let us regard AB as a relatively prime operator group on Q; then Q is irreducible on V, by Theorem 2 (c).

Since Q is faithful on V, q does not divide $|C_G(V)|$. Since $C_Q(B) \subseteq C_Q(A) \neq Q$, no conjugate of B is contained in $C_G(V)$. Therefore, $C_G(V)$ is an r-group. Thus, we may assume, henceforth, that G acts faithfully on V.

Let $\bar{Q} = Q/Q'$. By Lemma 6 (b), A has no non-identity fixed points on $\bar{Q}$. But A centralizes $Z(Q)$, and therefore $Z(Q) \subseteq Q'$. Since $Q' \subseteq Z(Q)$, we have that $Q' = Z(Q)$. Let q^n be the exponent of $\bar{Q}$. From (**10**, p. 150), we have that $(x, y)^r = (x^r, y) = (x, y^r)$ for all $x, y \in Q$ and all positive integers r. Thus, for arbitrary $x, y \in Q$,

$$(x, y)^{q^n} = (x^{q^n}, y) \in (Z(Q), Q) = 1.$$

Thus Q' has exponent at most q^n.

We claim that $n = 1$. Suppose that $n \geqq 2$. Let $k = q^{n-1}$; then $k^2 \geqq q^n$. For arbitrary $x, y \in Q$,

$$(x^k, y^k) = (x^k, y)^k = (x^{k^2}, y) \in (Z(Q), Q) = 1.$$

Thus, the elements x^k, $x \in Q$, generate an Abelian characteristic subgroup R of Q. Since $R \not\subseteq Z(Q)$, $(R, A) \neq 1$. As in our proof that the nilpotence class of Q is at most two, we may obtain the contradiction that R is not Abelian. Thus $n = 1$. Therefore, $D(Q) = Q' = Z(Q)$ and $Z(Q)$ has exponent q.

Now, $C_Q(A) = C_Q(B) = C_Q(AB) = Z(Q)$. Suppose that $\gamma \in AB$ and $C_Q(\gamma) \neq Z(Q)$. Take $\alpha \in A$ and $\beta \in B$ such that $\gamma = \alpha\beta$. Since α and β have relatively prime orders, α and β are both powers of γ. Therefore, $C_Q(\gamma) = C_Q(\alpha) \cap C_Q(\beta)$. Consequently, $\beta = 1$ and $\gamma = \alpha \in A$. Let C be the group generated by α, and let $R = (Q, C)$. Then C is a proper subgroup of A.

Suppose that $R \neq 1$. Since $Q' \subseteq C_Q(C)$ and $C_Q(C)/Q' \neq 1$, we have that $RQ'/Q' \neq Q/Q'$ by Lemma 6 (b). Hence, R is a proper subgroup of Q. Obviously, A and B normalize R. By Lemma 6 (b),

$$R \supseteq (R, A) \supseteq (R, C) = ((Q, C), C) = (Q, C) = R.$$

By the induction hypothesis applied to RAB and V, A is a 2-group and every proper subgroup of A centralizes R. Thus $1 = (R, C) = R$. This contradiction shows that $R = 1$, that is, C centralizes Q.

Now let $A_0 = C_A(Q)$. By the above paragraph, if $\gamma \in AB$ and $\gamma \notin A_0$, then $C_Q(\gamma) = C_Q(AB) = Z(Q)$. Obviously, every irreducible character of $Z(Q)$ has degree one. Let η be the character of QAB on V, and let $\zeta = \eta|_Q$. Since Q is faithful and non-Abelian, $\zeta(1) > 1$. By Theorem 2 (c) and Corollary 5 (b), there exists $\epsilon_1 = \pm 1$ such that $\zeta(1) - \epsilon_1$ is divisible by $|AB/A_0|$. Therefore,

$$|AB/A_0| \leqq \zeta(1) - \epsilon_1. \tag{18}$$

Suppose that $A_0 \neq 1$. Since $A_0 \subseteq Z(G)$ and G is faithful on V, $C_V(A_0) \neq V$ and $C_V(A_0)$ is fixed by G. Consequently, $C_V(A_0) = 1$. Therefore, $C_V(B) \subseteq C_V(A) \subseteq C_V(A_0) = 1$. Applying Corollary 6 (b) to QB, we obtain a sign $\epsilon_2 = \pm 1$ and an irreducible character θ_2 of B such that

$$\eta|_B = \epsilon_2\theta_2 + |B|^{-1}(\zeta(1) - \epsilon_2)\rho_B.$$

Since $C_V(B) = 0$, $(\eta|_B, 1_B) = 0$. Thus,

$$0 = (\eta|_B, 1_B) = \epsilon_2(\theta_2, 1_B) + (1/s)(\zeta(1) - \epsilon_2).$$

Since $\zeta(1) > 1$,

$$\epsilon_2 = -1, \quad \theta_2 = 1_B, \quad \text{and} \quad \zeta(1) = s + 1.$$

By (18), $s + 1 - \epsilon_1 \geqq |AB/A_0| \geqq r|B| = rs$. Therefore, $(r - 1)s \leqq 2$. This is impossible, since r and s are distinct primes. Thus $A_0 = 1$.

We have showed that whenever $\gamma \in AB$ and $\gamma \neq 1$, $C_Q(\gamma) = C_Q(AB) = Z(Q)$. By Corollary 6 (b), there exist $\epsilon = \pm 1$ and an irreducible character θ of AB with the properties that

$$|AB| \text{ divides } \zeta(1) - \epsilon \quad \text{and} \quad \eta|_{AB} = \epsilon\theta + |AB|^{-1}(\zeta(1) - \epsilon)\rho_{AB}.$$

Let $w = |AB|^{-1}(\zeta(1) - \epsilon)$. By hypothesis, $C_V(B) \subseteq C_V(A)$. Consequently, $C_V(B) = C_V(AB)$ and $(\eta|_{AB}, 1_{AB}) = (\eta|_B, 1_B)$. Thus

$$\epsilon(\theta, 1_{AB}) + w = \epsilon(\theta|_B, 1_B) + w|A|;$$

Thus $(\theta, 1_{AB}) \neq (\theta|_B, 1_B)$. Since θ has degree one, $\theta|_B = 1_B$ and $\theta \neq 1_{AB}$. Therefore, $\epsilon = -1$, $w = 1$, and $|A| = 2$. Hence $\zeta(1) = |AB|w + \epsilon = 2s - 1$. Since ζ is an irreducible complex character of Q, $\zeta(1)$ divides $|Q|$. Hence $2s - 1$ is a power of q. If $\alpha \in A$, then $\alpha^2 = 1$; thus, α^2 centralizes Q.

To complete the proof of Lemma 10, we need only verify that Q has exponent q. Since $r = 2$ and $r \neq q$, $q \geqq 3$. Let $x, y \in Q$. By an induction argument we may verify that

$$(xy)^i = x^i y^i (x, y)^{i(i-1)/2}, \qquad i = 1, 2, 3, \ldots .$$

In particular, $(xy)^q = x^q y^q$. Let α be a generator of A and let $z = \alpha^{-1}x\alpha$. Then $(x^{-1}z)^q = (x^{-1})^q z^q = (x^q)^{-1}(\alpha^{-1}x^q\alpha) = (x^q)^{-1}x^q = 1$. Thus, (Q, A) is contained in the kernel of the homomorphism of Q given by $g \to g^q$. Since $(Q, A) = Q$, Q has exponent q.

LEMMA 11. *Let G be a finite solvable group and let q be a prime. Suppose that G has no normal q-subgroup except the identity subgroup. Let H be the largest normal subgroup of G of order relatively prime to q. Then $C_G(H) \subseteq H$.*

Proof. This is a special case of (**11**, Lemma 1.2.3).

Proof of Theorem 6. Let us assume the hypothesis and notation of Theorem 6, as stated at the beginning of this section. Clearly, $F(G) \cap H \subseteq F(H)$. Since H is a normal subgroup of G and $F(H)$ is a characteristic nilpotent subgroup of H, we have that $F(H) \subseteq F(G)$. Thus $F(H) = F(G) \cap H$. By considering the natural mapping of G onto $G/F(G)$, we see that $F_2(H) = F_2(G) \cap H$. By Lemma 6(a), $H = (H, A)$. Hence, we may assume that $G = H$, that is, that $G = (G, A)$. We may also assume that G is not nilpotent.

Let q be a prime for which G has no normal Sylow subgroup. Let Q_0 be the largest normal Sylow q-subgroup of G and take K such that K/Q_0 is the largest normal subgroup of G/Q_0 of order relatively prime to q. By Lemma 6 (c), AB normalizes some Sylow q-subgroup Q_1 of G. Let $\bar{G} = G/Q_0$, $\bar{K} = K/Q_0$, and $\bar{Q}_1 = Q_1/Q_0$. Then $\bar{Q}_1 \neq 1$. Consider AB as an operator group on $\bar{G}$. Then

$(\bar{G}, A) = \bar{G}$. Since q divides $|\bar{G}|$, A does not centralize $\bar{Q}_1$, by Lemma 8. Let $Q = (Q_1, A)$ and $\bar{Q} = QQ_0/Q_0$. Then $\bar{Q} = (\bar{Q}, A)$.

Consider the semi-direct product $\bar{Q}AB$ as an operator group on $\bar{K}$. By Lemma 11, $\bar{Q}$ does not centralize $\bar{K}$. Let $\bar{L} = (\bar{K}, \bar{Q})$. Then $\bar{L} \neq 1$ and, by Lemma 6 (a), $\bar{L} = (\bar{L}, \bar{Q})$. Furthermore, $\bar{Q}AB$ fixes $\bar{L}$. Let $\bar{M}$ be a proper normal subgroup of $\bar{L}$ which is maximal, subject to the condition of being fixed by $\bar{Q}AB$. Since $\bar{L}$ is solvable, there exists a prime p such that $\bar{L}/\bar{M}$ is an elementary Abelian p-group. Let $V = \bar{L}/\bar{M}$ and $R = \bar{Q}/C_{\bar{Q}}(V)$. Since $\bar{L} = (\bar{L}, \bar{Q})$, $V = (V, Q)$. Hence $R \neq 1$. By Lemma 3 (c),

$$C_{\bar{Q}}(B) = C_Q(B)Q_0/Q_0 \subseteq C_Q(A)Q_0/Q_0 = C_{\bar{Q}}(A),$$

and, similarly, $C_R(B) \subseteq C_R(A)$ and $C_V(B) \subseteq C_V(A)$. Let F be the field of p elements. Since R acts faithfully on V, the hypothesis of Lemma 10 is satisfied. Hence $r = 2$, and $2s - 1$ is a power of q. Since $2s - 1$ can be a power of only one prime, q is unique. Hence, for every prime distinct from q, the Sylow subgroups of G are normal.

Thus $G/F(G)$ is a q-group. Furthermore, Q_0 is a Sylow q-subgroup of $F(G)$. Since $G/F(G)$ is a q-group, $F(G) = K$. Then $\bar{Q}_1$ is isomorphic to $G/F(G)$. Since $G = (G, A)$, $\bar{Q}_1 = (\bar{Q}_1, A) = \bar{Q}$. Let

$$\bar{K} = \bar{K}_0 \supset \bar{K}_1 \supset \ldots \supset \bar{K}_n = 1$$

be a composition series of $\bar{K}$ with respect to $\bar{Q}AB$. Let $g \in Q$ and $\alpha \in A$, and let $S = ((\bar{Q}, \bar{Q}), \bar{Q})$. For $i = 1, 2, \ldots, n$, either $\bar{Q}$ centralizes $\bar{K}_{i-1}/\bar{K}_i$ or Lemma 10 applies. In both cases, $(AB, Z(\bar{Q}))$ and S, and g^q and (g, α^2) all centralize $\bar{K}_{i-1}/\bar{K}_i$. By Lemma 3 (c) and an easy induction argument, $(AB, Z(\bar{Q})) = S = 1$ and $g^q = (g, \alpha^2) = 1$. Hence $D(\bar{Q}) = \bar{Q}' \subseteq Z(\bar{Q})$. Since $\bar{Q} = (\bar{Q}, A)$, $Z(\bar{Q}) = \bar{Q}'$, by Lemma 6 (b). Hence $\bar{Q}$ is a non-Abelian special q-group. This completes the proof of Theorem 6.

References

1. W. Burnside, *Theory of groups of finite order*, 2nd ed. (Dover, New York, 1955).
2. C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras* (Wiley, New York, 1962).
3. J. D. Dixon, *The fitting subgroup of a linear solvable group*, J. Austral. Math. Soc. *7* (1967), 417–428.
4. ——— *Normal p-subgroups of solvable linear groups*, J. Austral. Math. Soc. *7* (1967), 545–551.
5. W. Feit, *Characters of finite groups* (Mimeographed notes, Yale University, 1965).
6. W. Feit and J. G. Thompson, *Solvability of groups of odd order*, Pacific J. Math. *13* (1963), 775–1029.
7. P. X. Gallagher, *Group characters and normal Hall subgroups*, Nagoya Math. J. *21* (1962), 223–230.
8. G. Glauberman, *Correspondence of characters in relatively prime automorphism groups*, Notices Amer. Math. Soc. *11* (1964), 128–129.
9. ——— *Fixed points in groups with operator groups*, Math. Z. *84* (1964), 120–125.
10. M. Hall, Jr., *The theory of groups* (Macmillan, New York, 1959).

11. P. Hall and G. Higman, *On the p-length of p-solvable groups and reduction theorems for Burnside's problem*, Proc. London Math. Soc. (3) *6* (1956), 1–42.
12. I. Schur, *Über eine Klasse von endlichen Gruppen linearer Substitutionen*, S.-B. Preuss. Akad. Wiss. (1905), 77–91.
13. A. Speiser, *Die Theorie der Gruppen von endlicher Ordnung*, 4th ed. (Birkhäuser, Basel, Stuttgart, 1956).
14. J. G. Thompson, *Automorphisms of solvable groups*, J. Algebra *1* (1964), 259–267.
15. B. L. van der Waerden, *Modern algebra*, Volume I (Ungar, New York, 1953).
16. H. Zassenhaus, *The theory of groups*, 2nd Engl. ed. (Chelsea, New York, 1958).

University of Chicago,
Chicago, Illinois