UNIFORM O-ESTIMATES OF CERTAIN ERROR FUNCTIONS CONNECTED WITH k-FREE INTEGERS

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1. Introduction and notation

Let k be a fixed integer $\geqq 2$. A positive integer m is called k-free if m is not divisible by the k'th power of any integer > 1. Let $q_k(m)$ be the characteristic function of the set of k-free integers; that is, $q_k(m) = 1$ or 0 according as m is k-free or not. It can be easily shown that $q_k(m) = \sum_{d^k\delta=m} \mu(d)$, where $\mu(n)$ is the Möbius function. Let $x \geqq 1$ denote a real variable and n be a positive integer. Let $Q_k(x, n)$ and $Q'_k(x, n)$ be the number and the sum of the reciprocals of the k-free integers $\leqq x$ which are prime to n respectively.

Let $\sigma_t^*(n)$ be the sum of the t'th powers of the squarefree divisors of n and $\psi_k(n)$ be the arithmetical function defined by

$$\psi_k(n) = n \prod_{p|n} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{k-1}}\right), \tag{1.1}$$

the product being extended over all prime divisors p of n. It is clear that

$$\psi_k(n) = \frac{J_k(n)}{n^{k-2}\varphi(n)}, \tag{1.2}$$

where $\varphi(n)$ is the Euler totient function and $J_k(n)$ is the Jordan totient function (cf. [4], p. 147) which have the following arithmetical forms:

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right), \qquad J_k(n) = n^k \prod_{p|n} \left(1 - \frac{1}{p^k}\right). \tag{1.3}$$

It has been stated by R. L. Robinson ([6], lemma 2) that

$$Q_k(x, n) \equiv \sum_{\substack{m \leqq x \\ (m,n)=1}} q_k(m) = \frac{nx}{\zeta(k)\psi_k(n)} + O(\theta(n)x^{1/k}), \tag{1.4}$$

the O-estimate being uniform in n and x; where $\theta(n) = \sigma_0^*(n)$, the number of squarefree divisors of n and $\zeta(k)$ is the Riemann Zeta function. In case $k = 2$, the result (1.4) has already been established by E. Cohen (cf. [2], lemma 5.2).

The object of this paper is to improve the error term in (1.4) to $O(\sigma^*_{-s}(n)x^{1/k})$, where s is any number with $o \leqq s < 1/k$ and to establish an asymptotic formula for $Q'_k(x, n)$ with a corresponding uniform O-estimate (See Theorems 1 and 2 below).

2. Preliminaries

In this section we mention some of the known results which are needed in our discussion and prove some lemmas. Throughout the following s denotes a non-negative real number. The following elementary estimates are well-known:

$$\sum_{n \leqq x} \frac{1}{n^s} = O(x^{1-s}) \qquad \text{if } o \leqq s < 1. \tag{2.1}$$

$$\sum_{n > x} \frac{1}{n^s} = O\left(\frac{1}{x^{s-1}}\right) \qquad \text{if } s > 1. \tag{2.2}$$

Let $\varphi(x, n)$ denote the number of integers $\leqq x$ which are prime to n. Then we have

Lemma 1. (cf. [3], (4)). *For each s, with $o \leqq s < 1$,*

$$\varphi(x, n) = \frac{x\varphi(n)}{n} + O(x^s\sigma^*_{-s}(n)), \tag{2.3}$$

uniformly.

Lemma 2. (cf. [8], lemma 2.1).

$$\sum_{\substack{m \leqq x \\ (m, n) = 1}} \frac{1}{m} = \frac{\varphi(n)}{n}(\log x + \gamma + \alpha(n)) + O\left(\frac{\theta(n)}{x}\right), \tag{2.4}$$

uniformly, where $\alpha(n)$ is given (cf. [1]) by the following:

$$\alpha(n) \equiv -\frac{n}{\varphi(n)} \sum_{d \mid n} \frac{\mu(d) \log d}{d} = \sum_{p \mid n} \frac{\log p}{p-1} \tag{2.5}$$

and γ is Euler's constant.

Lemma 3.

$$\alpha_k(n) \equiv -\frac{n^k}{J_k(n)} \sum_{d \mid n} \frac{\mu(d) \log d}{d^k} = \sum_{p \mid n} \frac{\log p}{p^k - 1}. \tag{2.6}$$

Proof. This can be proved by the same method adopted in [1] for proving (2.5).

LEMMA 4. (cf. [8], lemma 2.3). *For* $s > 1$,

$$\sum_{\substack{m=1 \\ (m,n)=1}}^{\infty} \frac{\mu(m)}{m^s} = \frac{n^s}{\zeta(s)J(s,n)}, \tag{2.7}$$

where $J(s, n)$ *is defined for all* $s > 1$ *by*

$$J(s,n) = n^s \prod_{p|n} \left(1 - \frac{1}{p^s}\right). \tag{2.8}$$

In particular, for $s = k$ *by* (1.3),

$$J(k,n) = J_k(n). \tag{2.9}$$

LEMMA 5. (cf. [8], lemma 2.5). *For* $s > 1$,

$$\sum_{\substack{m=1 \\ (m,n)=1}}^{\infty} \frac{\mu(m)\log m}{m^s} = \frac{n^s}{\zeta(s)J(s,n)} \left\{\alpha(s,n) + \frac{\zeta'(s)}{\zeta(s)}\right\}, \tag{2.10}$$

where $\zeta'(s)$ *is the derivative of* $\zeta(s)$, *and*

$$\alpha(s,n) = \sum_{p|n} \frac{\log p}{p^s - 1}. \tag{2.11}$$

In particular, for $s = k$ *by* (2.6),

$$\alpha(k,n) = \alpha_k(n). \tag{2.12}$$

LEMMA 6. *For any arbitrary* q *and* $x \geqq 2$,

$$M_n(x) \equiv \sum_{\substack{m \leqq x \\ (m,n)=1}} \mu(m) = O\left(\frac{\theta(n)x}{\log^q x}\right), \tag{2.13}$$

uniformly.

PROOF. It is known (cf. [5], p. 594) that

$$M_1(x) = \sum_{m \leqq x} \mu(m) = O\left(\frac{x}{\log^q x}\right) \quad \text{for any arbitrary } q.$$

Since $x/\log^q x$ is monotonically increasing, we have for any given $t \geqq 1$,

$$M_1\left(\frac{x}{t}\right) = O\left(\frac{x}{\log^q x}\right). \tag{2.14}$$

We have

$$\begin{aligned} M_n(x) &= \sum_{d|n} \sum_{jd \leqq x} \mu(d)\mu(jd) = \sum_{d|n} \mu(d) \sum_{\substack{jd \leqq x \\ (j,d)=1}} \mu(jd) \\ &= \sum_{d|n} \mu^2(d) \sum_{\substack{j \leqq x/d \\ (j,d)=1}} \mu(j), \end{aligned}$$

so that

$$M_n(x) = \sum_{d|n} \mu^2(d) M_d\left(\frac{x}{d}\right) \tag{2.15}$$

Now, if p is a prime and $(p, n) = 1$, then

$$\begin{aligned} M_{p_n}(x) &= M_n(x) + M_{p_n}\left(\frac{x}{p}\right) \\ &= \sum_{i=0}^{c} M_n\left(\frac{x}{p^i}\right), \qquad \text{where } c = \left[\frac{\log x}{\log p}\right] \end{aligned} \tag{2.16}$$

In particular, taking $n = 1$ in (2.16),

$$\begin{aligned} M_p(x) &= \sum_{i=0}^{c} M_1\left(\frac{x}{p^i}\right) = O\left(\frac{cx}{\log^q x}\right), \qquad \text{by (2.14)} \\ &= O\left(\frac{x}{\log^q x}\right), \qquad \text{since } q \text{ is arbitrary.} \end{aligned} \tag{2.17}$$

Again, if p_1 and p_2 are primes, then by (2.16), taking $p = p_1$ and $n = p_2$,

$$\begin{aligned} M_{p_1p_2}(x) &= \sum_{i=0}^{c_1} M_{p_2}\left(\frac{x}{p_1^i}\right), \qquad \text{where } c_1 = \left[\frac{\log x}{\log p_1}\right] \\ &= O\left(\frac{c_1 x}{\log^q x}\right), \qquad \text{by (2.17)} \\ &= O\left(\frac{x}{\log^q x}\right), \qquad \text{since } q \text{ is arbitrary.} \end{aligned}$$

Similarly, if $p_1, p_2, \cdots p_r$ are distinct primes, then for any given $t \geqq 1$,

$$M_{p_1p_2\cdots p_r}\left(\frac{x}{t}\right) = O\left(\frac{x}{\log^q x}\right).$$

Hence for any square-free divisor d of n,

$$M_d\left(\frac{x}{d}\right) = O\left(\frac{x}{\log^q x}\right),$$

so that the lemma follows by (2.15).

Lemma 7. *For any arbitrary q, $x \geqq 2$ and $s > 1$,*

$$\sum_{\substack{m > x \\ (m,n)=1}} \frac{\mu(m)}{m^s} = O\left(\frac{\theta(n)}{x^{s-1}\log^q x}\right) \tag{2.18}$$

$$\sum_{\substack{m>x \\ (m,n)=1}} \frac{\mu(m)\log m}{m^s} = O\left(\frac{\theta(n)}{x^{s-1}\log^q x}\right), \tag{2.19}$$

uniformly.

PROOF. Let $\varepsilon(n) = 1$ or 0 according as $n = 1$ or $n > 1$, so that $M_n(x)$ in (2.13) turns out to be $\sum_{m \leqq x} \mu(m)\varepsilon((m, n))$. Putting $f(m) = 1/m^s$ and $g(m) = \log m/m^s$, it has been shown by the author (cf. [7], lemmas 3.1 and 3.2) that

$$f(m+1)-f(m) = O\left(\frac{1}{m^{s+1}}\right) \quad \text{and} \quad g(m+1)-g(m) = O\left(\frac{\log m}{m^{s+1}}\right).$$

We give the proof of (2.19) only, since (2.18) can be proved more easily following the same line of arguement.

By partial summation and (2.13),

$$\begin{aligned}\sum_{m>x} \mu(m)\varepsilon((m,n))g(m) &= -M_n(x)g([x]+1) \\ &\quad - \sum_{m>x} M_n(m)[g(m+1)-g(m)] \\ &= O\left(\frac{\theta(n)}{x^{s-1}\log^q x}\right) + O\left(\sum_{m>x} \frac{\theta(n)}{m^s \log^q m}\right),\end{aligned}$$

since q is arbitrary.

The second O-term is $O(\theta(n)/\log^q x)\sum_{m>x} 1/m^s$ which is $O(\theta(n)/x^{s-1}\log^q x)$, by (2.2).

Hence the lemma follows.

LEMMA 8. For any arbitrary q, $x \geqq 2$ and $s > 1$,

$$\sum_{\substack{m \leqq x \\ (m,n)=1}} \frac{\mu(m)}{m^s} = \frac{n^s}{\zeta(s)J(s,n)} + O\left(\frac{\theta(n)}{x^{s-1}\log^q x}\right) \tag{2.20}$$

$$\sum_{\substack{m \leqq x \\ (m,n)=1}} \frac{\mu(m)\log m}{m^s} = \frac{n^s}{\zeta(s)J(s,n)}\left\{\alpha(s,n) + \frac{\zeta'(s)}{\zeta(s)}\right\} + O\left(\frac{\theta(n)}{x^{s-1}\log^q x}\right), \tag{2.21}$$

uniformly.

PROOF. (2.20) follows by (2.7) and (2.18). (2.21) follows by (2.10) and (2.19).

3. Main results

We are now in a position to prove the following:

THEOREM 1. *For* $0 \leqq s < 1/k$,

$$Q_k(x, n) \equiv \sum_{\substack{m \leqq x \\ (m, n) = 1}} q_k(m) = \frac{nx}{\zeta(k)\psi_k(n)} + O(\sigma^*_{-s}(n)x^{1/k}), \tag{3.1}$$

uniformly.

PROOF. We have $q_k(m) = \sum_{d^k \delta = m} \mu(d)$. Hence

$$Q_k(x, n) = \sum_{\substack{m \leqq x \\ (m, n) = 1}} \sum_{d^k \delta = m} \mu(d) = \sum_{\substack{d^k \delta \leqq x \\ (d, n) = (\delta, n) = 1}} \mu(d)$$

$$= \sum_{\substack{d \leqq \sqrt[k]{x} \\ (d, n) = 1}} \mu(d) \sum_{\substack{\delta \leqq x/d^k \\ (\delta, d) = 1}} 1 = \sum_{\substack{d \leqq \sqrt[k]{x} \\ (d, n) = 1}} \mu(d) \varphi'\left(\frac{x}{d^k}, n\right).$$

By lemma 1,

$$Q_k(x, n) = \sum_{\substack{d \leqq \sqrt[k]{x} \\ (d, n) = 1}} \mu(d) \left\{ \frac{x}{d^k} \frac{\varphi(n)}{n} + O\left(\frac{x^s}{d^{sk}} \sigma^*_{-s}(n)\right) \right\}$$

$$= \frac{x\varphi(n)}{n} \sum_{\substack{d = 1 \\ (d, n) = 1}}^{\infty} \frac{\mu(d)}{d^k} + O(x \sum_{d > \sqrt[k]{x}} d^{-k})$$

$$+ O(x^s \sigma^*_{-s}(n) \sum_{d \leqq \sqrt[k]{x}} d^{-sk}).$$

The first O-term is $O(x^{1/k})$ by (2.2) and the second O-term is $O(\sigma^*_{-s}(n)x^{1/k})$ by (2.1), restricting s to the range $0 \leqq s < 1/k$.

Hence Theorem 1 follows by (2.7), (2.9) and (1.2).

COROLLARY 1. ($k = 2$). *For* $0 \leqq s < \frac{1}{2}$, *we have*

$$Q(x, n) \equiv \sum_{\substack{m \leqq x \\ (m, n) = 1}} \mu^2(m) = \frac{nx}{\zeta(2)\psi(n)} + O(\sigma^*_{-s}(n)\sqrt{x}), \tag{3.2}$$

where $\psi(n)$ *is Dedekind's* ψ-*function defined by* $\psi(n) = \sum_{d\delta = n} \mu^2(d)\delta$.

THEOREM 2. *For* $0 \leqq s < 1/k$,

$$Q'_k(x, n) \equiv \sum_{\substack{m \leqq x \\ (m, n) = 1}} \frac{q_k(m)}{m} = \frac{n}{\zeta(k)\psi_k(n)} \left(\log x + \gamma - \frac{k\zeta'(k)}{\zeta(k)} + \alpha(n) - k\alpha_k(n) \right) + O\left(\frac{\sigma^*_{-s}(n)}{x^{1-1/k}}\right), \tag{3.3}$$

uniformly, where $\alpha(n)$ *is given by* (2.5) *and* $\alpha_k(n)$ *is given by* (2.6).

PROOF.

$$Q'_k(x, n) = \sum_{\substack{m \leqq x \\ (m, n) = 1}} \frac{1}{m} \sum_{d^k \delta = m} \mu(d) = \sum_{\substack{d^k \delta \leqq x \\ (d, n) = (\delta, n) = 1}} \frac{\mu(d)}{d^k \delta}$$

$$= \sum_{\substack{d \leqq \sqrt[k]{x} \\ (d, n) = 1}} \frac{\mu(d)}{d^k} \sum_{\substack{\delta \leqq x/d^k \\ (\delta, n) = 1}} \frac{1}{\delta},$$

so that by lemma 2,

$$Q'_k(x, n) = \sum_{\substack{d \leq \sqrt[k]{x} \\ (d, n) = 1}} \frac{\mu(d)}{d^k} \left\{ \frac{\varphi(n)}{n} \left(\log \frac{x}{d^k} + \gamma + \alpha(n) \right) + O\left(\frac{\theta(n) d^k}{x} \right) \right\}$$

$$= \frac{\varphi(n)}{n} (\log x + \gamma + \alpha(n)) \sum_{\substack{d \leq \sqrt[k]{x} \\ (d, n) = 1}} \frac{\mu(d)}{d^k}$$

$$- \frac{k\varphi(n)}{n} \sum_{\substack{d \leq \sqrt[k]{x} \\ (d, n) = 1}} \frac{\mu(d) \log d}{d^k} + O\left(\frac{\theta(n)}{x^{1-1/k}} \right).$$

By lemma 8, (2.9), (2.12) and (1.2), since q is arbitrary,

$$Q'_k(x, n) = \frac{n}{\zeta(k)\psi_k(n)} (\log x + \gamma + \alpha(n)) + O\left(\frac{\theta(n)}{x^{1-1/k} \log^q x} \right)$$

$$- \frac{kn}{\zeta(k)\psi_k(n)} \left(\alpha_k(n) + \frac{\zeta'(k)}{\zeta(k)} \right) + O\left(\frac{\theta(n)}{x^{1-1/k} \log^q x} \right)$$

$$+ O\left(\frac{\theta(n)}{x^{1-1/k}} \right).$$

Hence

$$Q'_k(x, n) = \frac{n}{\zeta(k)\psi_k(n)} \left(\log x + \gamma - \frac{k\zeta'(k)}{\zeta(k)} + \alpha(n) - k\alpha_k(n) \right) + O\left(\frac{\theta(n)}{x^{1-1/k}} \right). \tag{3.4}$$

Again,

$$Q'_k(x, n) = \sum_{m \leq x} \frac{q_k(m)\varepsilon((m, n))}{m}, \qquad \text{where } \varepsilon(1) = 1 \qquad \text{and } \varepsilon(n) = 0 \text{ if } n > 1.$$

By partial summation, we have

$$Q'_k(x, n) = \frac{Q_k(x, n)}{x} - \sum_{m \leq x-1} Q_k(m, n) \left\{ \frac{1}{m+1} - \frac{1}{m} \right\}$$

$$= \frac{Q_k(x, n)}{x} + \int_1^x \frac{Q_k(t, n)}{t^2} \, dt.$$

If

$$\Delta_k(x, n) = Q_k(x, n) - \frac{nx}{\zeta(k)\psi_k(n)},$$

then by Theorem 1,

$$\Delta_k(x, n) = O(\sigma^*_{-s}(n) x^{1/k}).$$

Hence

$$Q_k'(x,n) = \frac{n}{\zeta(k)\psi_k(n)} + \frac{\Delta_k(x,n)}{x} + \int_1^x \left\{\frac{n}{\zeta(k)\psi_k(n)t} + \frac{\Delta_k(t,n)}{t^2}\right\} dt$$

$$= \frac{n}{\zeta(k)\psi_k(n)} + \frac{\Delta_k(x,n)}{x} + \frac{n \log x}{\zeta(k)\psi_k(n)} + \int_1^\infty \frac{\Delta_k(t,n)}{t^2}\, dt$$

$$- \int_x^\infty \frac{\Delta_k(t,n)}{t^2}\, dt$$

$$= \frac{n}{\zeta(k)\psi_k(n)}(\log x + c_k(n)) + O\left(\frac{\sigma_{-s}^*(n)}{x^{1-1/k}}\right), \tag{3.5}$$

where $c_k(n)$ is independent of x.

Now, keeping n fixed and taking the limit as $x \to \infty$ of the difference between (3.4) and (3.5) we get that

$$c_k(n) = \gamma - \frac{k\zeta'(k)}{\zeta(k)} + \alpha(n) - k\alpha_k(n).$$

Substituting this value of $c_k(n)$ in (3.5), we get Theorem 2.

COROLLARY 2. ($k = 2$). *For* $0 \leqq s < \frac{1}{2}$, *we have*

$$Q'(x,n) \equiv \sum_{m \leqq x} \frac{\mu^2(m)}{m} = \frac{n}{\zeta(2)\psi(n)}\left(\log x + \gamma - \frac{2\zeta'(2)}{\zeta(2)} + \alpha(n) - 2\beta(n)\right)$$

$$+ O\left(\frac{\sigma_{-s}^*(n)}{\sqrt{x}}\right), \tag{3.6}$$

where $\alpha(n)$ *is given by* (2.5) *and*

$$\beta(n) = -\frac{n^2}{J(n)} \sum_{d|n} \frac{\mu(d)\log d}{d^2},$$

$J(n)$ *being the Jordan totient function of order* 2.

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