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ON THE MEASURE OF THE ONE-SKELETON OF THE SUM OF CONVEX COMPACT SETS

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Abstract

For any two compact convex sets in a Euclidean space, the relation between the volume of the sum of the two sets and the volume of each of them is given by the Brünn-Minkowski inequality. In this note we prove an analogous relation for the one-dimensional Hausdorff measure of the one-skeleton of the above sets. Also, some counterexamples are given which show that the above results are the best possible in some special cases.

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1. Introduction

When K is a convex compact subset of a Euclidean space E^d, then for $\nu = 0, 1, \ldots, d$, the *ν-skeleton* $\operatorname{skel}_\nu K$ of K consists of those points of K which are not centres of $(\nu + 1)$-dimensional balls contained in K.

It is well known (see Larman and Rogers [4]) that the ν-skeleton of a compact convex set in E^d is a measurable set with respect to the ν-dimensional Hausdorff measure, denoted by $\mathscr{H}^\nu(\cdot)$. We define $n_\nu(K) = \mathscr{H}^\nu(\operatorname{skel}_\nu K)$. If K_0 and K_1 are compact convex subsets of E^d, then it is known that the dth root of $n_d(\cdot)$ is a concave function, i.e. for any $0 \leqslant t \leqslant 1$, if $K_t = (1 - t)K_0 + tK_1$, then

$$\left(n_d(K_t)\right)^{1/d} \geqslant (1 - t)\left(n_d(K_0)\right)^{1/d} + t\left(n_d(K_1)\right)^{1/d}$$

for any K_0, K_1. This inequality is known as the Brünn-Minkowski inequality.

In this note we prove that $n_1(\cdot)$ is a concave function, i.e.

$$n_1(K_t) \geqslant (1-t)n_1(K_0) + tn_1(K_1)$$

for any compact convex sets K_0, K_1 of E^d.

In the course of the proof of the above property, we establish an inequality between $\mathscr{H}^s(\operatorname{ext} K_t)$ and $\mathscr{H}^s(\operatorname{ext} K_i)$, $i = 0, 1$, for any $s \geqslant 0$, where $\operatorname{ext} K$ denotes the set of extreme points of K.

We also prove, constructing appropriate counterexamples in E^3, that the two inequalities cannot be reversed.

2. The results

We quote first a lemma which is to be used in the proofs that follow.

LEMMA 2.1. *Let K_1, K_2 be convex, compact subsets of E^d and let $K = \lambda K_1 + \mu K_2$, $\lambda, \mu \geqslant 0$, be a Minkowski linear combination of K_1 and K_2, where λ, μ are fixed but arbitrary. Then the following hold.*

(i) *For any point e belonging to the set* $\operatorname{ext} K$ *of the extreme points of K, there exist uniquely defined points e_1, e_2, where $e_i \in \operatorname{ext} K_i$, $i = 1, 2$, such that $e = \lambda e_1 + \mu e_2$.*

(ii) *For any point $e_1 \in \operatorname{ext} K_1$, there exists a point $e_2 \in \operatorname{ext} K_2$ such that $(\lambda e_1 + \mu e_2) \in \operatorname{ext} K$. A similar property holds for the extreme points of K_2.*

PROOF. If either $\lambda = 0$ or $\mu = 0$, the results are obvious. Suppose now that λ, $\mu \neq 0$. We consider part (i). Let $e \in \operatorname{ext} K$, with $e = \lambda e_1 + \mu e_2$ for some $e_i \in K_i$, $i = 1, 2$. If $e_1 \notin \operatorname{ext} K_1$, then $e_1 = (x_1 + y_1)/2$ for some $x_1, y_1 \in K_1$ with $x_1 \neq y_1$, and so $e = (\lambda x_1 + \mu e_2)/2 + (\lambda y_1 + \mu e_2)/2$. But $\lambda x_1 + \mu e_2$ and $\lambda y_1 + \mu e_2$ are distinct points of K, which is a contradiction, as $e \in \operatorname{ext} K$. Hence $e_1 \in \operatorname{ext} K_1$. In a similar way $e_2 \in \operatorname{ext} K_2$.

Suppose now that there exist another pair e_1', e_2', where $e_i' \in \operatorname{ext} K_i$, $= 1, 2$, such that $e = \lambda e_1 + \mu e_2 = \lambda e_1' + \mu e_2'$. Then $e = \lambda(e_1 + e_1')/2 + \mu(e_2 + e_2')/2$, which implies that $(e_i + e_i')/2 \in \operatorname{ext} K_i$, $i = 1, 2$. This, in turn, implies that $e_i = e_i'$, $i = 1, 2$, which proves part (i).

For part (ii), consider $e_1 \in \operatorname{ext} K_1$. If u is a unit vector, let $K_u^{(1)}$ denote the intersection of K_1 with its support hyperplane with outer normal u. If $U = (u_1, \ldots, u_k)$ is a k-frame of orthogonal unit vectors, then $K_U^{(1)}$ is defined recursively by $K^{(1)}_{(u_1, \ldots, u_k)} = (K_{(u_1, \ldots, u_{k-1})})_{u_k}$.

Now for the point e_1 there exists a k-frame $U = (u_1, \ldots, u_k)$, $1 \leqslant k \leqslant d$, such that $\{e_1\} = K_U^{(1)}$. If $K_U^{(2)}$, K_U are the corresponding sets for K_2 and K, then $K_U = \lambda K_U^{(1)} + \mu K_U^{(2)}$ (see Eggleston [3, Theorem 38]). Hence $K_U = \lambda\{e_1\} + \mu K_U^{(2)}$, and from part (i) we have

$$\operatorname{ext} K_U \subseteq \lambda\{e_1\} + \mu \operatorname{ext} K_U^{(2)}.$$

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Consider $z \in \operatorname{ext} K_U \subseteq \operatorname{ext} K$. Then $z = \lambda e_1 + \mu e_2$ for some $e_2 \in \operatorname{ext} K_U^{(2)} \subseteq \operatorname{ext} K_2$. Therefore, for $e_1 \in \operatorname{ext} K_1$, there exists $e_2 \in \operatorname{ext} K_2$ with $\lambda e_1 + \mu e_2 \in \operatorname{ext} K$. This concludes the proof of (ii).

Now we quote and prove the following propositions

PROPOSITION 2.1. *Let K_1, K_2 and K be defined as in Lemma 2.1. Then $\mathscr{H}^s(\operatorname{ext} K) \geqslant \max\{\lambda^s \mathscr{H}^s(\operatorname{ext} K_1), \mu^s \mathscr{H}^s(\operatorname{ext} K_2)\}$ for any non-negative number s.*

PROOF. As $\mathscr{H}^s(\operatorname{ext}(\lambda K_1)) = \lambda^s \mathscr{H}^s(\operatorname{ext} K_1)$, to prove the inequality, it is sufficient to prove it for $\lambda = \mu = 1$. If $e \in \operatorname{ext} K$, then the cap-neighbourhoods of e form a basis for the neighborhoods of e (see G. Choquet [2], page 107). Therefore $\mathscr{H}^s(\operatorname{ext} K) = \sup_{\varepsilon > 0} \inf\{\sum_{n=1}^{\infty} d^s(C_n) : C_n,\ n = 1, 2, \ldots,$ are caps; $\operatorname{ext} K \subseteq \bigcup_{n=1}^{\infty} C_n;\ d(C_n) < \varepsilon\}$.

Let C_n, $n = 1, 2, \ldots,$ be a sequence of caps of K covering $\operatorname{ext} K$, where $C_n = \{x \in K : a_n - t_n \leqslant x \cdot u_n \leqslant a_n\}$, where $a_n = \sup_{x \in K} x \cdot u_n$, and where $x \cdot u_n$ denotes the inner-product of x with a unit vector u_n. We define $C_n^{(i)} = \{x \in K_i : b_n^{(i)} - t_n \leqslant x \cdot u_n \leqslant b_n^{(i)}\}$, $i = 1, 2$, where $b_n^{(i)} = \sup_{x \in K_i} x \cdot u_n$, $i = 1, 2$. Then $a_n = b_n^{(1)} + b_n^{(2)}$. We shall prove that $\operatorname{ext} K_i \subseteq \bigcup_{n=1}^{\infty} C_n^{(i)}$, $i = 1, 2$. Let $e_1 \in \operatorname{ext} K_1$. Then by part (ii) of Lemma 2.1 there exists $e_2 \in \operatorname{ext} K_2$ such that $(e_1 + e_2) \in \operatorname{ext} K$. Let $e_1 + e_2 \in C_n$ for some $n \in N$. Then $e_i \in C_n^{(i)}$, $i = 1, 2$. For, if not, then $e_1 \notin C_n^{(1)}$, say. Then $e_1 \cdot u_n < b_n^{(1)} - t_n$, so $(e_1 + e_2) \cdot u_n < (b_n^{(1)} - t_n) + b_n^{(2)} = a_n - t_n$. This is impossible since $e_1 + e_2 \in C_n$. Hence, for any $e_1 \in \operatorname{ext} K_1$, there exists a cap $C_n^{(1)}$ such that $e_1 \in C_n^{(1)}$, and so $\operatorname{ext} K_1 \subseteq \bigcup_{n=1}^{\infty} C_n$. We also have $d(C_n^{(i)}) \leqslant d(C_n)$, $i = 1, 2$, $n \in \mathbb{N}$. Indeed, for $\beta_2 \in K_2$ with $\beta_2 \cdot u_n = b_n^{(2)}$, we have $C_n^{(1)} + \beta_2 \subseteq C_n$, and so $d(C_n^{(1)}) = d(C_n^{(1)} + \beta_2) \leqslant d(C_n)$. Then $\inf\{\sum_{n=1}^{\infty} d(S_n)$: $\operatorname{ext} K_i \subseteq \bigcup_{n=1}^{\infty} S_n$, $d(S_n) \leqslant \varepsilon\} \leqslant \inf\{\sum_{n=1}^{\infty} d^s(C_n)$: $\operatorname{ext} K \subseteq \bigcup_{n=1}^{\infty} C_n$, $d(C_n) < \varepsilon$, C_n cap, $n \in \mathbb{N}\}$ for any $\varepsilon > 0$. Therefore $\mathscr{H}^s(\operatorname{ext} K_i) \leqslant \mathscr{H}^s(\operatorname{ext} K)$, $i = 1, 2$. This concludes the proof of the proposition.

We note that in general no kind of reverse inequality holds.

More precisely, we show, by constructing a counterexample, that there does not exist a positive constant M such that the inequality

$$\mathscr{H}^1(\operatorname{ext} K) \leqslant M\big(\max\{\mathscr{H}^1(\operatorname{ext} K_1), \mathscr{H}^1(\operatorname{ext} K_2)\}\big)$$

holds for any compact convex sets K_1, K_2 in E^3. Indeed, take $K_1 = \{(x, 0, z) \in \mathbb{R}^3 : x \geqslant 0,\ z \geqslant 0,\ (x^2 + z^2)^{1/2} \leqslant 1\}$ and $K_2 = \{(0, y, z) \in \mathbb{R}^3 : y \geqslant 0,\ z \geqslant 0,\ (y^2 + z^2)^{1/2} \leqslant 1\}$. Then $\mathscr{H}^1(\operatorname{ext} K_1) = \mathscr{H}^1(\operatorname{ext} K_2) = \pi/2 < +\infty$. The sum of K_1 and K_2 is the set $K = \{(x, y, z) \in \mathbb{R}^3 : 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1,\ 0 \leqslant z \leqslant (1 - x^2)^{1/2} + (1 - y^2)^{1/2}\}$, and $\operatorname{ext} K = \{(x, y, z) \in \mathbb{R}^3 : 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1, z = (1 - x^2)^{1/2} + (1 - y^2)^{1/2}\} \cup \{(0,0,0)\} \cup \{(1,0,0)\} \cup \{(0,1,0)\}$. Therefore $\mathscr{H}^2(\operatorname{ext} K) > 0$. But then $\mathscr{H}^1(\operatorname{ext} K) = +\infty$, and in fact $\operatorname{ext} K$ is not σ-finite with respect to $\mathscr{H}^1$.

Proposition 2.2. *Let* K_1, K_2 *and* K *be as in Lemma* 2.1. *Then* $\mathscr{H}^1(\operatorname{skel}_1 K) \geqslant \lambda\mathscr{H}^1(\operatorname{skel}_1 K_1) + \mu\mathscr{H}^1(\operatorname{skel}_1 K_2)$.

Proof. As in Proposition 2.1, it is sufficient to prove the inequality for $\lambda = \mu = 1$. Then $K = K_1 + K_2$. If $\mathscr{H}^1(\operatorname{skel}_1 K) = +\infty$, we have nothing to prove.

Assume now that $\mathscr{H}^1(\operatorname{skel}_1 K) < \infty$. It is known, (see Burton [4, Theorems 1 and 3]), that $\operatorname{skel}_1 K$ is the union of $\operatorname{ext} K$ with countably many exposed edges F_n ($n = 1, 2, \dots$), and that $\mathscr{H}^1(\operatorname{ext} K) = 0$. Hence $\mathscr{H}^1(\operatorname{skel}_1 K) = \sum_{n=1}^{\infty}\mathscr{H}^1(F_n)$ and, by Proposition 2.1, $\mathscr{H}^1(\operatorname{ext} K_i) = 0$, $i = 1, 2$.

Now $F_n = K \cap H = K_1 \cap H_1 + K_2 \cap H_2$, where H is the support hyperplane of K at F_n, and where H_1, H_2 are the corresponding suport hyperplanes of K_1, K_2. As $\dim(F_n) = 1$, we conclude that $F_n = l_1 + l_2$, where l_1 and l_2 are parallel line segments which are edges of K_1 and K_2; or $F_n = l_1 + \{e_2\}$, where l_1 is an edge of K_1 and e_2 an exposed point of K_2; or $F_n = \{e_1\} + l_2$, where l_2 is an edge of K_2 and e_1 an exposed point of K_1. The above expression is uniquely determined. Suppose, for example, that $F_n = l_1 + l_2 = l_1' + \{e_2'\}$, where l_1, l_1' are edges of K_1, where l_2 is an edge of K_2, and where e_2' is an exposed point of K_2. Then $F_n = (l_1 + l_1')/2 + (l_2 + \{e_2'\})/2$, which implies that $\frac{1}{2}(l_1 + l_1')$ is an edge of K_1; but since $(l_1 + l_1')/2 \subset \operatorname{conv}(l_1, l_1')$, we have $l_1 = l_1'$. Therefore, $F_n = l_1 + l_2 = l_1 + \{e_2'\}$, and hence $\{e_2'\} = l_2$. Similar arguments apply to the other possible expressions for F_n.

Let l_1 be an edge of K_1. We denote by $\operatorname{pr}(\cdot)$ the projection onto E^{d-1} which maps in the direction of l_1. Then $\operatorname{pr}(K) = \operatorname{pr}(K_1) + \operatorname{pr}(K_2)$, and $\operatorname{pr}(l_1)$ is an extreme point of $\operatorname{pr}(K_1)$. Then, from Proposition 2.1, there exists an extreme point, say, e_2, of $\operatorname{pr}(K_2)$ such that $\operatorname{pr}(l_1) + e_2 = e$, where $e \in \operatorname{ext}\operatorname{pr}(K)$. Then $\operatorname{pr}^{-1}(e) \cap K = l_1 + \operatorname{pr}^{-1}(e_2) \cap K_2$. From the last relation and from the fact that e is an extreme point of $\operatorname{pr}(K)$, we conclude that $\operatorname{pr}^{-1}(e) \cap K$ must be an edge of K, and that $\operatorname{pr}^{-1}(e_2) \cap K_2$ must be an extreme point or an edge of K_2. Hence, for each edge l_1 of K_1, there exists an extreme point e_2 or an edge l_2 of K_2 such that either $l_1 + l_2$ or $l_1 + e_2$ is an edge of K. From the above we conclude that a given edge l_i of K_i could give rise to more than one edge of K. So the edges of K_1 and K_2 are countable, and $\operatorname{skel}_1 K_i = \bigcup_{n=1}^{\infty}(l_n^i \cup \operatorname{ext} K_i)$, $i = 1, 2$. Hence

$$\mathscr{H}^1(\operatorname{skel}_1 K) \geqslant \sum_{n=1}^{\infty}\mathscr{H}^1(l_n^1) + \sum_{n=1}^{\infty}\mathscr{H}^1(l_n^2) = \mathscr{H}^1(\operatorname{skel}_1 K_1) + \mathscr{H}^1(\operatorname{skel}_1 K_2),$$

as $\mathscr{H}^1(\operatorname{ext} K_i) = 0$, $i = 1, 2$. This concluces the proof of the proposition.

An immediate consequence of Proposition 2.2 is the following corollary, whose proof is obvious.

Corollary 2.1. *The function* $n_1(\cdot)$ *is a concave function.*

In the same way as in Proposition 2.1, we assert that there does not exist a positive number M such that $\mathscr{H}^1(\text{skel}_1 K) \leqslant M[\lambda \mathscr{H}^1(\text{skel}_1 K_1) + \mu \mathscr{H}^1(\text{skel}_1 K_2)]$ for any compact convex sets K_1, K_2 in E^3. To show this, we construct two convex compact sets A_1 and A_2 in E^3 such that $\mathscr{H}^1(\text{skel}_1 A_i) < +\infty$, $i = 1, 2$, while $\mathscr{H}^1(\text{skel}_1(A_1 + A_2)) = +\infty$. Let $u_1(0,2,0)$, $u_2 = (0,-2,0)$, $\beta_0 = (2,2,0)$, $\gamma_0 = (2,-2,0)$, $\alpha_0 = (2,0,1)$ and $\delta_0 = (0,0,1)$. Define K_0 to be convex hull of these points and let $l = [\alpha_0, \delta_0]$. We consider a plane H_1 such that $(0,0,0) \in H_1^+$, $\alpha_0 \in H_1^-$, and $K_0 \cap H_1$ is an isosceles triangle $T_1 = \text{conv}(\alpha_1, \beta_1, \gamma_1)$ with $|\alpha_1 - \beta_1| = |\alpha_1 - \gamma_1|$, with diameter$(T_1) = 2^{-1}$, with $\alpha_1 \in l$, and with the line segment $[\beta_1, \gamma_1]$ parallel to $[\beta_0, \gamma_0]$. Define $K_1 = H_1^+ \cap K_0$. We now proceed inductively. Assuming that we have constructed K_n $(n \geqslant 1)$, we choose the plane H_{n+1} in such a way that $(0,0,0) \in H_{n+1}^+$, that $\alpha_n \in H_{n+1}^-$, and that $K_n \cap H_{n+1}$ is an isosceles triangle $T_{n+1} = \text{conv}(\alpha_{n+1}, \beta_{n+1}, \gamma_{n+1})$ with $|\alpha_{n+1} - \beta_{n+1}| = |\alpha_{n+1} - \gamma_{n+1}|$, with diameter$(T_{n+1}) = 2^{-(n+1)}$, with $\alpha_{n+1} \in l$, and with the line segment $[\beta_{n+1}, \gamma_{n+1}]$ parallel to $[\beta_0, \gamma_0]$. Then $K_{n+1} = H_{n+1}^+ \cap K_n$.

Now let $A_1 = \lim_{n \to \infty} K_n = \bigcap_{n=0}^{\infty} K_n = K_0 \cap \bigcap_{n=1}^{\infty} H_n^+$. Then $A_1 = \text{clconv}\{\{u_1\} \cup \{u_2\} \cup \{\delta_0\} \cup \bigcup_{n=0}^{\infty}\{\beta_n\} \cup \bigcup_{n=0}^{\infty}\{\gamma_n\}\}$, and $\text{skel}_1 A_1 = [u_1, u_2] \cup [u_1, \delta_0] \cup [u_2, \delta_0] \cup [u_1, \beta_0] \cup [u_2, \gamma_0] \cup \bigcup_{n=0}^{\infty}[\beta_n, \gamma_n] \cup \bigcup_{n=0}^{\infty}[\beta_n, \beta_{n+1}] \cup \bigcup_{n=0}^{\infty}[\gamma_n, \gamma_{n+1}] \cup [\delta_0, \delta_1]$, where $\delta_1 = \lim_{n \to \infty} \beta_n = \lim_{n \to \infty} \gamma_n$. Then $\mathscr{H}^1(\text{skel}_1 A_1) \leqslant \mathscr{H}^1(\text{skel}_1 K_0) + \sum_{n=1}^{\infty} \mathscr{H}^1([\beta_n, \gamma_n]) \leqslant \mathscr{H}^1(\text{skel}_1 K_0) + \sum_{n=1}^{\infty} 2^{-n} < +\infty$. On the other hand, we define A_2 to be the orthogonal parallelogram with vertices u_1, u_2, β_0, γ_0, $u_1 + \delta_0$, $u_1 + \beta_0$, $\beta_0 + \delta_0$ and $\gamma_0 + \delta_0$. Obviously $\mathscr{H}^1(\text{skel}_1 A_2) < +\infty$. But the sum $A_1 + A_2$ has in its 1-skeleton countably many edges with length greater than 4. Hence $\mathscr{H}^1(\text{skel}_1(A_1 + A_2)) = +\infty$. From this the assertion follows.

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