CORANKS OF A QUASI-PROJECTIVE MODULE AND ITS ENDOMORPHISM RING

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Recently several authors have studied dualizing Goldie dimension of a module: spanning dimension in [**2**], codimension in [**13**], corank in [**16**] and also [**9**, **17**, **12**, **5**, **11**, **6**, **4**, **7**] ([**13**] may be read in comparison with the others). In the present note we prove the equality corank ${}_RP = $ corank ${}_SS$, where P is a quasi-projective left R-module and S is its endomorphism ring. This result is an answer to the question [**12**, p. 1898] and an extension of [**3**, Corollary 4.3] which shows the above equality for a Σ-quasi-projective left R-module P.

Throughout what follows R denotes an associative ring with identity and ${}_RP$, ${}_RM$ left R-modules. Let S be the endomorphism ring of ${}_RP$. Then P is an (R,S)-bimodule: $P = {}_RP_S$, $P^* := \mathrm{Hom}_R(P, M)$ is a left S-module and

$$P^*(A) := \{f \in P^* \mid Pf \subset A\}$$

is a submodule of ${}_SP^*$ for a submodule ${}_RA$ of ${}_RM$. For the definitions and properties we refer to [**13**] on coindependency of a set of submodules (which accords with meet-independency in [**3**]), to [**16**] on corank of a module and to [**10**] on a Σ-quasi-projective module. A set $\mathcal{S}$ of proper submodules of M is said to be *coindependent* if $\bigcap_{i=1}^{n-1} A_i + A_n = M$ for any finite subset $\{A_1, A_2, \ldots, A_n\}$ of $\mathcal{S}$ with $n \geqq 2$. Every single-element set is coindependent. We say corank ${}_RM = n$, a positive integer, if there exists an epimorphism from M to a direct product of n nonzero modules but there is no epimorphism from M to one of $n+1$ nonzero modules. Then corank ${}_RM = n$ if and only if there exists a coindependent set of n proper submodules of M but there is no coindependent set of $n+1$ proper ones. A module M is said to be Σ-*quasi-projective* if the direct sum of any number of copies of M is quasi-projective. It is said that P (*finitely*) *generates* M, or M is (*finitely*) P-*accessible*, if M is a homomorphic image of a (finite) direct sum of copies of P. It is also said that P is *finitely* M-*projective* if for every homomorphism f of P into N and every epimorphism h of M onto N, with N an arbitrary finitely cogenerated left R-module, there exists a homomorphism g in P^* such that $gh = f$ (see [**15**]).

As the first half of our preparation we borrow the following from [**3**, Theorem 4.2i)].

PROPOSITION 1. *If P is quasi-projective and P finitely generates M, then* corank ${}_SP^* \leqq$ corank ${}_RM$.

Proof. See [**3**, p. 104] or [**1**, Section 4].

The second half of our preparation is a continuation of [**14**] and [**15**].

LEMMA 2. *Let P be finitely M-projective and let $\{B_1, B_2, \ldots, B_n\}$ be a coindependent set of proper submodules of M such that each M/B_i is finitely cogenerated. Then for any*

$f_1, f_2, \ldots, f_n$ in P^* there exists a homomorphism g in P^* such that $g - f_i$ belongs to $P^*(B_i)$ for each i.

Proof. Set $\bar{M} := \bigoplus_{i=1}^{n} M/B_i$ and define homomorphisms f, h as follows:

$$f : P \to \bar{M}, \qquad pf = (pf_i + B_i) \quad (p \in P),$$
$$h : M \to \bar{M}, \qquad xh = (x + B_i) \quad (x \in M).$$

Then coindependency $B_i + \bigcap_{j \neq i} B_j = M$ implies that h is an epimorphism. Since $\bar{M}$ is finitely cogenerated and P is finitely M-projective, there exists a homomorphism g in P^* such that $gh = f$. Thus we have $P(g - f_i) \subset B_i$ for each i.

Proposition 3. *If P is finitely M-projective and P generates M, then* corank $_RM \leqq$ corank $_SP^*$.

Proof. Assume that $\{A_1, A_2, \ldots, A_n\}$ is a coindependent set of proper submodules of M. Then it is easily deduced from [**8**, Lemma 1.1], or proved directly that there exist proper submodules B_i, $A_i \subset B_i$, of M such that each M/B_i is finitely cogenerated. Evidently $\{B_1, B_2, \ldots, B_n\}$ is a coindependent set. Since B_i is proper in $M = PP^*$, we know that each $_SP^*(B_i)$ is a proper submodule of $_SP^*$. Let f be in P^*. Then Lemma 2 implies that for each i there exists a homomorphism g in P^* such that

$$g - f \in P^*(B_i) \quad \text{and} \quad g - 0 \in P^*(B_j) \qquad (j \neq i).$$

Thus $P^*(B_i) + \bigcap_{j \neq i} P^*(B_j) = P^*$ for each i and hence $\{P^*(B_1), P^*(B_2), \ldots, P^*(B_n)\}$ is a coindependent set of proper submodules of P^*. This gives the conclusion.

Corollary 4 ([**3**, Theorem 4.2ii)]). *If P is Σ-quasi-projective and P generates M, then* corank $_RM \leqq$ corank $_SP^*$.

Proof. This is clear since every Σ-quasi-projective module that generates M is M-projective.

Combining Propositions 1 and 3, we have the following.

Theorem 5. *If P is quasi-projective and P finitely generates M, then* corank $_RM =$ corank $_SP^*$.

Corollary 6. *If P is quasi-projective, then* corank $_RP =$ corank $_SS$.

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