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REGULAR AND STRONGLY FINITARY STRUCTURES OVER STRONGLY ALGEBROIDAL CATEGORIES

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Introduction. Most properties an algebraist needs in categories are reflected by regular functors, introduced in [**6**]. If $G : \mathscr{K} \to \mathscr{S}$ is a regular and strongly finitary functor and $\mathscr{S}$ has some nice properties, it can be shown that the left adjoint functor of G helps to characterize finitary and strongly finitary objects of $\mathscr{K}$. The property of being algebroidal can be lifted from $\mathscr{S}$ to $\mathscr{K}$ if a certain condition holds in $\mathscr{K}$. As an application, the implicational hull of subcategories can be constructed with the help of reduced products.

1. Regular categories and regular functors. In the following we often need regular categories with direct limits. $\mathscr{S}$ is a regular category with direct limits, if and only if the following two conditions hold in $\mathscr{S}$:

(A) $\mathscr{S}$ is regular (cf. [**6**]), i.e. every source $\langle f_i : X \to X_i | i \in I\rangle$ has a factorization

$$X \xrightarrow{f_i} X_i = X \xrightarrow{e} Y \xrightarrow{m_i} X_i$$

where e is a regular epimorphism and $\langle m_i : Y \to X_i | i \in I\rangle$ is a monosource. (I may be a proper class; also, I may be empty).

(B) $\mathscr{S}$ has direct limits, i.e. for any direct family $\langle f_{ik} : A_i \to A_k | i \leqq k \in I\rangle$ there exists a colimit $\langle f_i : A_i \to A | i \in I\rangle$, which is called a *direct limit.*

1.1 *Definitions* (cf. [**6**, Def. 2.1]). Let $\mathscr{S}$ be a regular category and $G : \mathscr{K} \to \mathscr{S}$ be a functor. G is called a *regular functor*, if and only if the following two conditions hold:

(A) G has a left adjoint functor.
(B) G creates regular factorizations.

1.2 *Convention.* In the following context we will suppose that $\mathscr{S}$ is a regular category, $G : \mathscr{K} \to \mathscr{S}$ is a regular functor with left adjoint F, front adjunctions $\eta_X : X \to GFX$ and back adjunctions $\epsilon_A : FGA \to A$.

1.3 *Definition.* (cf. [**7**, p. 184]) Let $g : X \to GA$ be an $\mathscr{S}$-morphism. We say that g *extremely G-generates* A, if and only if

(A) for all $r, s : A \to \bullet$, $Gr \cdot g = Gs \cdot g \Rightarrow r = s$, and
(B) for every monomorphism $m : B \to A$ with $g = Gm \cdot f$, m is an isomorphism.

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It is clear that $g : X \to GA$ extremely G-generates A if and only if $\epsilon_A \cdot Fg : FX \to A$ is extremal epi. In the following we shall always omit the reference to the functor and simply say "g generates A" instead of "g extremely G-generates A".

Since $\mathscr{K}$ is a regular category [**6**, Prop. 2.3] and since for regular categories extremal epimorphisms are regular epimorphisms, it is a simple task to prove the following proposition.

1.4 *If $G : \mathscr{K} \to \mathscr{S}$ is a regular functor, for a morphism $g : X \to GA$ the following conditions are equivalent:*

(i) *g extremely G-generates A.*

(ii) *$FX \xrightarrow{\epsilon_A \cdot Fg} A$ is a regular epimorphism.*

For an object $A \in \mathrm{Ob}\,\mathscr{K}$ we will call a diagram

$$R \underset{s}{\overset{r}{\rightrightarrows}} FX \xrightarrow{e} A$$

a *presentation* (*of* A), if and only if $\langle e, A\rangle$ is a coequalizer of $\langle r, s\rangle$.

By Proposition 1.4 and since $1_{GA} : GA \to GA$ extremely G-generates A, we have the following.

1.5 THEOREM. *Every object $A \in \mathrm{Ob}\,\mathscr{K}$ has a presentation.*

2. Finitary and strongly finitary functors.

2.1 *Definitions.* Let G be a functor.

1) G is *finitary* if and only if it for any direct limit $\langle f_i : A_i \to A\rangle$ the sink $\langle Gf_i : GA_i \to GA\rangle$ is an epi-sink.

2) G is *strongly finitary* if and only if it preserves direct limits.

3) An object A of a category $\mathscr{L}$ is (*strongly*) *finitary* if and only if the functor hom $(A, \bullet) : \mathscr{L} \to \mathrm{Set}$ is (strongly) finitary.

4) A category $\mathscr{L}$ is (*strongly*) *algebroidal*, if and only if for any object A there is a direct limit $\langle f_i : A_i \to A\rangle$ where all the objects A_i are (strongly) finitary.

In [**2**] finitary objects are called $\aleph_0$-*small*; algebroidal categories are called $\aleph_0$-*algebroidal.*

The proofs of the following lemmas are straightforward:

2.2 LEMMA.

1) *An object A is finitary if and only if for every direct limit $\langle f_i : B_i \to B\rangle$ and every morphism $f : A \to B$ there exists an index k and a morphism $g : A \to B_k$ such that $f = f_k \cdot g$.*

2) *An object A is strongly finitary if and only if it is finitary and for every pair of morphisms $g_1, g_2 : A \to B_i$ with $f_i \cdot g_1 = f_i \cdot g_2$* there exists an index $k \geqq i$ *and $f_{ik} \cdot g_1 = f_{ik} \cdot g_2$.*

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2.3 LEMMA. *Let A be (strongly) finitary and $r : A \to B$ be a retraction. Then B is (strongly) finitary.*

2.4 COROLLARY. *Let $\mathscr{L}$ be any category. The following conditions are equivalent:*
(i) *$\mathscr{L}$ is algebroidal and every finitary object is strongly finitary.*
(ii) *$\mathscr{L}$ is strongly algebroidal.*

Proof. (i) $\Rightarrow$ (ii) is obviously true. (ii) $\Rightarrow$ (i): Let B be finitary and $\langle f_i : B_i \to B \rangle$ be a direct limit of strongly finitary objects. There exists an index i and $g : B \to B_i$ such that $f_i \cdot g = 1_B$. f_i is a retraction and thus B is strongly finitary.

Convention. For the rest of this section the convention 1.2 holds. Moreover we assume that $\mathscr{S}$ has direct limits and $G : \mathscr{K} \to \mathscr{S}$ is strongly finitary.

2.5 LEMMA. *If $X \in \mathrm{Ob}\,\mathscr{S}$ is (strongly) finitary, then $FX \in \mathrm{Ob}\,\mathscr{K}$ is (strongly) finitary.*

Proof. The functors hom $(FX, \bullet)$ and hom $(X, G\bullet)$ are naturally equivalent and G preserves direct limits.

For objects of K we need the following definition.

2.6 *Definition.* An object $A \in \mathrm{Ob}\,\mathscr{K}$ is
1) *(strongly) finitarily generated*, if and only if there exists a finitary (resp. strongly finitary) object $X \in \mathrm{Ob}\,\mathscr{S}$ and a morphism $g : X \to GA$ extremely G-generating A;
2) *(strongly) finitarily presented*, if and only if there exists a presentation

$$R \underset{s}{\overset{r}{\rightrightarrows}} FX \xrightarrow{e} A,$$

where R is (strongly) finitarily generated and X is (strongly) finitary.

For the following theorem and for later use we need the following lemma whose proof is straightforward.

2.7 LEMMA. *Let $\mathscr{L}$ be a category, let H be a class of $\mathscr{L}$-objects such that for every $\mathscr{L}$-object A there exists an epi-sink $\langle f_i : A_i \to A \mid i \in I \rangle$ with $A_i \in H$ for $i \in I$. For a source $\langle g_i : B \to B_i \mid i \in I \rangle$ the following conditions are equivalent:*
(i) *$\langle g_i : B \to B_i \mid i \in I \rangle$ is a mono-source; and*
(ii) *for every pair of morphisms*

$$C \underset{s}{\overset{r}{\rightrightarrows}} B \quad \textit{with } C \in H,$$

$$(\forall i\ g_i \cdot r = g_i \cdot s) \Rightarrow r = s.$$

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As a consequence in (strongly) algebroidal categories when testing whether a source is a mono-source you need only consider pairs of morphisms with a (strongly) finitary domain.

The next lemma is due to I. Németi (personal correspondence) and is crucial for some theorems which follow.

2.8 LEMMA.

1) *If* $\mathscr{S}$ *is strongly algebroidal, direct limits of monomorphisms are monomorphisms, which means the following: If* $\langle f_{ik} : A_i \rightarrow A_k | i \leqq k \in I\rangle$ *and* $\langle g_{ik} : B_i \rightarrow B_k | i \leqq k \in I\rangle$ *are direct families and* $\langle m_i : A_i \rightarrow B_i | i \in I\rangle$ *is a family of monomorphisms such that for every* $i \leqq k$, $m_k \cdot f_{ik} = g_{ik} \cdot m_i$, *and if* $\langle f_i : A_i \rightarrow A | i \in I\rangle$ *and* $\langle g_i : B_i \rightarrow B | i \in I\rangle$ *are direct limits of the families, then the morphism* $m : A \rightarrow B$ *for which* $m \cdot f_i = g_i \cdot m_i$ *for all* $i \in I$, *is a morphism.*

2) *If* $\mathscr{S}$ *is algebroidal, direct unions of subobjects are subobjects, i.e. we consider only the case that for* $i \in I$, $B_i = B$ *and for* $i \leqq k$, $g_{ik} = 1_B$ *(Then the morphisms* f_{ik} *are automatically monomorphisms).*

Proof. 1) Let $m \cdot r = m \cdot s$.

$$
\begin{tikzcd}
 & & C \arrow[dll, bend right, "\hat{r}"'] \arrow[dll, bend right, "\hat{s}"] \arrow[d, shift right, "r"'] \arrow[d, shift left, "s"] \\
A_i \arrow[r, "f_{ik}"] \arrow[d, "m_i"] & A_k \arrow[r, "f_k"] \arrow[d, "m_k"] & A \arrow[d, "m"] \\
B_i \arrow[r, "g_{ik}"'] & B_k \arrow[r, "g_k"'] & B
\end{tikzcd}
$$

By Lemma 2.7 we can assume that C is strongly finitary. Thus there exists an index i and morphisms $\hat{r}, \hat{s}$ such that $r = f_i \cdot \hat{r}, s = f_i \cdot \hat{s} \Rightarrow m \cdot f_i \cdot \hat{r} = m \cdot f_i \cdot \hat{s} \Rightarrow g_i \cdot m_i \cdot \hat{r} = g_i \cdot m_i \cdot \hat{s} \Rightarrow$ for some $k \geqq i$, $g_{ik} \cdot m_i \cdot \hat{r} = g_{ik} \cdot m_i \cdot \hat{s} \Rightarrow m_k \cdot f_{ik} \cdot \hat{r} = m_k \cdot f_{ik} \cdot \hat{s} \Rightarrow f_{ik} \cdot \hat{r} = f_{ik} \cdot \hat{s} \Rightarrow r = f_i \cdot \hat{r} = f_k \cdot f_{ik} \cdot \hat{r} = f_k \cdot f_{ik} \cdot \hat{s} = f_i \cdot \hat{s} = s$.

2) In a similar way, we get by diagram chasing that $m \cdot r = m \cdot s \Rightarrow r = s$, assuming that C is a finitary object.

$$
\begin{tikzcd}
 & & C \arrow[dll, bend right, "\hat{r}"'] \arrow[dll, bend right, "\hat{s}"] \arrow[d, shift right, "r"'] \arrow[d, shift left, "s"] \\
A_i \arrow[r, "f_{ik}"] \arrow[drr, "m_i"'] & A_k \arrow[r, "f_k"] \arrow[dr, "m_k"] & A \arrow[d, "m"] \\
 & & B
\end{tikzcd}
$$

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2.9 THEOREM. *If $\mathscr{S}$ is (strongly) algebroidal, every object A of $\mathscr{K}$ is a direct union of (strongly) finitarily generated objects.*

Proof. Let $\langle f_i : B_i \to GA | i \in I \rangle$ be a direct limit of $\langle f_{ik} : B_i \to B_k | i \leqq k \in I \rangle$, where the objects $B_i \in \operatorname{Ob} \mathscr{S}$ are (strongly) finitary. For every $i \in I$ let $\langle A_i, m_i \rangle$ be the subobject of A generated by B_i, i.e. there exists a morphism $g_i : B_i \to GA_i$ generating A_i such that $f_i = Gm_i \cdot g_i$ (cf. [**6**, Prop. 2.13]).

For $i \leqq k$ we define m_{ik} in the following way:

$$\begin{tikzcd}
B_i \arrow[r, "g_i"] \arrow[d, "f_{ik}"'] \arrow[dr, phantom, "1"] & GA_i \arrow[dd, "Gm_i"] \arrow[ddl, dashed, "Gm_{ik}"'] \\
B_k \arrow[d, "g_k"'] & \arrow[d, phantom, "2"] \\
GA_k \arrow[r, "Gm_k"] & GA
\end{tikzcd}$$

g_i generates A_i, m_k is a monomorphism and the square commutes. By Prop. 2.15 of [**6**] there exists a unique morphism m_{ik} such that 1 and 2 commute.

We show that $\langle m_i : A_i \to A | i \in I \rangle$ is a direct limit of

$$M := \langle m_{ik} : A_i \to A_k | i \leqq k \in I \rangle. \text{ Let } \langle n_i : A_i \to C | i \in I \rangle$$

be a direct limit of M. There exists $m : C \to A$ such that for all $i \in I$, $m \cdot n_i = m_i$.

$$\begin{tikzcd}
B_i \arrow[r, "f_{ik}"] \arrow[d, "g_i"'] & B_k \arrow[r, "f_k"] \arrow[d, "g_k"'] & GA \arrow[d, shift left, "g"] \\
GA_i \arrow[r, "Gm_{ik}"] & GA_k \arrow[r, "Gn_k"] \arrow[ur, "Gm_k"] & GC \arrow[u, shift left, "Gm"]
\end{tikzcd}$$

Since the left squares commute (they are 1 in the first diagram of this proof) there exists a morphism $g : GA \to GC$ such that for all $i \in I$, $Gn_i \cdot g_i = g \cdot f_i$. Since $\langle f_i : B_i \to GA | i \in I \rangle$ is an epi-sink and for all $i \in I$,

$$Gm \cdot g \cdot f_i = Gm \cdot Gn_i \cdot g_i = Gm_i \cdot g_i = f_i = 1_{GA} \cdot f_i,$$

we get $Gm \cdot g = 1_{GA}$, i.e. Gm is a retraction. By Lemma 2.8 we see that Gm is a monomorphism. So Gm is an isomorphism and since G reflects isomorphisms m is an isomorphism, too. Thus $\langle m_i : A_i \to A | i \in I \rangle$ is a direct limit.

In [**5**] "$\aleph_0$-*erzeugbare Objekte*" are objects for which the functor hom $(A, \bullet)$ preserves direct unions. In the next theorem we show that under the assumption that $\mathscr{S}$ is (strongly) algebroidal the $\aleph_0$-erzeugbaren Objekte are just the (strongly) finitarily generated objects.

2.10 THEOREM. *Let the category $\mathscr{S}$ be (strongly) algebroidal. For $A \in \operatorname{Ob} \mathscr{K}$ the following conditions are equivalent:*

(i) *A is (strongly) finitarily generated.*

(ii) *For every direct union $\langle m_i : B_i \to B \rangle$ and every morphism $f : A \to B$ there exists an index i and a morphism $g : A \to B_i$ such that $m_i \cdot g = f$.*

Proof. (i) $\Rightarrow$ (ii): Let $\langle m_i : B_i \to B \rangle$ be any direct union and $f : A \to B$ be any morphism and

$$\begin{tikzcd} R \arrow[r, shift left, "r"] \arrow[r, shift right, "s"'] & FX \arrow[r, "e"] & A \end{tikzcd}$$

be a presentation of A where FX is finitary which exists by 1.4. There exists an index i and $h : FX \to B_i$ such that $f \cdot e = m_i \cdot h$.

$$\begin{tikzcd} & B_i \arrow[rr, "m_i"] & & B \\ R \arrow[r, shift left, "r"] \arrow[r, shift right, "s"'] & FX \arrow[u, "h"] \arrow[rr, "e"] & & A \arrow[u, "f"'] \arrow[llu, dashed, "g"'] \end{tikzcd}$$

We have $m_i \cdot h \cdot r = f \cdot e \cdot r = f \cdot e \cdot s = m_i \cdot h \cdot s$ and since m_i is a monomorphism, $h \cdot r = h \cdot s$. Since e is a coequalizer of $\langle r, s \rangle$, there exists $g : A \to B_i$ with $g \cdot e = h$. Thus we have $m_i \cdot g \cdot e = m_i \cdot h = f \cdot e$, and since e is an epimorphism, $m_i \cdot g = f$.

(ii) $\Rightarrow$ (i): By Theorem 2.9 there exists a direct union $m_i : A_i \to A$ where the objects A_i are (strongly) finitarily generated. By assumption there exists an index i and a morphism $g : A \to A_i$ such that $m_i \cdot g = 1_A \cdot m_i$ is a retraction and a monomorphism, thus an isomorphism. Therefore A is (strongly) finitarily generated.

2.11 THEOREM. *Let $A \in \mathrm{Ob}\mathscr{K}$ be any object. If $\mathscr{S}$ is strongly algebroidal the conditions are equivalent:*

(i) *A is finitarily presented.*

(ii) *A is strongly finitarily presented.*

(iii) *A is strongly finitary.*

(iv) *A is finitary.*

Proof. (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (iv) are trivial and (i) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (iii) are consequences of Corollary 2.4.

(ii) $\Rightarrow$ (iv) : Let

$$\begin{tikzcd} R \arrow[r, shift left, "r"] \arrow[r, shift right, "s"'] & FX \arrow[r, "e"] & A \end{tikzcd}$$

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be a presentation where FX is strongly finitary and let R be strongly finitarily generated. Let $e' : FZ \to R$ be a regular epimorphism where $Z \in \mathrm{Ob}\,\mathscr{S}$ is strongly finitary. Let $\langle f_i : B_i \to B \rangle$ be a direct limit of $\langle f_{ik} : B_i \to B_k \rangle$ and let $g : A \to B$ be any morphism.

$$\begin{tikzcd}
 & & B_i \arrow[rr, "f_i"] \arrow[dr, "f_{ik}"] & & B \\
 & & & B_k \arrow[ur, "f_k"] & \\
FZ \arrow[r, "e'"] & R \arrow[r, shift left, "r"] \arrow[r, shift right, "s"'] & FX \arrow[uu, "h"] \arrow[rr, "e"] & & A \arrow[uu, "g"'] \arrow[ul, "d"']
\end{tikzcd}$$

Since FX is finitary there exists an index i and $h : FX \to B_i$ such that $g \cdot e = f_i \cdot h$. Thus we have $f_i \cdot h \cdot r \cdot e' = g \cdot e \cdot r \cdot e' = g \cdot e \cdot s \cdot e' = f_i \cdot h \cdot s \cdot e'$ and since FZ is strongly finitary there exists an index k with $f_{ik} \cdot h \cdot r \cdot e' = f_{ik} \cdot h \cdot s \cdot e'$. e' is an epimorphism and therefore $f_{ik} \cdot h \cdot r = f_{ik} \cdot h \cdot s$. $\langle e, A \rangle$ is a coequalizer of $\langle r, s \rangle$, thus there exists $d : A \to B_k$ with $d \cdot e = f_{ik} \cdot h$. Summarizing the equalities we get $g \cdot e = f_i \cdot h = f_k \cdot f_{ik} \cdot h = f_k \cdot d \cdot e$ and since e is an epimorphism $g = f_k \cdot d$.

(iv) $\Rightarrow$ (ii): Let A be finitary. A satisfies the condition (ii) of 2.10 and is therefore strongly finitarily generated. Let

$$\begin{tikzcd}
R \arrow[r, shift left, "r"] \arrow[r, shift right, "s"'] & FX \arrow[r, "e"] & A.
\end{tikzcd}$$

be a presentation where X is strongly finitary. Let $\langle f_i : R_i \to R \rangle$ be a direct limit with strongly finitarily generated objects R_i. For every index i let $\langle e_i, A_i \rangle$ be a coequalizer of $\langle r \cdot f_i, s \cdot f_i \rangle$, i.e. A_i is strongly finitarily presented. Since colimits commute, A is a direct limit of the objects A_i.

$$\begin{tikzcd}
R_i \arrow[r, "f_{ik}"] \arrow[d, shift right, "r \cdot f_i"'] \arrow[d, shift left, "s \cdot f_i"] & R_k \arrow[r, "f_k"] \arrow[d, shift right, "r \cdot f_k"'] \arrow[d, shift left, "s \cdot f_k"] & R \arrow[d, shift right, "r"'] \arrow[d, shift left, "s"] \\
FX \arrow[r, "1"] \arrow[d, "e_i"] & FX \arrow[r, "1"] \arrow[d, "e_k"] & FX \arrow[d, "e"] \\
A_i \arrow[r, "g_{ik}"] & A_k \arrow[r, "g_k"] & A
\end{tikzcd}$$

Since A is finitary there exists an index i and $h : A \to A_i$ such that $g_i \cdot h = 1_A$. So we have $g_i \cdot h \cdot g_i \cdot e_i = g_i \cdot e_i$, and since FX is strongly finitary there exists an index k and $g_{ik} \cdot h \cdot g_i \cdot e = g_{ik} \cdot e_i$. $g_{ik} \cdot h$ is an epimorphism, being a left

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factor of $g_{ik} \cdot e_i = e_k$, and a section because of $g_k \cdot g_{ik} \cdot h = g_i \cdot h = 1_A$; thus $g_{ik} \cdot h$ is an isomorphism. Since A_k is strongly finitarily presented the same holds for A.

2.12 THEOREM. *If $\mathscr{S}$ is strongly algebroidal and in $\mathscr{K}$ every free object is a direct union of finitary free objects, then $\mathscr{K}$ is strongly algebroidal.*

Proof. Because of Corollary 2.4 we need not distinguish between finitary and strongly finitary objects. Let A be any object of $\mathscr{K}$ with a presentation

$$R \underset{s}{\overset{r}{\rightrightarrows}} FX \xrightarrow{e} A.$$

By Theorem 2.9 R is a direct union of finitarily generated subobjects of R; let $\langle t_i : R_i \rightarrow R | i \in I\rangle$ be a colimit of $\langle t_{ik} : R_i \rightarrow R_k | i \leqq k \in I\rangle$ where all the morphisms t_i are monomorphisms. Furthermore, let $\langle f_\mu : FX_\mu \rightarrow FX | \mu \in \mathrm{M}\rangle$ be a colimit of $\langle f_{\mu\nu} : FX_\mu \rightarrow FX_\nu | \mu \leqq \nu \in \mathrm{M}\rangle$ where the morphisms f_μ are monomorphisms and FX_μ are finitary objects. By Theorem 2.10 there exists for every $i \in I$ a $\mu \in \mathrm{M}$ and a morphism $r_{i\mu} : R_i \rightarrow FX_\mu$ such that $r \cdot t_i = f_\mu \cdot r_{i\mu}$. By the same reason there exists $\hat{\mu} \in \mathrm{M}$ and $s_{i\hat{\mu}} : R_i \rightarrow FX_{\hat{\mu}}$ such that $s \cdot t_i = f_{\hat{\mu}} \cdot s_{i\hat{\mu}}$. These morphisms $r_{i\mu}$ and $s_{i\hat{\mu}}$ are unique because f_μ and $f_{\hat{\mu}}$ are monomorphisms.

Let $\Lambda := \{\langle i, \lambda\rangle | \exists r_{i\lambda} \exists s_{i\lambda}\, r \cdot t_i = f_\lambda \cdot r_{i\lambda} \wedge s \cdot t_i = f_\lambda \cdot s_{i\lambda}\}$. Let $\langle i, \lambda\rangle \leqq \langle j, \kappa\rangle : \Leftrightarrow i \leqq j \wedge \lambda \leqq \kappa$. Let $P_I : \Lambda \rightarrow I$ and $P_\mathrm{M} : \Lambda \rightarrow \mathrm{M}$ be the projections. P_I is surjective and P_M maps Λ on a cofinal subset of M because $f_{\mu\nu} \cdot s_{i\mu} = s_{i\nu}$ and $f_{\mu\nu} \cdot r_{i\mu} = r_{i\nu}$. P_I and P_M are monotone mappings. By the following lemma $\langle t_i : R_i \rightarrow R | \langle i, \lambda\rangle \in \Lambda\rangle$ and $\langle f_\lambda : FX_\lambda \rightarrow FX | \langle i, \lambda\rangle \in \Lambda\rangle$ are direct limits. For $\langle i, \lambda\rangle \in \Lambda$ let $\langle e_{i\lambda}, A_{i\lambda}\rangle$ be a coequalizer of $\langle r_{i\lambda}, s_{i\lambda}\rangle$. Since colimits commute, A is a direct limit of the objects $A_{i\lambda}$.

$$\begin{tikzcd}
R_i \arrow[r, "t_{ik}"] \arrow[d, shift right, "r_{i\mu}"'] \arrow[d, shift left, "s_{i\mu}"] & R_k \arrow[r, "t_k"] \arrow[d, shift right, "r_{k\nu}"'] \arrow[d, shift left, "s_{k\nu}"] & R \arrow[d, shift right, "r"'] \arrow[d, shift left, "s"] \\
FX_\mu \arrow[r, "f_{\mu\nu}"] \arrow[d, "e_{i\mu}"] & FX_\nu \arrow[r, "f_\nu"] \arrow[d, "e_{k\nu}"] & FX \arrow[d, "e"] \\
A_{i\mu} \arrow[r] & A_{k\nu} \arrow[r] & A
\end{tikzcd}$$

2.13 LEMMA. *Let I and J be direct partially ordered sets and $\varphi : I \rightarrow J$ be a monotone and surjective mapping. Let $D := \langle f_{ik} : A_i \rightarrow A_k | i \leqq k \in J\rangle$ be a direct family and let $\langle f_i : A_i \rightarrow A | i \in J\rangle$ a colimit of D. Then $\langle f_{\varphi(\iota)} : A_{\varphi(\iota)} \rightarrow A | \iota \in I\rangle$ is a colimit of $D\varphi := \langle f_{\varphi(\iota),\varphi(\kappa)} : A_{\varphi(\iota)} \rightarrow A_{\varphi(\kappa)} | \iota \leqq \kappa \in I\rangle$.*

Remark. In the proof of this lemma you need only that I is a direct category (i.e. for objects ι, κ there exists an object λ and morphisms $\iota \rightarrow \lambda$, $\kappa \rightarrow \lambda$) and that J is a partially ordered category and that $\varphi : I \to J$ is a functor which is surjective on the objects.

3. Reduced products and ultraproducts.

3.1 *Definition* (cf. [**4**]). Let $\mathscr{L}$ be any category with products and direct limits and let $\langle A_i | i \in I\rangle$ be a family of $\mathscr{L}$-objects; let $\mathscr{F}$ be a filter basis on the power set $\mathscr{P}I$. For $J \in \mathscr{F}$ let $\Pi_J A := \Pi\langle A_i | i \in J\rangle$ and for $K \subseteq J$, $K \in \mathscr{F}$, $J \in \mathscr{F}$ let $p_{JK} : \Pi_J A \to \Pi_K A$ be the induced projection. Let $\langle p_J : \Pi_J A \to \Pi A/\mathscr{F} | J \in \mathscr{F}\rangle$ be a direct limit of

$$\langle p_{JK} : \Pi_J A \to \Pi_K A | J \supseteq K \in \mathscr{F}\rangle.$$

The object $\Pi A/\mathscr{F}$ is called a *reduced product.* If $\mathscr{F}$ generates an ultrafilter, $\Pi A/\mathscr{F}$ is called an *ultraproduct.*

Remark 1) We admit that the empty set is an element of the filter basis—in which case we yield terminal objects.

2) If the filter basis consists solely of I, a reduced product is a product.

3) If we substitute the filter basis by the filter generated by it, the reduced product does not change. Thus we might assume that $\mathscr{F}$ is a filter.

In the set theoretic model theory it is well known that the closedness under the following constructions is equivalent:

(i) subobjects, products, direct limits

(ii) subobjects, reduced products

(iii) subobjects, products, ultraproducts.

Before we show that these conditions are equivalent in regular, strongly algebroidal categories with direct limits we need a lemma which holds in any category and whose proof is straightforward.

3.2 Lemma (cf. Def. 3.1). *Let $\langle A_i | i \in I\rangle$ be a family of objects and $\mathscr{G} \subseteq \mathscr{P}I$ be any family of subsets of I, let $G := \bigcup \mathscr{G}$. For a pair of morphisms*

$$C \underset{s}{\overset{r}{\rightrightarrows}} \Pi_I A$$

the following conditions are equivalent:

(i) $\forall K \in \mathscr{G}$, $p_{IK} \cdot r = p_{IK} \cdot s$.

(ii) $p_{IG} \cdot r = p_{IG} \cdot s$.

3.3 Theorem. *Let $\mathscr{K}$ be a regular, strongly algebroidal category with direct limits and products. For a subcategory $\mathscr{L}$ the following conditions are equivalent:*

(i) *$\mathscr{L}$ is closed under subobjects, products and direct limits.*

(ii) *$\mathscr{L}$ is closed under subobjects and reduced products.*

(iii) *$\mathscr{L}$ is closed under subobjects, products and ultraproducts.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is obvious.

(ii) $\Rightarrow$ (i) We must show that $\mathscr{L}$ is closed under direct limits. Let $\langle f_i : A_i \rightarrow A | i \in I\rangle$ be a colimit of $\langle f_{ik} : A_i \rightarrow A_k | i \leqq k \in I\rangle$ where $A_i \in \mathrm{Ob}\,\mathscr{L}$ for $i \in I$. For $i \in I$ let $I_i : = \{k \in I | k \geqq i\}$ and $P_i : = \prod_{I_i} A$ with the projections $pr_{ik} : P_i \rightarrow A_k$ for $i \leqq k$. The set $\{I_i | i \in I\}$ is a filter basis and let $P \in \mathscr{L}$ be the reduced product. For $i \in I$, let $m_i : A_i \rightarrow P_i$ be the morphism which factors $\langle f_{ik} | k \in I_i\rangle$ through $\langle pr_{ik} | k \in I_i\rangle$, i.e. $f_{ik} = pr_{ik} \cdot m_i$.

$$\begin{tikzcd}
A_i \arrow[r, "f_{ik}"] \arrow[d, shift right, "m_i"'] & A_k \arrow[r, "f_k"] \arrow[d, shift right, "m_k"'] & A \arrow[d, "m"] \\
P_i \arrow[r, "p_{ik}"'] \arrow[u, shift right, "pr_{ii}"'] \arrow[ur, "pr_{ik}"] & P_k \arrow[r, "p_k"'] \arrow[u, shift right, "pr_{kk}"'] & P
\end{tikzcd}$$

Especially $1 = f_{ii} = pr_{ii} \cdot m_i$ for every $i \in I$, i.e. m_i is a section and thus a monomorphism. For $i \leqq k$ and every $l \geqq k$ we have $pr_{kl} \cdot p_{ik} \cdot m_i = pr_{il} \cdot m_i = f_{il} = f_{kl} \cdot f_{ik} = pr_{kl} \cdot m_k \cdot f_{ik}$ and therefore $p_{ik} \cdot m_i = m_k \cdot f_{ik}$. Thus there exists a unique morphism $m : A \rightarrow P$ for which $m \cdot f_i = m_i \cdot p_i$ for all $i \in I$. As a direct limit of monomorphisms m is a monomorphism and thus $A \in \mathrm{Ob}\mathscr{L}$.

(iii) $\Rightarrow$ (ii): Let $\langle A_i | i \in I\rangle$ be a family of $\mathscr{L}$-objects and $\mathscr{F}$ be a filter on $\mathscr{P}I$. Let $\{E_\mu | \mu \in \mathrm{M}\}$ be the set of all ultrafilters on $\mathscr{P}I$ for which $\mathscr{E}_\mu \supseteq \mathscr{F}$. For each $\mu \in \mathrm{M}$ let $\langle p^\mu{}_J : \prod_J A \rightarrow \prod A/\mathscr{E}_\mu | J \in \mathscr{E}_\mu\rangle$ be a direct limit. Since $\mathscr{E}_\mu \supseteq \mathscr{F}$, we have exactly one homomorphism $q^\mu : \prod A/\mathscr{F} \rightarrow \prod A/\mathscr{E}_\mu$ with $p^\mu{}_J = q^\mu \cdot p_J$ for all $J \in \mathscr{F}$. We claim that $\langle q^\mu : \prod A/\mathscr{F} \rightarrow \prod A/\mathscr{E}_\mu | \mu \in \mathrm{M}\rangle$ is a mono-source, which completes the proof.

$$\begin{tikzcd}
 & \prod_{L_\mu} A \arrow[r, "q_{L_\mu}{}^\mu"] & \prod A/\mathscr{E}_\mu \\
\prod_J A \arrow[ur, "p_{JL_\mu}"] \arrow[urr, "q_J{}^\mu"'] \arrow[r, "p_J"] & \prod A/\mathscr{F} \arrow[ur, "q^\mu"'] & \\
 & B \arrow[ul, shift left, "\hat{s}"'] \arrow[ul, shift right, "\hat{r}"] \arrow[u, shift left, "r"] \arrow[u, shift right, "s"'] &
\end{tikzcd}$$

Let

$$B \overset{r}{\underset{s}{\rightrightarrows}} \prod A/\mathscr{F}$$

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be a pair of K-morphisms and $q^\mu \cdot r = q^\mu \cdot s$ for all $\mu \in \mathrm{M}$. By Lemma 2.7 we assume that B is strongly finitary. There exists $J \in \mathscr{F}$ and $\hat{r}, \hat{s} : B \to \prod_J A$ such that $p_J \cdot \hat{r} = r$ and $p_J \cdot \hat{s} = s$. Thus we have for all $\mu \in \mathrm{M}$,

$$q^\mu{}_J \cdot \hat{r} = q^\mu \cdot p_J \cdot \hat{r} = q^\mu \cdot r = q^\mu \cdot s = \ldots = q^\mu{}_J \cdot \hat{s}.$$

Since B is strongly finitary there exists for every $\mu \in \mathrm{M}$, $L_\mu \in \mathscr{E}_\mu$ such that $p_{JL_\mu} \cdot \hat{r} = p_{JL_\mu} \cdot \hat{s}$. Let $L := \bigcup \langle L_\mu | \mu \in \mathrm{M} \rangle$. L is an element of $\mathscr{F}$, because $L \in \bigcap \langle \mathscr{E}_\mu | \mu \in \mathrm{M} \rangle = \mathscr{F}$.

For a subcategory $\mathscr{L} \subseteq \mathscr{K}$ let $\mathbf{S}\mathscr{L}$ be the class of subobjects of $\mathscr{L}$ and $\underset{\rightarrow}{\mathbf{P}}\mathscr{L}$ be the class of reduced products of $\mathscr{L}$. $\mathbf{S}$ is trivially a closure operator while in [**1**] it is shown that $\underset{\rightarrow}{\mathbf{P}}$ is a closure operator if $\mathscr{K}$ is strongly algebroidal. The next proposition shows that $\underset{\rightarrow}{\mathbf{P}}\mathbf{S}\mathscr{L} \subseteq \mathbf{S}\underset{\rightarrow}{\mathbf{P}}\mathscr{L}$.

3.4 PROPOSITION. *Let $\mathscr{L}$ be a subcategory of the regularly finitary category $\mathscr{K}$ which has products, and let $\langle A_i | i \in I \rangle$ be a family of $\mathscr{L}$-objects, for $i \in I$, let $m_i : B_i \to A_i$ be a monomorphism and let $\mathscr{F}$ be a filter basis on $\mathscr{P}\, I$. The object $\prod B/\mathscr{F}$ is a subobject of $\prod A/\mathscr{F}$.*

Proof. Since products preserve monomorphisms, for any $J \in \mathscr{F}$ $\prod_J B$ is a subobject of $\prod_J A$. The proposition follows because direct limits of monomorphisms are monomorphisms (Lemma 2.8.1).

4. Applications. In [**2**], $\aleph_0$-implicational subcategories are defined. A subcategory L of K is called *$\aleph_0$-implicational* if and only if there exists a class H of regular epimorphisms with finitary domains such that the objects of $\mathscr{L}$ are exactly the H-injective objects of $\mathscr{K}$. If $\mathscr{K}$ is strongly algebroidal the $\aleph_0$-implicational classes can be described by closure properties.

The following theorem is well known in the case of algebras (cf. for example [**8**]).

4.1 THEOREM. *Let $\mathscr{K}$ be a regular, strongly algebroidal category with products and direct limits. For a subcategory $\mathscr{L}$ of $\mathscr{K}$ the following conditions are equivalent:*

(i) *$\mathscr{L}$ is $\aleph_0$-implicational.*

(ii) *$\mathscr{L}$ is closed under subobjects, products and direct limits.*

(iii) *$\mathscr{L}$ is closed under subobjects and reduced products.*

(iv) *$\mathscr{L}$ is closed under subobjects, products and ultraproducts.*

Proof. (i) $\Leftrightarrow$ (ii) is proved in [**2**] and (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) is Theorem 3.3.

With help of Proposition 3.4 we immediately get the following result.

4.2 THEOREM. *Let $\mathscr{K}$ be a regular, strongly algebroidal category with products and direct limits. Let $\mathscr{L}$ be a subcategory of $\mathscr{K}$. The $\aleph_0$-implicational hull of $\mathscr{L}$ is $\mathbf{S}\underset{\rightarrow}{\mathbf{P}}\mathscr{L}$.*

4.3 Let Ω be an operator domain (cf. [**3**]) and $\mathscr{K}_\Omega$ be the category of Ω-algebras and Ω-homomorphisms, let $U : \mathscr{K}_\Omega \to \mathrm{Set}$ be the forgetful functor.

U is regular and strongly finitary and Set is regular and strongly algebroidal and has direct limits. Moreover the left adjoint of U preserves monomorphisms and thus every free object (in $\mathscr{K}_\Omega$) is a direct union of finitarily generated free objects. Therefore the finitary objects of $\mathscr{K}_\Omega$ are exactly the finitarily U-presented algebras and $\mathscr{K}_\Omega$ is algebroidal and regularly finitary.

The same holds for *categories of heterogeneous algebras* (cf. [**9**]). By the propositions in Section 2 one needs only show that the forgetful functor is regular and strongly finitary and that the category Set^H is strongly algebroidal, regular and has direct limits. The left adjoint of the forgetful functor $U : \mathscr{K}_\Omega \to \mathrm{Set}^H$ preserves monomorphisms—but there exist equational classes where not every free object is a direct union of finitary free objects.

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