THE TEMPORAL CONTINUUM

MOHAMMAD ARDESHIR
Department of Mathematics, Sharif University of Technology
and

RASOUL RAMEZANIAN
Department of Economics, University of Lausanne

Abstract. The continuum has been one of the most controversial topics in mathematics since the time of the Greeks. Some mathematicians, such as Euclid and Cantor, held the position that a line is composed of points, while others, like Aristotle, Weyl, and Brouwer, argued that a line is not composed of points but rather a matrix of a continued insertion of points. In spite of this disagreement on the structure of the continuum, they did distinguish the *temporal line* from the *spatial line*. In this paper, we argue that there is indeed a difference between the intuition of the spatial continuum and the intuition of the temporal continuum. The main primary aspect of the temporal continuum, in contrast with the spatial continuum, is the notion of *orientation*.

The continuum has usually been mathematically modeled by *Cauchy* sequences and the *Dedekind* cuts. While in the first model, each point can be approximated by rational numbers, in the second one, that is not possible constructively. We argue that points on the temporal continuum cannot be approximated by rationals as a temporal point is a *flow* that sinks to the past. In our model, the continuum is a collection of constructive *Dedekind* cuts, and we define two topologies for temporal continuum: 1. *oriented* topology and 2. the *ordinary* topology. We prove that every total function from the *oriented* topological space to the *ordinary* one is continuous.

§1. Introduction. Mathematicians have long been divided into two philosophical camps regarding the structure of the continuum. Some, like Euclid [12] and Cantor [7, 11], view the continuum as a composition of points. On the other hand, others, including Aristotle [2], Weyl [4, 19], and Brouwer [5, 6], consider it as a *whole*, not composed of points.

In this paper, we aim to highlight another distinction, namely, between what we refer to as *the spatial continuum* and *the temporal continuum*. We will model the temporal continuum using a specific type of Dedekind cuts that illustrate this difference.

The main distinction between the spatial continuum and the temporal continuum lies in the notion of *orientation*: the temporal continuum is oriented, moving exclusively from the past to the future, and it is impossible to move in the *opposite* direction[1]. Our

Received: August 25, 2023.
2020 *Mathematics Subject Classification*: Primary 00A30, 03A05, 03A10.
Key words and phrases: intuitionism, continuity, constructive mathematics.

[1] In this paper, we *assume* that 'time is directed' without delving into detailed arguments. Numerous works have explored the direction of time from both physical and philosophical perspectives; for further reading, refer to [15, 16].

doi:10.1017/S1755020324000078

subjective experience reinforces this clear differentiation between the future and the past. We can vividly remember the past, whereas the future remains uncertain. Our ignorance of future events, including our own choices and actions, contributes to the concept of *free will*. Conversely, we lack the ability to alter the past due to the absence of free will in that direction.

> We can affect the future by our actions: so why can we not by our actions affect the past? The answer that springs to mind is this: you cannot change the past; if a thing has happened, it has happened, and you cannot make it not to have happened [10].

The primary objective of this paper is to propose a model for the continuum that incorporates the concept of orientation. To achieve this, we will introduce a formalization of orientation through a specific type of Dedekind cuts, which we will refer to as oriented Dedekind cuts.

Brouwer and Weyl emphasized two essential aspects of the intuitive continuum, which are *inexhaustibility* and *non-discreteness* (also referred to as *non-atomicity*) (see [3, p. 86]). The concept of choice sequences is motivated by these characteristics. Non-lawlike sequences signify the continuum's inexhaustibility, and identifying points with unfinished sequences of nested intervals reflects its non-discreteness (see [3, p. 87]). Defining points as choice sequences of nested intervals captures the notion that a point on the continuum is not a dimension-less atom but rather a *halo*.

As choice sequences are non-predetermined and unfinished entities, the value of every total function from choice sequences to natural numbers relies on an initial segment of sequences. Brouwer leverages this property to demonstrate that every total function over the continuum is continuous.

Both the spatial continuum and the temporal continuum share the aforementioned aspects. However, a defining characteristic of the temporal continuum is the concept of *orientation*. "Duration" is a continuous *becoming*, always moving towards the future. In the spatial continuum, movement is possible in both directions. To introduce witnesses for points on the spatial line, one may proceed by introducing witnesses for all points in the segment $[-1, 0]$, then for points in the segment $[1, 2]$, and subsequently for points in the segment $[0, 1]$, and so on. There is no requirement to adhere to any specific direction. However, for the temporal line, this approach is not feasible. Let the locations of a moving ball in space serve as witnesses for moments. If we are at the moment t_0, and t_1 is a future moment, we cannot determine the location of the ball at time t_1 at the present moment. We must wait until we reach that moment, and only then will the locations of the ball in all moments before t_1 be determined. *We are constantly moving towards the future*. The following *assertion* may provide further clarity regarding our understanding of the notion of orientation. We leverage notion of orientation to demonstrate that every total function over the temporal continuum is continuous.

In considering how we experience a point on the temporal continuum, we can conceptualize a point t as the moment of occurrence of an event E:

$$t := \text{“The moment of the occurrence of the event } E.\text{”}$$

For instance, assume it is currently 8:00 am, and I am aware that the event E will occur sometime after 8:00 am but before 12:00 pm. To express this more formally, we treat rational numbers as states and utilize modal operators:

https://doi.org/10.1017/S1755020324000078 Published online by Cambridge University Press

$\Diamond$ for 'sometimes in the future', and
$\Diamond^{-1}$ for 'sometimes in the past',

in LTL, linear temporal logic. Also let

$K\phi$ means 'the subject has evidence that ϕ is true'.

In this context:

$$s_{8:00} \models K\Diamond E \wedge K(s_{12:00} \models \Diamond^{-1} E),$$

the statement says that at 8:00 am, the event E is expected to occur at some point in the future, and at 8:00 am, the subject is aware that by 12:00 pm, the event E has already taken place.

As time passes, I (the subject) experience the point t in the following manner: I examine the occurrence of E at several consecutive times. To do this, I plan a strategy and choose specific moments to observe the event. The choice of these moments is up to me. Note that, given our assumption of using rational numbers as states, and since rational numbers do not have successors, consequently, there are no 'next states' within our model.

Suppose I (the subject) am in state s. Since s is a rational number, it has no successors within the rational numbers. However, I must choose a next state s' to examine the occurrence of E. As soon as I choose the state s', I forego all the states in the interval (s, s') because time is oriented, and I cannot revert.

For example, I may decide to examine the occurrence of E at times like 10:15 am, 10:20 am, 10:40 am, 11:30 am, and so on. Suppose it is now 11:30 am, and the last time I checked the occurrence of E was at 10:40 am, where I found no evidence of its happening yet, so,

$$s_{10:40} \models K\Diamond E.$$

Also, suppose it is 11:30 am, and I observe the event and find evidence of its occurrence,

$$s_{11:30} \models K\Diamond^{-1} E.$$

At this point, I *cannot move to the past* to examine occurrence of E at any previous moments.

All the information I have about t is that it lies *sometime* within the interval $(10:40, 11:30]$. I cannot distinguish the moment t from other moments within this interval. In other words, *the point t has "sunk back" to the past, and I cannot experience it any further*. It is *absolutely undecidable* whether t is before 11:00 am or after it. The moment t cannot be estimated more accurately, and consequently, I cannot approximate t using rational moments.

Furthermore, if I have a constructive method to introduce witnesses for all moments on the temporal line, then the witness for the moment t is also the witness for all moments within the interval $(10:40, 11:30]$. Thus, a point on the temporal continuum is like a *flow* that sinks back to the past eventually.

We consider the above *explanation* as our understanding of the notion of *orientation* and attempt to formalize it in this paper. In our perspective, we regard t as a *flow* from the past, meaning it encompasses all moments that have occurred before t. If t is a past moment, we cannot obtain any additional information about which moments belong to this collection and which ones do not. If t is a future moment, we still have time to gather data about the elements of the collection before t eventually sinks back to the past. During this time, we may examine the membership status of certain moments. However, as soon as t sinks back to the past, we can no longer acquire any new data.

This passage discusses the experience of a point on the temporal continuum, emphasizing the concept of *orientation* and how it is related to the passage of time and our ability to observe and distinguish moments. It also introduces the idea of a point being a *flow* from the past, capturing the continuous and irreversible nature of time.

Brouwer's perspective on the continuum is that it is intuitively given as a flowing medium of cohesion between two events, not comprised of points (events) itself, but rather an inexhaustible matrix allowing for a continued insertion of points. Originally, there are no points on the continuum, but we can construct points on it or indicate a position within it. Brouwer emphasized that the intuition of the continuum is the intuition of the medium of cohesion between two events. He distinguishes between two things: the medium of cohesion (first thing) and the continuum (second thing), or *primum* and *secundum* as he puts it (See [14, p. 70]). Brouwer utilized the category of choice sequences and the continuity principle to provide a mathematical analysis of continua. Using choice sequences to define points on the continuum, a point is modeled as a *halo*. Consequently, to construct a witness for a point on the continuum, the witness is constructed for a halo around it. Brouwer proposed his famous continuity principle for choice sequences and used it to prove that every total function over the real line is continuous (see [17, p. 305]).

Motivated by the characteristic aspect of the temporal continuum, i.e., the *orientation*, as explained above, we introduce oriented cuts to model points on the temporal continuum as *flows*. The traditional modeling of the continuum using the *Cauchy* fundamental sequences of rationals [17] allows every real number to be approximated by rational numbers. However, since moments on the temporal continuum sink back to the past, they *cannot* be approximated by rationals. Consequently, the *Cauchy* sequences appear unsuitable for modeling the temporal continuum. Instead, Dedekind cuts prove to be more appropriate for this purpose. Among the constructive Dedekind lines introduced in [17], only $\mathbb{R}^d$ can be positively approximated by rationals due to its *locatedness* property (see [17, p. 270]), while $\mathbb{R}^e$ and $\mathbb{R}^{be}$ can be approximated by rationals only up to the double negation via the *strong monotonicity* property. We demonstrate that the collection of oriented Dedekind cuts cannot be approximated by rationals, as desired in this case. As discussed earlier, points on the temporal continuum are like flows that sink back to the past. Once they have sunk back, we cannot acquire new data about them, and thus, they cannot be approximated by rationals.

In addition to our main objective of introducing a mathematical model for the temporal continuum, we also aim to demonstrate that every total function over the temporal line is continuous, similar to the Brouwerian real line. For this purpose, we require a *continuity principle* for the Dedekind cuts. Hence, we propose a principle called the *oriented continuity principle* (**OCP**) and define two topologies for the temporal continuum: 1. the *oriented topology* and 2. the *ordinary topology*. Utilizing **OCP**, we establish that every total function from the oriented topological space to the ordinary one is continuous.

The sequel of the paper is organized as follows. In Section 2, we define the notion of the oriented cuts, and justify a continuity principle for them. The oriented continuity principle, **OCP**, expresses formally the feature we have in mind about a continuity principle. We show that the oriented reals cannot be approximated by rationals. In Section 3, the *oriented topology* and the *ordinary topology* are defined, and then some consequences of **OCP** for the temporal continuum are demonstrated.

We argue in the context of constructive logic. As far as possible, the standard notations in [17] are used in this paper.

§2. Oriented cuts. In this section, we introduce a type of left cuts of $\mathbb{Q}$, named *oriented cuts*, and state the oriented continuity principle.

Definition 2.1. *We let $\mathbb{R}^o$ be the set of all strictly increasing bounded sequences of rational numbers, i.e., $\alpha \in \mathbb{R}^o$ if and only if $\exists M \forall n(\alpha(n) < \alpha(n+1) < M)$. For all $\alpha, \beta \in \mathbb{R}^o$ we define*:

- $\alpha \leq \beta$ *if and only if* $\forall m \exists n(\alpha(m) < \beta(n))$,
- $\alpha < \beta$ *if and only if* $\exists n \forall m(\alpha(m) < \beta(n))$, *and*
- $\alpha =^o \beta$ *if and only if* $\alpha \leq \beta \wedge \beta \leq \alpha$.

We call the set $\mathbb{R}^o$, regarding equality $=^o$, the set of all oriented reals (*cuts*).

Oriented reals, bounded strictly increasing sequences of rationals, are supposed to model the passing of time. The increase of sequences reflects the passing of time and the strictness of increase ensures that time cannot rest.

Definition 2.2. *For a rational number $q \in \mathbb{Q}$, let $\hat{q}$ be the oriented real defined by $\hat{q}(n) = q - \frac{1}{n+1}$, for all* n. *Let $q \to \hat{q}$ be the mapping which assigns the oriented real $\hat{q}$ to* q. *For each $q \in \mathbb{Q}$ and $\alpha \in \mathbb{R}^o$, we say*:

a. $q < \alpha$ *if and only if* $\hat{q} < \alpha$,
b. $\alpha \leq q$ *if and only if* $\alpha \leq \hat{q}$,
c. $\alpha < q$ *if and only if* $\alpha < \hat{q}$, *and*
d. $q \leq \alpha$ *if and only if* $\hat{q} \leq \alpha$.

Proposition 2.3. *For $q \in \mathbb{Q}$ and $\alpha \in \mathbb{R}^o$*,

(1) $q < \alpha$ *if and only if* $\exists n(q < \alpha(n))$,
(2) $\alpha \leq q$ *if and only if* $\forall n(\alpha(n) < q)$,
(3) $\alpha < q$ *if and only if* $\exists p \in \mathbb{Q}(p < q \wedge \alpha \leq p)$,
(4) $q \leq \alpha$ *if and only if* $\forall p \in \mathbb{Q}(p < q \to p < \alpha)$.

Proof. For each $q \in \mathbb{Q}$ consider its map $\hat{q}$ in oriented reals. Items (1) and (2) are straight forward.

(3). If $\alpha < \hat{q}$ then by Definition 2.1, there exists m such that for all n, $\alpha(n) < q - \frac{1}{m+1}$. Let $p = q - \frac{1}{m+1}$. By (2), we have $\alpha \leq p$. The converse is straight forward.

(4). If $\hat{q} \leq \alpha$ then for all m there exists n_m such that $q - \frac{1}{m+1} < \alpha(n_m)$. For $p < q$, there exists m_0 such that $p < q - \frac{1}{m_0+1} < \alpha(n_{m_0})$. By (1), $p < \alpha$. For the converse assume $\forall p \in \mathbb{Q}(p < q \to p < \alpha)$. Then for all m, $q - \frac{1}{m+1} < \alpha$. By (1), there exists n_m such that $q - \frac{1}{m+1} < \alpha(n_m)$. □

For each $\alpha \in \mathbb{R}^o$, let $A_\alpha = \{q \in \mathbb{Q} \mid \exists n(q < \alpha(n))\}$ be the *cut* specified by α. Note that $\alpha =^o \beta$ if and only if $A_\alpha = A_\beta$ as sets.

PROPOSITION 2.4. *For $\alpha \in \mathbb{R}^o$,*

1. $\forall q \in \mathbb{Q}\,(q \in A_\alpha \to \exists p \in \mathbb{Q}\,(p > q \wedge p \in A_\alpha))$ (openness),
2. $\forall p, q \in \mathbb{Q}\,(p < q \wedge q \in A_\alpha \to p \in A_\alpha)$ (monotonicity),
3. $\exists p, q \in \mathbb{Q}\,(p \in A_\alpha \wedge q \notin A_\alpha)$ (boundedness).

Proof. It is straightforward. □

LEMMA 2.5. *For all $\alpha, \beta \in \mathbb{R}^o$, there exists $\gamma \in \mathbb{R}^o$, which specifies the cut $A_\alpha \cap A_\beta$.*

Proof. The proof is easy. □

LEMMA 2.6. *Let $\gamma : \mathbb{N} \to \mathbb{Q}$ be an upper bounded sequence, i.e., $\exists M \in \mathbb{Q}\forall n(\gamma(n) < M)$, then there exists $\alpha \in \mathbb{R}^o$ which specifies $A = \{q \in \mathbb{Q} \mid \exists n \in \mathbb{N}(q < \gamma(n))\}$.*

Proof. Let $\alpha(n) = max\{\gamma(0), \gamma(1), \dots, \gamma(n)\} - \frac{1}{n+1}$. The sequence α is strictly increasing and the set A would be equal to $\{q \in \mathbb{Q} \mid \exists n \in \mathbb{N}(q < \alpha(n))\}$. □

The main difference between the collection of oriented reals $\mathbb{R}^o$ and other collections of Dedekind reals such as $\mathbb{R}^d$, extended reals $\mathbb{R}^{be}$ and classical reals $\mathbb{R}^e$ is that the former satisfies the *monotonicity* property $(\forall \alpha \in \mathbb{R}^o \forall p, q \in \mathbb{Q}(p < q \wedge q \in A_\alpha \to p \in A_\alpha))$, whereas the others satisfy *strong monotonicity* (for each Dedekind cut A, $\forall p, q \in \mathbb{Q}(p < q \wedge \neg\neg q \in A \to p \in A)$). It is known that $\mathbb{R}^d \subset \mathbb{R}^{be} \subset \mathbb{R}^e$ (see [17, p. 270]). In the following proposition, by using the Markov Principle (see [17, p. 204]):

$$\textbf{MP} \quad \forall \beta \in \mathbb{N}^{\mathbb{N}}(\neg\neg\exists k \beta(k) = 0 \to \exists k \beta(k) = 0),$$

we show that $\mathbb{R}^o \subseteq \mathbb{R}^{be}$.

PROPOSITION 2.7 (**MP**). *For $\alpha \in \mathbb{R}^o$, the cut A_α satisfies* strong monotonicity.

Proof. We show that for every $\alpha \in \mathbb{R}^o$, A_α satisfies *strong monotonicity*, i.e., $\forall p, q \in \mathbb{Q}(p < q \wedge \neg\neg q \in A_\alpha \to p \in A_\alpha)$. Since $q \in A_\alpha \leftrightarrow \exists k \in \mathbb{N}(q < \alpha(k))$, we have $\neg\neg q \in A_\alpha \leftrightarrow \neg\neg\exists k \in \mathbb{N}(q < \alpha(k))$. By **MP**, $\neg\neg\exists k \in \mathbb{N}(q < \alpha(k)) \leftrightarrow \exists k \in \mathbb{N}(q < \alpha(k))$, (assume $\beta(k) = \{\begin{smallmatrix} 0 & q<\alpha(k) \\ 1 & otherwise \end{smallmatrix}$). So $\neg\neg q \in A_\alpha \leftrightarrow q \in A_\alpha$, for any arbitrary $q \in \mathbb{Q}$. Then $p < q \wedge \neg\neg q \in A_\alpha \leftrightarrow p < q \wedge q \in A_\alpha$, and since A_α satisfies *monotonicity*, we derive $p \in A_\alpha$. □

For choice sequences, if Φ is a total function from the collection of choice sequences to natural numbers, the value of Φ for a sequence α just depends on a finite segment of α. Now this question seems natural in constructive mathematics:

*how can we **construct** a **total** mapping $\Phi : \mathbb{R}^o \to \mathbb{N}$?*

Assume we have a strategy to construct a total mapping $\Phi : \mathbb{R}^o \to \mathbb{N}$. Then for any arbitrary sequence $\alpha \in \mathbb{R}^o$, we must be able to construct a witness for α. Since Φ is well-defined, for any sequence $\beta =^o \alpha$, we will have $\Phi(\beta) = \Phi(\alpha)$. Therefore, our strategy cannot depend on any finite segment of α. Because of this obstacle, one may expect that it is not possible construct Φ unless Φ happens to be constant. We fulfill this intention, using Brouwer's weak continuity principle (see [17, p. 209]):

$$\textbf{WCN} \quad \forall \alpha \in \mathbf{T}\exists y(\Phi(\alpha) = y) \to \forall \alpha \in \mathbf{T}\exists x \forall \beta \in \mathbf{T}(\bar{\beta}x = \bar{\alpha}x \to \Phi\beta = \Phi\alpha),$$

where $\mathbf{T}$ is a spread and for each sequence α, $\bar{\alpha}x = \langle \alpha(0), \alpha(1), \dots, \alpha(x-1)\rangle$.

PROPOSITION 2.8 (**WCN**). *Any (constructive) total function $\Phi : \mathbb{R}^o \to \mathbb{N}$ is constant.*

Proof. Let $\alpha, \beta \in \mathbb{R}^o$ be arbitrary, $\alpha \leq \beta$, $M \in \mathbb{Q}$ be an upper-bound for β. The set $\mathbb{R}^o_M = \{\gamma \in \mathbb{R}^o \mid \gamma < M\}$ is a spread. Since Φ is a total, by **WCN**, there exists t such that, for all $\gamma \in \mathbb{R}^o_M$, if $\bar{\gamma}t = \bar{\alpha}t$ then $\Phi(\gamma) = \Phi(\alpha)$. Find $\gamma \in \mathbb{R}^o_M$ passing through $\bar{\alpha}t$ such that $\gamma =^o \beta$. By well-definedness of Φ we conclude $\Phi(\beta) = \Phi(\alpha)$. So, for all $\alpha, \beta \in \mathbb{R}^o$, if $\alpha \leq \beta$ then $\Phi(\beta) = \Phi(\alpha)$. Now, for arbitrary $\alpha, \beta \in \mathbb{R}^o$, define $\gamma(n) = min(\alpha(n), \beta(n))$. Here, $\gamma \leq \alpha$ and $\gamma \leq \beta$. Then $\Phi(\gamma) = \Phi(\alpha)$ and $\Phi(\gamma) = \Phi(\beta)$. Consequently $\Phi(\beta) = \Phi(\alpha)$. □

We showed that any total function from $\mathbb{R}^o$ to natural numbers is constant in the presence of the Weak Continuity Principle, that is, if one has a constructive method that for each oriented cut is able to introduce a witness, a natural number, then the witness is unique. To avoid this, as a replacement for natural numbers, we consider another category of objects called *almost natural numbers*, and construct witnesses for oriented reals from this category.

Definition 2.9. *We let $\mathbb{N}^*$ be the set of all functions ξ from $\mathbb{N}$ to $\mathbb{N}$ such that, for some* k, *for all* n, $\xi(n) \leq \xi(n+1) \leq k$. *For all ξ, ν in $\mathbb{N}^*$ we define*:

$\xi \leq \nu$ *if and only if* $\forall m \exists n(\xi(m) \leq \nu(n))$, *and*
$\xi =^* \nu$ *if and only if* $(\xi \leq \nu) \wedge (\nu \leq \xi)$.

We call $\mathbb{N}^*$, regarding equality $=^*$, the set of almost natural numbers.

It is clear that $\mathbb{N}^*$ is classically isomorphic to $\mathbb{N}$ as a set. In fact, classically, elements in $\mathbb{N}^*$ are increasing sequences that converge.

Lemma 2.10. *For every $\xi \in \mathbb{N}^*$,*

1. $\neg\neg\exists n \forall m > n[\xi(m) = \xi(n)]$,
2. $\neg\forall n \exists m > n\ (\xi(m) \neq \xi(m+1))$.

Proof.

1. To show $\forall \xi \in \mathbb{N}^* \neg\neg\exists n \forall m > n[\xi(m) = \xi(n)]$, it is enough to show $\neg\exists \xi \in \mathbb{N}^* \neg\exists n \forall m > n[\xi(m) = \xi(n)]$, by the intuitionistic valid statement $\neg\exists x A(x) \leftrightarrow \forall x \neg A(x)$. So assume for some $\xi_0 \in \mathbb{N}^*$, $\neg\exists n \forall m > n[\xi_0(m) = \xi_0(n)]$. Then, $\forall n \neg\forall m > n[\xi_0(m) = \xi_0(n)]$.
   Since $\xi_0 \in \mathbb{N}^*$, there exists $k_0 \in \mathbb{N}$ such that $\xi_0(n) \leq \xi_0(n+1) \leq k_0$, for all $n \in \mathbb{N}$.
   We prove that for every $k \in \mathbb{N}$, $(\forall n(\xi_0(n) \leq k)) \to (\forall n(\xi_0(n) < k))$ (1).
   Assume there exists $t \in \mathbb{N}$ such that $\xi_0(t) = k$. Since ξ_0 is nondecreasing and $\forall n(\xi_0(n) \leq k))$, we have $\forall n > t[\xi_0(n) = k]$. It contradicts with $\forall n \neg\forall m > n[\xi_0(m) = \xi_0(n)]$. Hence $\forall n(\xi_0(n) < k)$.
   Now let $k = k_0$. By (1), we derive $\forall n(\xi_0(n) \leq k_0 - 1)$. Repeating using (1), we have got $\forall n(\xi_0(n) = 0)$. It contradicts with $\forall n \neg\forall m > n[\xi_0(m) = \xi_0(n)]$.
2. Let $\xi \in \mathbb{N}^*$, there exists $k \in \mathbb{N}$ such that for all n, $\xi(n) < k$. Assume

$$\forall n \exists m > n\ (\xi(m) \neq \xi(m+1)).\ (2)$$

Define $f : \mathbb{N} \to \mathbb{N}$ as

$$f(n) = \min\{m \mid (m > n \wedge \xi(m) \neq \xi(m+1))\} + 1.$$

https://doi.org/10.1017/S1755020324000078 Published online by Cambridge University Press

Due to the assumption (2), the function f is a constructive well defined function. It is easy to see that $n < f(n)$ and $\xi(n) < \xi(f(n))$. Therefore,

$$\xi(1) < \xi(f(1)) < \xi(f(f(1))) < \xi(f(f(f(1)))) < \dots .$$

It contradicts with the fact that for all n, $\xi(n) < k$. $\square$

It is worth mentioning that if γ is an almost natural number then the set $I = \{n \in \mathbb{N} \mid \exists k\ (n \leq \gamma(k))\}$ is not necessary finite, but it is *quasi-finite* in the sense of [18], i.e., it is a subset of a finite set. Moreover, its complement is *almost full* [18], meaning that for every strictly increasing sequence α, one may find a natural number n such that $\alpha(n)$ belongs to the complement of I.

Until now, the two notions of oriented reals and almost natural numbers have been defined. The following proposition (accompanied by its proof) shows that how these two notions are related to our intuition of orientation. In contrast with Proposition 2.8, we have the following.

PROPOSITION 2.11. *There exists a non-constant total function* $\Phi : \mathbb{R}^o \to \mathbb{N}^*$.

Proof. Choose $d_1 < d_2 < \cdots < d_j$ from $\mathbb{Q}$ for some fixed $j \in \mathbb{N}$. For any $\beta \in \mathbb{R}^o$, we define:

$\Phi(\beta)(0) = 0$, and
$\Phi(\beta)(n) = max(\{i \mid i \leq j \wedge (d_i \leq \beta(n))\} \cup \{0\})$ for all $n \geq 1$.

One may easily check that the followings hold true for Φ:

1. $\Phi(\beta) \in \mathbb{N}^*$.
2. Φ is well-defined, i.e., if $\alpha =^o \beta$ then $\Phi(\alpha) =^* \Phi(\beta)$.
3. Φ is non-constant. Assume two different oriented cuts α, β, such that $d_1 \notin A_\alpha$ and $d_1 \in A_\beta, d_2 \notin A_\beta$. Then $\forall n \in \mathbb{N}\ \Phi(\alpha)(n) = 0$, whereas, $\exists k \forall n > k\ \Phi(\beta)(n) = 1$.

The function Φ can be computed by the following algorithm as well:

1. Put $\Phi(\beta)(0) := 0$;
2. Put $t = 1$;
3. For i=1 to j do:
   {
   while $(\beta(t) < d_i)$ do:
      {
      define $\Phi(\beta)(t) := i - 1$;
      put $t = t + 1$;
      }
   }
4. For k=t+1 to ∞ do:
   {
   Put $\Phi(\beta)(k) := j$;
   }

The above algorithm checks the membership status of d_i's in A_β, respectively, in an ordered manner. As soon as, the algorithm detects that d_i is in A_β, the future value of $\Phi(\beta)$ changes. We defined Φ using the above algorithm, in order to evoke our sense of the orientation explained in the introduction; the oriented real number β is assumed to be a moment in future. Rational numbers $d_1 < d_2 < \cdots < d_j$ are assumed as future moments which we examine the occurrence of β at them, as time passing. Using the information attained by the examination, the value of $\Phi(\beta) \in \mathbb{N}^*$ is determined. □

Therefore there exists a total function from $\mathbb{R}^o$ to $\mathbb{N}^*$ which is not constant. *How a constructive* total *function form $\mathbb{R}^o$ to $\mathbb{N}^*$ can be? or in other word, how can we construct a* total *mapping* $\Phi : \mathbb{R}^o \to \mathbb{N}^*$? We answer this question soon in Section 2.2.

In the following, we illustrate that, although the oriented reals can not be approximated by rationals, they can be approximated by *almost rational numbers*, defined below.

Definition 2.12. *Let* $\mathbf{Q}$ *be the set of all rational sequences* $\zeta : \mathbb{N} \to \mathbb{Q}$ *in which*:

1. ζ *is increasing, and*
2. *the image of* ζ *is a subset of a finite set*.

For all $\zeta, \zeta' \in \mathbf{Q}$, *we define*:

$\zeta \leq \zeta'$ *if and only if* $\forall m \exists n (\zeta(m) \leq \zeta'(n))$, *and*
$\zeta = \zeta'$ *if and only if* $(\zeta \leq \zeta') \wedge (\zeta' \leq \zeta)$.

We call $\mathbf{Q}$ *the set of* almost rational numbers.

Note that $\mathbf{Q}$ is classically isomorphic to $\mathbb{Q}$ as a set. The set of almost rational numbers $\mathbf{Q}$ is embedded into $\mathbb{R}^o$, by $\zeta \mapsto \hat{\zeta}$ where $\hat{\zeta}(n) = \zeta(n) - \frac{1}{n+1}$, for $n \in \mathbb{N}$, is an oriented real specifying the cut $\{q \in \mathbb{Q} \mid \exists k (q < \zeta(k))\}$.

Definition 2.13. *For every* $\alpha \in \mathbb{R}^o$, *and* $r \in \mathbb{Q}$, *define* $(\alpha + r) \in \mathbb{R}^o$, *by* $(\alpha + r)(n) := \alpha(n) + r$, *for every* n.

Definition 2.14. *We say that an oriented real* α *can be approximated by rationals, if for each* n, *there exists* $q \in \mathbb{Q}$ *such that* $(q \leq \alpha \wedge \alpha \leq q + 2^{-n})$.

As we explained in the introduction, since moments of time sink back into the past, it is not possible to approximate them using rationals. The following proposition shows that oriented reals cannot be approximated by rational numbers either.

Proposition 2.15 ($\neg\exists$-PEM). *It is false that every oriented real can be approximated by rationals.*

Proof. For an arbitrary bounded increasing sequence α of rationals, let β_α be an oriented real defined by $\beta_\alpha(n) = \alpha(n) - \frac{1}{n+1}$, for $n \in \mathbb{N}$, which specifies the cut $A = \{q \in \mathbb{Q} \mid \exists k (q < \alpha(k))\}$. If β_α can be approximated by rationals, then β_α is a Cauchy sequence. Consequently, the sequence α is also Cauchy and so it converges. Since α was arbitrary, it would imply that every bounded monotone sequence has a limit in Brouwerian real line. That contradicts with $\neg\exists$-PEM (see [17, p. 268]). □

Definition 2.16. *We say that an oriented real* α *can be approximated by almost rationals, if* $\forall n \exists \zeta \in \mathbf{Q}(\hat{\zeta} \leq \alpha \wedge \alpha \leq (\hat{\zeta} + 2^{-n}))$.

Proposition 2.17. *Every oriented real can be approximated by almost rationals numbers.*

Proof. Let $\beta \in \mathbb{R}^o$, and $M \in \mathbb{Q}$ be an upper bound for β. We define ζ by induction. Let $\zeta(0) = \beta(0)$, and assume we have defined $\zeta(k)$. We define

$$\zeta(k+1) = \zeta(k) + 2^{-n}t, \text{ where } t \in \mathbb{N} \text{ and} \zeta(k) + 2^{-n}t \leq \beta(k+1) < \zeta(k) + 2^{-n}(t+1).$$

Note that $\zeta \in \mathbf{Q}$, since $image(\zeta) \subseteq \{\beta(0) + 2^{-n}t \mid \beta(0) + 2^{-n}t < M, t \in \mathbb{N}\}$ and the latter is not infinite. We claim that $\hat{\zeta} \leq \beta \wedge \beta \leq \hat{\zeta} + 2^{-n}$. For $\hat{\zeta} \leq \beta$, we have for each $k \in \mathbb{N}$ there exists m such that $\zeta(k) < \beta(m)$. For the second clause, i.e., $\beta \leq \hat{\zeta} + 2^{-n}$, we have $\beta(k) < \zeta(k) + 2^{-n}$. □

A subset B of $\mathbb{N}$ is called (intuitionistically) enumerable if it is the image of a function on natural numbers (see [1]). Assuming the Kripke schema (see [17, p. 236]):

$$\mathbf{KS} \quad \forall X \exists \alpha \forall n[n \in X \leftrightarrow \exists m(\alpha(n,m) = 0)],$$

every inhabited[2] subset of natural numbers is (intuitionistically) enumerable [1].

Lemma 2.18. *If $B \subseteq \mathbb{Q}$ is (intuitionistically) enumerable and bounded from above, then there exists an oriented real number α such that $A_\alpha = C(B) = \{q \in \mathbb{Q} \mid \exists p(p \in B \wedge q < p)\}$.*

Proof. As $B \subseteq \mathbb{Q}$ is (intuitionistically) enumerable, there exists a sequence $\gamma : \mathbb{N} \to \mathbb{Q}$ such that $\forall q \in \mathbb{Q}[q \in B \leftrightarrow \exists m(\gamma(m) = q)]$. Since B is bounded from above, the sequence γ has an upper bound. By Lemma 2.6, there exists an oriented number α such that $A_\alpha = C(B)$. □

Assume $B \subseteq \mathbb{Q}$ is (intuitionistically) enumerable and upper bounded. We define the supremum of B, $sup(B)$, to be the oriented real number α that $A_\alpha = C(B)$. Assuming the Kripke schema, **KS**, every upper bounded subset of $\mathbb{Q}$ has supremum.

Proposition 2.19 (**KS**). *Assume $B \subseteq \mathbb{Q}$ is upper bounded. Then for $\alpha = sup(B)$, the following hold.*

1. $\forall q \in \mathbb{Q}(q \in B \to q \leq \alpha)$,
2. $\forall p \in \mathbb{Q}(p < \alpha \to \exists q \in \mathbb{Q}(q \in B \wedge p < q))$.

Proof. (1.) Assume $p \in B$. For any $q < p$, we have $q \in C(B) = A_\alpha$. Then there exists n such that $q < \alpha(n)$. By items (1) and (4) of Proposition 2.3, we have $p \leq \alpha$.

(2.) Assume $p < \alpha$. Then $p \in A_\alpha$. As $A_\alpha = C(B)$, there exists $q \in B$ such that $p < q$. □

Definition 2.20. *Let $D \subseteq \mathbb{Q}$ is lower bounded, and $B = \{p \in \mathbb{Q} \mid \forall q(q \in D \to q \geq p)\}$. We define the infimum of D, $inf(D)$, to be the supremum of B.*

Proposition 2.21 (**KS**). *Assume $D \subseteq \mathbb{Q}$ is lower bounded. Then for $\alpha = inf(D)$, the following hold.*

1. $\forall q \in \mathbb{Q}(q \in D \to \alpha \leq q)$,
2. $\forall p \in \mathbb{Q}[(\forall q \in \mathbb{Q}(q \in D \to p \leq q)) \to p \leq \alpha]$.

Proof. (1.) $\alpha = inf(D)$ is an oriented real such that $A_\alpha = C(B) = \{q \in \mathbb{Q} \mid \exists p(p \in B \wedge q < p)\}$, where $B = \{p \in \mathbb{Q} \mid \forall q(q \in D \to q \geq p)\}$. Let $q \in D$. Since $A_\alpha = \{q \in \mathbb{Q} \mid \exists n(q < \alpha(n))\}$ and α is strictly increasing, we have $\alpha(n) \in A_\alpha$, for every n. By the

[2] A set B is inhabited if $\exists x(x \in B)$. We indicate that by $B\#\emptyset$.

equality $A_\alpha = C(B)$, it is derived that for each n, there exists $p \in B$ such that $\alpha(n) < p$. Then by definition of B, $\alpha(n) < q$.

(2.) Assume $p \in \mathbb{Q}$ is such that for all $q \in D$, $p \leq q$. Then $p \in B$, and thus for all $p_0 < p$, $p_0 \in C(B) = A_\alpha$. According to definition of A_α, there exists n such that $p_0 < \alpha(n)$. Then, $(\forall p_0 < p)\exists n(p_0 < \alpha(n))$. By items (1) and (4) of Proposition 2.3, we have $p \leq \alpha$. □

THEOREM 2.22 (**KS**. The monotone convergence theorem). *For every upper bounded nondecreasing sequence* $(\alpha_n)_{n\in\mathbb{N}}$ *of oriented reals, there exists an oriented real* α *such that*:

1. $\forall n(\alpha_n \leq \alpha)$,
2. $(\forall p \in \mathbb{Q})[(p < \alpha) \to (\exists m \forall n(n \geq m \to p < \alpha_n))]$.

Proof. Let $B = \{p \in \mathbb{Q} \mid \exists n(p < \alpha_n)\}$. The set B is upper bounded, so let $\alpha = sup(B)$. For all $n, m \in \mathbb{N}$, we have $\alpha_n(m) \in B$, since α_n is a strictly decreasing sequence of rationals. By Proposition 2.19, we have for each n, for all m, $\alpha_n(m) < \alpha_n(m+1) \leq \alpha$. So, by items (1) and (4) of Proposition 2.3, for each n, $\alpha_n \leq \alpha$.

If $p < \alpha$, then by Proposition 2.19, there exists $q \in B$ such that $p < q$. By definition of B, there exists $m \in \mathbb{N}$, $p < q < \alpha_m$. As $(\alpha_n)_{n\in\mathbb{N}}$ is a non-decreasing sequence, we have for all $n \geq m$, $p < \alpha_n$. □

Among the Dedekind lines $\mathbb{R}^e$, $\mathbb{R}^{be}$, and $\mathbb{R}^d$ introduced in [17], the line $\mathbb{R}^{be}$ is much similar to $\mathbb{R}^o$. The only difference between the cuts of $\mathbb{R}^{be}$ and the cuts of $\mathbb{R}^o$ is that the first one satisfies *strong monotonicity*, whereas the second one just fulfils monotonicity. On the other hand, as we have already noted, the reals in $\mathbb{R}^{be}$ cannot be approximated by rationals, as desired as a requirement for the temporal line. Hence, the line $\mathbb{R}^{be}$ could be assumed as an appropriate mathematical model for the temporal line if there was a *continuity principle* for it, like Brouwer's continuity principle for *Cauchy* reals $\mathbb{R}$. We note that the line $\mathbb{R}$ is the *Cauchy* completion of $\mathbb{Q}$ and the line $\mathbb{R}^{be}$ is the *order completion* of $\mathbb{Q}$.

Let us define a mapping $\Psi : \mathbb{R}^o \to \mathbb{R}^{be}$ as follows:

$$\Psi(\alpha) := \{r \in \mathbb{Q} \mid (\exists s \in \mathbb{Q})[r < s \wedge \neg\neg\exists n(s < \alpha(n))]\}.$$

One can easily check that Ψ is well-defined and for all $\alpha \in \mathbb{R}^o$, $\Psi(\alpha) \in \mathbb{R}^{be}$.

PROPOSITION 2.23 (**KS**). *The mapping* $\Psi : \mathbb{R}^o \to \mathbb{R}^{be}$ *is surjective.*

Proof. Assuming the Kripke schema, it can be shown that every inhabited subset of natural numbers is (intuitionistically) enumerable [1].

Let $S \in \mathbb{R}^{be}$, and $\gamma : \mathbb{N} \to S$ enumerates S, i.e., $\forall q \in \mathbb{Q}[q \in S \leftrightarrow \exists m(\gamma(m) = q)]$. Define

$$\alpha(n) = max\{\gamma(0), \gamma(1), \dots, \gamma(n)\} - \frac{1}{n+1}.$$

The sequence α is in $\mathbb{R}^o$. We claim $\Psi(\alpha) = S$.

Assume $r \in \Psi(\alpha)$. Then there exists $s \in \mathbb{Q}$ such that $r < s$ and $\neg\neg\exists n(s < \alpha(n))$. For the sake of argument, assume $\exists n(s < \alpha(n))$. Note that for all m, $\alpha(m) \in S$. By the *monotonicity* of S, we have $s \in S$. Thus $\neg\neg s \in S$. By the *strong monotonicity*, we have $r \in S$.

https://doi.org/10.1017/S1755020324000078 Published online by Cambridge University Press

For the converse, assume $r \in S$. Let k be such that $\gamma(k) = r$. By openness, there exists $r' \in S$ such that $r < r'$, and let $\gamma(m) = r'$ for some m. Choose $n > m$ such that $r' - r < \frac{1}{n+1}$. Then we have $r < \alpha(n)$ and thus $r \in \Psi(\alpha)$. □

PROPOSITION 2.24. *For all* $\alpha, \beta \in \mathbb{R}^o$

$$\alpha < \beta \textit{ implies } \Psi(\alpha) < \Psi(\beta).$$

Proof. The relation "$<$" for cuts S, T is defined as follows: $S < T := \exists r > 0(S + r \subset T)$ (see [17, defn. 5.4]). Suppose $\alpha < \beta$. Then for some n, we have $\forall m(\alpha(m) < \beta(n))$. Let $r = \beta(n+1) - \beta(n)$. It is easy to check that $\Psi(\alpha) + r \subset \Psi(\beta)$. □

2.1. Arithmetic in $\mathbb{R}^o$***.*** As explained in the introduction, we propose $\mathbb{R}^o$ as a model for the temporal line. Each oriented real illustrates a moment. Then what does it mean to add or multiply two moments? What is a proper arithmetic of the temporal line?

The real line $\mathbb{R}^{be}$ is a *field* due to the *strong monotonicity* property of its elements. A field $\langle F, +, \cdot \rangle$ is an algebraic structure with two *functions* $+$ and $\cdot$ from $F \times F$ to F, where F equipped with the functions $+$ or $\cdot$, is a group, and $\cdot$ is distributed over $+$. In our model, $\mathbb{R}^o$, the operations $+$ and $\cdot$ are *relations* instead of being functions, i.e., it is an algebraic structure known as a *hyperstructure*[3].

DEFINITION 2.25. *A* h-field *is a tuple* $\langle F, +, *, 0, 1 \rangle$ *where* $+ \subseteq F \times F \times F$ *and* $* \subseteq F \times F \times F$ *satisfy the following axioms*:

- Inhabitance.
$$(\forall x, y)(\exists z) + (x, y, z) \qquad (\forall x, y)(\exists z) * (x, y, z).$$
- Identity.
$$(\forall x) + (x, 0, x) \qquad (\forall x) * (x, 1, x).$$
- Inverse.
$$(\forall x \exists y) + (x, y, 0) \qquad (\forall x \exists y) * (x, y, 1).$$
- Commutativity.
$$(\forall x, y, z)(+(x, y, z) \leftrightarrow +(y, x, z)) \qquad (\forall x, y, z)(*(x, y, z) \leftrightarrow *(y, x, z)).$$
- Associativity.
$$(\forall x, y, z, w, v, u)((+(x, y, w) \wedge +(w, z, v) \wedge +(y, z, u)) \rightarrow +(x, u, v))$$
$$(\forall x, y, z, w, v, u)((+(x, y, w) \wedge +(x, u, v) \wedge +(y, z, u)) \rightarrow +(w, z, v))$$
$$(\forall x, y, z, w, v, u)((*(x, y, w) \wedge *(w, z, v) \wedge *(y, z, u)) \rightarrow *(x, u, v))$$
$$(\forall x, y, z, w, v, u)((*(x, y, w) \wedge *(x, u, v) \wedge *(y, z, u)) \rightarrow *(w, z, v)).$$
- Distributivity.
$$(\forall x, y, z, w, v, u, r)((*(x, v, w) \wedge +(y, z, v) \wedge *(x, y, u) \wedge *(x, z, r)) \rightarrow +(u, r, w))$$
$$(\forall x, y, z, w, v, u, r)((+(u, r, w) \wedge +(y, z, v) \wedge *(x, y, u) \wedge *(x, z, r)) \rightarrow *(x, v, w)).$$

[3] For similar definitions and applications of hyper-algebraic structures as arithmetic, see [8, 9, 13].

We define an *addition relation* $+$, and a *multiplication relation* $*$ on $\mathbb{R}^o$ as follows: For $\alpha, \beta, \gamma \in \mathbb{R}^o$, we let:

1. $+(\alpha, \beta, \gamma)$ if and only if $\Psi(\alpha) + \Psi(\beta) = \Psi(\gamma)$, and
2. $*(\alpha, \beta, \gamma)$ if and only if $\Psi(\alpha) \cdot \Psi(\beta) = \Psi(\gamma)$.

PROPOSITION 2.26 (**KS**). *$\langle \mathbb{R}^o, +, *, \hat{0}, \hat{1} \rangle$ is a h-field.*

Proof. The proof is straightforward by using the fact that $\mathbb{R}^{be}$ is a field. □

2.2. The oriented continuity principle. In this part, we propose the oriented continuity principle which expresses formally our sense of the notion of orientation. To do this, we study total functions Φ from $(0,1]^o$ to $\mathbb{N}^*$, where $(0,1]^o = \{\alpha \in \mathbb{R}^o \mid \hat{0} < \alpha \leq \hat{1}\}$. Similarly, we let $[0,1]^o = \{\alpha \in \mathbb{R}^o \mid \hat{0} \leq \alpha \leq \hat{1}\}$. We assume such function Φ is well-defined, i.e., $\alpha =^o \beta$ implies $\Phi(\alpha) =^* \Phi(\beta)$.

LEMMA 2.27 (**WCN**). *Let $\Phi \in (0,1]^o \to \mathbb{N}^*$ be total. Then $\forall \alpha, \beta \in (0,1]^o (\alpha \leq \beta \to \Phi(\alpha) \leq \Phi(\beta))$.*

Proof. For $\theta \in \mathbb{N}^*$ define $N_\theta = \{m \in \mathbb{N} \mid \exists n (m \leq \theta(n))\}$. We need to show that $\alpha \leq \beta$ implies $N_{\Phi(\alpha)} \subseteq N_{\Phi(\beta)}$. Assume $\Phi(\alpha)(n) = k$ for some $n, k \in \mathbb{N}$. The set $(0,1]^o$ is a spread and Φ is total, so by **WCN** we can find a t such that for each $\delta \in (0,1]^o$, $\bar{\delta}t = \bar{\alpha}t$ implies $\Phi(\delta)(n) = \Phi(\alpha)(n)$. We have, for each $\delta \in (0,1]^o$, if $\bar{\delta}t = \bar{\alpha}t$ then $k \in N_{\Phi(\delta)}$. Since $\alpha \leq \beta$ there exists $\lambda \in (0,1]^o$ such that $\bar{\lambda}t = \bar{\alpha}t$ and $\lambda =^o \beta$. Note that Φ is well-defined, therefore $k \in N_{\Phi(\beta)}$. Since we assumed $k \in \mathbb{N}$ to be arbitrary, we have $N_{\Phi(\alpha)} \subseteq N_{\Phi(\beta)}$. □

The following theorem is by W. Veldman from our correspondence with him.

THEOREM 2.28 (**WCN**). *Let Φ be a total well-defined function from $(0,1]^o$ to $\mathbb{N}^*$. Then*

$$\forall \alpha \in (0,1]^o \neg\neg\exists q \in \mathbb{Q}[q < \alpha \wedge \forall \beta \in (0,1]^o[(q < \beta \leq \alpha \to \Phi(\alpha) =^* \Phi(\beta)]].$$

Proof. Let $\alpha \in (0,1]^o$. Since $\Phi(\alpha) \in \mathbb{N}^*$, by Lemma 2.10, $\neg\neg\exists n \forall m > n[\Phi(\alpha)(m) = \Phi(\alpha)(n)]$. For the sake of the argument, assume $\exists n \forall m > n[\Phi(\alpha)(m) = \Phi(\alpha)(n)]$, and let n_0 be such that $\forall m > n_0[\Phi(\alpha)(m) = \Phi(\alpha)(n_0)]$. By **WCN**, there is a t such that for all β in the spread $(0,1]^o$, if $\bar{\beta}t = \bar{\alpha}t$ then $\Phi(\beta)(n_0) = \Phi(\alpha)(n_0)$. It follows that, for all $\beta \in (0,1]^o$ passing through $\bar{\alpha}t = \langle \alpha 0, \alpha 1, \dots, \alpha(t-1) \rangle$, $\Phi(\alpha) \leq \Phi(\beta)$.

Define $q := \alpha(t-1)$. Assume that $\beta \in (0,1]^o$ and $q < \beta \leq \alpha$. Find λ passing through $\bar{\alpha}t$ such that $\lambda =^o \beta$, and conclude $\Phi(\alpha) \leq \Phi(\lambda) =^* \Phi(\beta)$. On the other hand, by Lemma 2.27, since $\beta \leq \alpha$ we have $\Phi(\beta) \leq \Phi(\alpha)$. Hence, for every β satisfying $q < \beta \leq \alpha$, $\Phi(\beta) =^* \Phi(\alpha)$. This conclusion is obtained from the assumption: $\exists n \forall m > n[\Phi(\alpha)(m) = \Phi(\alpha)(n)]$. As we know $\neg\neg\exists n \forall m > n[\Phi(\alpha)(m) = \Phi(\alpha)(n)]$, we may conclude $\forall \alpha \in (0,1]^o \neg\neg\exists q \in \mathbb{Q}[q < \alpha \wedge \forall \beta \in (0,1]^o[(q < \beta \leq \alpha \to \Phi(\alpha) =^* \Phi(\beta)]]$. □

Note that for any $\Phi : (0,1]^o \to \mathbb{N}^*$, there exists $k \in \mathbb{N}$ such that for all n, $\Phi(\hat{1})(n) \leq k$, by definition of almost natural numbers.

PROPOSITION 2.29 (**WCN**). *Let $\Phi : (0,1]^o \to \mathbb{N}^*$ be total, $k \in \mathbb{N}$ be such that for all n, $\Phi(\hat{1})(n) \leq k$, and $T_i = \{q \in \mathbb{Q} \mid \Phi(\hat{q}) =^* \underline{i}\}$, for $0 \leq i \leq k$, where $\underline{i} = \langle i, i, i, \dots \rangle \in \mathbb{N}^*$. Then*:

https://doi.org/10.1017/S1755020324000078 Published online by Cambridge University Press

(a) *if* $i < j$, $q \in T_i$ *and* $p \in T_j$, *then* $q < p$,
(b) *for each* $\alpha \in (0,1]^o$, *if* $A_\alpha \cap T_i \# \emptyset$, *then* $\Phi(\alpha) \geq \underline{i}$,
(c) *for each* $\alpha \in (0,1]^o$, *if* $\Phi(\alpha) =^* \underline{i}$, *then* $\neg(A_\alpha \cap T_i = \emptyset)$,
(d) *for each* $\alpha \in (0,1]^o$, $\neg\neg\exists i[(0 \leq i \leq k) \wedge (\Phi(\alpha) =^* \underline{i})]$.

Proof. (a) and (b) are derived by Lemma 2.27. (c) follows from Theorem 2.28, and (d) is a consequence of the definition of almost natural numbers, and the fact $\forall\alpha \in (0,1]^*(\Phi(\alpha) \leq \Phi(\hat{1}) \leq \underline{k})$. □

THEOREM 2.30 (**WCN** + **KS**). *Assume* $\Phi : (0,1]^o \to \mathbb{N}^*$. *Let* $\delta_i = inf(T_i)$, *for* $T_i s$ $0 \leq i \leq k$, *defined above, and* $E = \{\delta_i \mid 0 \leq i \leq k\}$. *For each* $\alpha \in (0,1]^o$, *define* $L_\alpha = \{\gamma \in (0,1]^o \mid \gamma < \alpha\}$. *Then*

$$\forall\alpha, \beta \in (0,1]^o[(L_\alpha \cap E = L_\beta \cap E) \to \neg\neg(\Phi(\alpha) =^* \Phi(\beta))].$$

Proof. First, observe that for $\alpha \in (0,1]^o$ and $0 \leq i \leq k$,

(a) $\delta_i \in L_\alpha \to \neg(A_\alpha \cap T_i = \emptyset)$, and
(b) $A_\alpha \cap T_i \# \emptyset \to \delta_i \in L_\alpha$

To prove the theorem, let $\alpha, \beta \in (0,1]^o$ such that $L_\alpha \cap E = L_\beta \cap E$ and $\neg(\Phi(\alpha) =^* \Phi(\beta))$ (1). we want to derive a contradiction. By Proposition 2.29(d), we have $\neg\neg\exists i[(0 \leq i \leq k) \wedge (\Phi(\alpha) =^* \underline{i})]$ (2), and $\neg\neg\exists i[(0 \leq i \leq k) \wedge (\Phi(\beta) =^* \underline{i})]$ (3). Applying the intuitionistic valid statement $\neg\neg(\varphi \wedge \psi) \leftrightarrow \neg\neg\varphi \wedge \neg\neg\psi$ to (1), (2), and (3), gives $\neg\neg(\exists i, j[(0 \leq i, j \leq k) \wedge (\Phi(\alpha) =^* \underline{i}) \wedge (\Phi(\beta) =^* \underline{j}) \wedge \neg(\Phi(\alpha) =^* \Phi(\beta))])$, which implies

$$\neg\neg(\exists i, j[(0 \leq i, j \leq k) \wedge (\Phi(\alpha) =^* \underline{i}) \wedge (\Phi(\beta) =^* \underline{j}) \wedge \neg(\underline{i} =^* \underline{j})]).$$

Now, for the sake of argument, assume

$$\psi \equiv \exists i, j[(0 \leq i, j \leq k) \wedge (\Phi(\alpha) =^* \underline{i}) \wedge (\Phi(\beta) =^* \underline{j}) \wedge \neg(\underline{i} =^* \underline{j})].$$

Either $i < j$ or $j < i$, and assume the first case. By Proposition 2.29(b), $\neg(A_\alpha \cap T_j \# \emptyset)$, i.e., $A_\alpha \cap T_j = \emptyset$. So by (a), $\neg(\delta_j \in L_\alpha)$, and by the assumption $L_\alpha \cap E = L_\beta \cap E$, we have $\neg(\delta_j \in L_\beta)$. By (b), $\neg(A_\beta \cap T_j \# \emptyset)$, that is, $A_\beta \cap T_j = \emptyset$. But it contradicts with Proposition 2.29(c), and thus $\neg\neg(\Phi(\alpha) =^* \Phi(\beta))$. By assuming ψ, we derived a contradiction. By $(\psi \to \varphi) \to (\neg\neg\psi \to \neg\neg\varphi)$, assuming $\neg\neg\psi$ yield also a contraction. So $\neg\neg(\Phi(\alpha) =^* \Phi(\beta))$. □

The theorem says that if Φ is a total constructive function from the temporal interval $(0,1]^o$ to $\mathbb{N}^*$ then there exists a *non-infinite* subset E of $[0,1]^o$ such that

$$\forall\alpha, \beta \in (0,1]^o[(L_\alpha \cap E = L_\beta \cap E) \to \neg\neg(\Phi(\alpha) =^* \Phi(\beta))].$$

As mentioned in the introduction, a moment α on the temporal continuum is a *flow* from the past, i.e., the moment α is adhered to the collection of all moments that happened before it. We think up of this flow as L_α. To construct a witness for a moment α, the occurrence of α is compared with the occurrence of *non-infinitely* many moments in E. If the results of the comparison are the same for two moments α and β, then it is not false that the witness constructed for α is also a witness for β. In this way, the Theorem 2.30 formalizes our sense of the orientation discussed in the introduction.

We believe that the following form of Theorem 2.30 is plausible to be accepted as a principle, which we name the *oriented continuity principle*, **OCP**:

Oriented continuity principle:

For every $\Phi : (0,1]^o \to \mathbb{N}^*$,
if $\forall \alpha \in (0,1]^o\ \exists \xi \in \mathbb{N}^*\ [\Phi(\alpha) =^* \xi]$ then there exists a *non-infinite* subset $E \subseteq [0,1]^o$ such that $\forall \alpha, \beta \in (0,1]^o\ [(L_\alpha \cap E = L_\beta \cap E) \to (\Phi(\alpha) =^* \Phi(\beta))]$.

§3. Topologies for continuum. What is the proper topology of the intuitive temporal continuum? Clearly, the topology must display the notion of orientation. As is emphasized in the introduction, since t sinks back to the past, the moment t is not distinguishable from moments in $(10:40, 11:30]$. Therefore, if we have a constructive method to introduce witnesses for all moments on the temporal line then the witness for the moment t is also the witness for all moments in $(10:40, 11:30]$. Thus, for a suitable topology of the temporal continuum, it seems that every total function is continuous.

As Brouwer distinguishes between the intuitive continuum and the "full continuum" of the unfinished elements of the unit segment (intuitionistic real line) (see [14, p. 74]), we do not claim that our oriented line, equipped with topologies defined below is exactly the intuitive temporal continuum based on our intuition. We believe that the only difference between the intuitive continuum and the intuitive temporal continuum is taking into account the notion of *orientation* as a new aspect of the continuum besides *inexhaustibility* and *non-discreteness*. Our oriented line and the following topologies are our suggestions for modeling the temporal continuum.

3.1. The oriented topology.

Definition 3.1 (The oriented topology on the temporal interval $(0,1]^o$). *A subset U of $(0,1]^o$ is called open if and only if for every α in U there exists a* non-infinite *set $E \subseteq [0,1]^o$ such that $S_E(\alpha) \subseteq U$, where $S_E(\alpha) = \{\beta \in (0,1]^o \mid L_\alpha \cap E = L_\beta \cap E\}$. We indicate the set of all open subsets by $\mathcal{T}_1$, and refer to $((0,1]^o, \mathcal{T}_1)$ as the oriented topological space.*

Example 3.2. *The interval $(\frac{1}{4}, \frac{1}{2}]^o = \{\alpha \in (0,1]^o \mid \hat{\frac{1}{4}} < \alpha \leq \hat{\frac{1}{2}}\}$ is open. For $\alpha \in (\frac{1}{4}, \frac{1}{2}]^o$, let $E = \{\hat{\frac{1}{4}}, \hat{\frac{1}{2}}\}$. Then $L_\alpha \cap E = \{\hat{\frac{1}{4}}\}$, and $S_E(\alpha) \subseteq (\frac{1}{4}, \frac{1}{2}]$.*

We must show that the class of all open subsets of $(0,1]^o$ is a topology. It is obvious that the empty set and $(0,1]^o$ belong to $\mathcal{T}_1$. One may easily see that the topology $\mathcal{T}_1$ is closed under arbitrary union. The next proposition shows that it is also closed under finite intersection.

Proposition 3.3. *The class of open subsets of $(0,1]^o$ is closed under finite intersection.*

Proof. Let U_1 and U_2 be open, and $\alpha \in U_1 \cap U_2$. Let the non-infinite sets E_1, E_2 be such that $S_{E_1}(\alpha) \subseteq U_1$, and $S_{E_2}(\alpha) \subseteq U_2$. It is easily seen that $S_{E_1 \cup E_2}(\alpha) \subseteq U_1 \cap U_2$. $\square$

3.2. The ordinary topology. We first define the notion of semi-metric spaces, and then introduce the *ordinary* topology based on this notion.

Definition 3.4. *Assume $\mathbb{Q}^+ = \{q \in \mathbb{Q} \mid q > 0\}$. A* semi-metric space *is a couple $(I, \mathbf{d})$, where I is a set and $\mathbf{d}$ is a relation $\mathbf{d} \subseteq I \times I \times \mathbb{Q}^+$ satisfying the following properties:*

P1. $\forall x \in I \forall q \in \mathbb{Q}^+ \mathbf{d}(x, x, q)$.
P2. $\forall x, y \in I \exists q \in \mathbb{Q}^+ \mathbf{d}(x, y, q)$.

P3. $\forall x, y \in I \forall q, p \in \mathbb{Q}^+(\mathbf{d}(x, y, q) \wedge q < p \to \mathbf{d}(x, y, p))$.
P4. $\forall x, y \in I \forall q \in \mathbb{Q}^+(\mathbf{d}(x, y, q) \leftrightarrow \mathbf{d}(y, x, q))$.
P5. $\forall x, y, z \in I \forall q, p \in \mathbb{Q}^+(\mathbf{d}(x, y, q) \wedge \mathbf{d}(y, z, p) \to \mathbf{d}(x, z, p + q))$ (triangle inequality).

The intended meaning of $\mathbf{d}(x, y, q)$ is 'the distance of x from y is less than q'. Then by this intention, all properties P1–P5 are understood clearly. For $x \in I$ and $p \in \mathbb{Q}^+$, let $S_p(x) = \{y \in I \mid \mathbf{d}(x, y, p)\}$.

Remark 3.5. *Note that two notions of metric space and semi-metric space are* classically *the same. Assume $\mathbb{R}$ is the classical real line. For a semi-metric space $(I, \mathbf{d})$, we can define a metric function $dist : I \times I \to \mathbb{R}$, such that $dist(x, y) = inf\{p \mid \mathbf{d}(x, y, p)\}$. One may easily verify that dist is a (classical) metric. Also if $(I, dist)$ is a (classical) metric space, defining $\mathbf{d} \subseteq I \times I \times \mathbb{Q}^+$ by $\mathbf{d}(x, y, q) \leftrightarrow dist(x, y) < q$, would make $(I, \mathbf{d})$ a semi-metric space.*

Proposition 3.6. *If $(I, \mathbf{d})$ is semi-metric space, then the collection of subsets $U \subseteq I$ satisfying the following property is a topology*:

$$\forall x \in U \exists p \in \mathbb{Q}^+(S_p(x) \subseteq U) \ \ (*).$$

Proof. It is clear that both sets I and the empty set satisfy $(*)$. We only need to show that the collection is closed under arbitrary union and finite intersection. Being closed under arbitrary union is trivial. We prove that it is closed under finite intersection. Assume U_1 and U_2 satisfy the property and let $x \in U_1 \cap U_2$. Since both U_1 and U_2 satisfy the property, there exist $p_1, p_2 \in \mathbb{Q}^+$ such that $S_{p_1}(x) \subseteq U_1$ and $S_{p_2}(x) \subseteq U_2$. Let $q = min(p_1, p_2)$. Then $S_q(x) \subseteq U_1 \cap U_2$. To show this, assume $y \in S_q(x)$. Then $\mathbf{d}(x, y, q)$ holds, and by (P3), we have $\mathbf{d}(x, y, p_1)$ and $\mathbf{d}(x, y, p_2)$, which implies $y \in S_{p_1}(x) \cap S_{p_2}(x) \subseteq U_1 \cap U_2$. Hence $U_1 \cap U_2$ satisfies the property. □

Proposition 3.7. *Let $\mathbf{d} \subseteq \mathbb{R}^o \times \mathbb{R}^o \times \mathbb{Q}^+$, defined by*

$$\mathbf{d}(\alpha, \beta, q) := \exists \zeta \in \mathbf{Q} \exists p \in \mathbb{Q}^+(p \leq q \wedge (\hat{\zeta} \leq \alpha, \beta \leq \hat{\zeta} + p)).[4]$$

Then $(\mathbb{R}^o, \mathbf{d})$ is a semi-metric space.

Proof. We must show that $\mathbf{d}$ satisfies properties P1–P5.

- P1. It follows from Proposition 2.17.
- P2. Let α, β be two oriented reals. There exist rational numbers p_1, p_2, q_1, q_2 such that $p_1 < \alpha < p_2, q_1 < \beta < q_2$. Let $\zeta \in \mathbf{Q}$ defined by $\zeta(n) = min(p_1, q_1)$ for each $n \in \mathbb{N}$. Then
$$\hat{\zeta} \leq \alpha, \beta \leq (\hat{\zeta} + |min(p_1, q_1)| + max(p_2, q_2)).$$
- P3. Trivial.
- P4. Trivial.
- P5. Assume $\alpha, \beta, \delta \in \mathbb{R}^o$ and for some $p, q \in \mathbb{Q}$, $\mathbf{d}(\alpha, \beta, q)$ and $\mathbf{d}(\beta, \delta, p)$ hold. Then there exist $\zeta_1, \zeta_2 \in \mathbf{Q}$, $q_1 \leq q$, and $p_1 \leq p$, such that
$$\hat{\zeta}_1 \leq \alpha, \beta \leq (\hat{\zeta}_1 + q_1)$$

[4] We use expression $a \leq x, y \leq b$ as a short abbreviation of $a \leq x \leq b \wedge a \leq y \leq b$.

https://doi.org/10.1017/S1755020324000078 Published online by Cambridge University Press

and

$$\hat{\zeta_2} \leq \beta, \delta \leq (\hat{\zeta_2} + p_1).$$

Let $\zeta(n) = min(\zeta_1(n), \zeta_2(n))$, for each n. We claim that

$$\hat{\zeta} \leq \alpha, \delta \leq (\hat{\zeta} + (p_1 + q_1)).$$

From $\hat{\zeta_1} \leq \alpha$ and $\hat{\zeta_2} \leq \alpha$, we derive $\hat{\zeta} \leq \alpha$. We also derive $\hat{\zeta} \leq \delta$ similarly. Assume an arbitrary $t \in \mathbb{N}$. The fact $\alpha \leq \hat{\zeta_1} + q_1$ implies that there exists $k \in \mathbb{N}$ such that $\alpha(t) < \hat{\zeta_1}(k) + q_1$. On the other hand, since $\hat{\zeta_1} \leq \beta$ and $\beta \leq \hat{\zeta_2} + p_1$, there exists $k' \in \mathbb{N}$ such that $\hat{\zeta_1}(k) < \hat{\zeta_2}(k') + p_1$, and consequently $\alpha(t) < \hat{\zeta_2}(k') + p_1 + q_1$. Let $k'' = max(k, k')$. Since both of $\hat{\zeta_1}, \hat{\zeta_2}$ are strictly increasing, we have $\hat{\zeta_1}(k) < \hat{\zeta_1}(k'')$, $\hat{\zeta_2}(k') < \hat{\zeta_2}(k'')$, and hence $\alpha(t) < \hat{\zeta_1}(k'') + q_1 + p_1$ and $\alpha(t) < \hat{\zeta_2}(k'') + p_1 + q_1$. Then $\alpha(t) < (min(\zeta_1(k''), \zeta_2(k'')) - \frac{1}{k''+1}) + p_1 + q_1$. It is shown that $\alpha \leq \hat{\zeta} + p_1 + q_1$. Similar argument works for $\beta \leq \hat{\zeta} + p_1 + q_1$. □

Definition 3.8 (The ordinary topology on $\mathbb{R}^o$). *The* ordinary *topology on* $\mathbb{R}^o$ *is the topology induced by the semi-metric* $(\mathbb{R}^o, \mathbf{d})$, *where* **d** *is defined above. We show the* ordinary *topological space by* $(\mathbb{R}^o, \mathcal{T}_2)$.

3.3. A consequence of OCP. We use **OCP** to prove that:

Theorem 3.9. *Every total function from* $((0,1]^o, \mathcal{T}_1)$ *to* $(\mathbb{R}^o, \mathcal{T}_2)$ *is continuous.*

Proof. Let $f : ((0,1]^o, \mathcal{T}_1) \to (\mathbb{R}^o, \mathcal{T}_2)$ be total. We prove that for each $n \in \mathbb{N}$, for every $\alpha \in (0,1]^o$, there exists $S_{E_n}(\alpha) \in \mathcal{T}_1$ such that for every $\beta \in S_{E_n}(\alpha)$, $\mathbf{d}(f(\alpha), f(\beta), 2^{-n})$. Consider rational numbers $q_i = i2^{-n}$, $0 \leq i \leq 2^n$. For each $\delta \in (0,1]^o$, define $\phi(f(\delta))(0) = 0$ and $\phi(f(\delta))(k) = i$ if and only if $q_i \leq f(\delta)(k) < q_{i+1}$. The function $\Phi(\delta) = \varphi(f(\delta))$ from $(0,1]^o$ to $\mathbb{N}^*$ is total and well-defined. By **OCP**, there exists a non-infinite subset $E \subseteq [0,1]^o$, such that for every α, β, if $\beta \in S_E(\alpha)$ then $\Phi(\alpha) =^* \Phi(\beta)$. It easily seen that for $\zeta_1(i) = q_{\Phi(\alpha)(i)} \in \mathbf{Q}$ and $\zeta_2(i) = q_{\Phi(\beta)(i)} \in \mathbf{Q}$, we have $\zeta_1 = \zeta_2$ and

$$\hat{\zeta_1} \leq f(\alpha), f(\beta) \leq (\hat{\zeta_1} + 2^{-n}),$$

i.e., $\mathbf{d}(f(\alpha), f(\beta), 2^{-n})$. □

§4. Concluding remarks. For Brouwer, constructing a total function over the continuum that is not continuous seemed implausible. The continuum is a unified whole, not composed of discrete atoms. To address this, Brouwer introduced choice sequences to model points on the continuum, not as dimensionless atoms, but rather as *halos*. The proof that every total function over the continuum is continuous is grounded in the non-atomicity modeled by choice sequences and Brouwer's continuity principle. Choice sequences are non-predetermined and unfinished entities, and the value of every total function from choice sequences to natural numbers relies on an initial segment of sequences.

We modeled points on continuum, not as atoms, but rather as *flows* from past to future. We also demonstrated that every total function over the continuum is continuous, but our assertion is based on the orientation property modeled by oriented cuts and the oriented continuity principle, **OCP**. As oriented cuts (moments) recede into the past, the value of every total function from oriented cuts to almost natural numbers relies on the experience of the oriented cut (moment) before receding into the past.

In the spirit of Brouwer and Weyl, our modeling philosophy aligns with the idea that the continuum is not separable. Yet, what distinguishes our approach is the derivation of this conclusion from another intrinsic property of the temporal continuum – its orientation.

In Brouwer's intuitionism, the *'creative subject'* constructs choice sequences based on their free will. Due to the free will of the creative subject, always only a finite segment of a choice sequence is determined. It is through this premise that Brouwer justifies his continuity principle.

In our framework, the *'knowing subject'* interacts with time, which operates independently of them. The knowing subject experiences time, and orientation imposes a restriction on this interaction, thereby justifying our oriented continuity principle.

Our temporal continuum suggests a new framework for constructive analysis. We need to investigate the following questions as further work: Is every total function on $\mathbb{R}^o$ with *one* topology (either the oriented topology or the ordinary one) continuous, or do we need a third topology? What happens to the *uniform continuity theorem*? What does the *intermediate value theorem* looks like?

Acknowledgements. We are very thankful to Dirk van Dalen, Mark van Atten, and Wim Veldman for reading a draft of this paper and for their helpful comments and suggestions. Moreover, our correspondence with Wim Veldman resulted in modifications of some crucial claims in this paper.

BIBLIOGRAPHY

[1] Ardeshir, M., & Ramezanian, R. (2009). Decidability and Specker sequences in intuitionistic mathematics. *Mathematical Logic Quarterly*, **55**(6), 637–648.

[2] Aristotle. (1999). *Physics*. Oxford: Oxford University Press. Translation by R. Waterfield.

[3] van Atten, M. (2006). *Brouwer Meets Husserl. On the Phenomenology of Choice Sequences*. Berlin: Springer.

[4] van Atten, M., van Dalen, D., & Tieszen, R. (2002). Brouwer and Weyl: The phenomenology and mathematics of the intuitive continuumm. *Philosophia Mathematica (3)*, **10**, 203–226.

[5] Brouwer, L. E. J. (1907). *On the Foundation of Mathematics*. Ph.D. Thesis, University of Amsterdam, in [6].

[6] ———. (1975). *Collected Works*, Vol. 1, ed. by Heyting, A. Amsterdam: North-Holland.

[7] Cantor, G. (1883). *Foundations of a General Theory of Manifolds: A Mathematico-Philosophical Investigation into the Theory of the Infinite*, in [11].

[8] Connes, A., & Consani, C. (2011). The hyperring of adele classes, *Journal of Number Theory*, **131**(2), 159–194.

[9] ———. (2010). From monoids to hyperstructures: In search of an absolute arithmetic. Casimir Force, Casimir Operators and the Riemann Hypothesis:

Mathematics for Innovation in Industry and Science, edited by Gerrit van Dijk and Masato Wakayama, Berlin, New York: De Gruyter, 2010, pp. 147–198. https://doi.org/10.1515/9783110226133.147.

[10] Dummett, M. (1964). Bringing about the past. *Philosophical Review*, **73**(3), 338–359.

[11] Ewald, W. (1996). ***From Kant to Hilbert: A Source Book in the Foundations of Mathematics***, Vol. 2. Oxford: Oxford Science Publications.

[12] Heath, T. L. (1956). ***The Thirteen Books of Euclids Elements***, Vol. 1. New York: Dover. Unabridged republication of the second edition from 1925.

[13] Krasner, M. (1983). A class of hyperrings and hyperfields. *International Journal of Mathematics and Mathematical Sciences*, **6**(2), 307311.

[14] Kuiper, J. J. C. (2004). IDEAS AND EXPLORATIONS, Brouwers Road to Intuitionism. Ph.D. Thesis, Utrecht University.

[15] Martel, I. (2000). Probabilistic Empiricism: in Defence of a Reichenbachian Theory of Causality and the Direction of Time. Ph.D. Thesis, University of Colorado at Boulder.

[16] Reichenbach, H. (1956). ***The Direction of Time***. Berkeley: University of California Press.

[17] Troelstra, A. S., & van Dalen, D. (1988). ***Constructivism in Mathematics: An Introduction***, Vol. 1. Amsterdam: North-Holland.

[18] Veldman, W. (1995). Some intuitionisitic variations on the notion of a finite set of natural numbers. In deSwart, H. C. M. and Bergmans, L. J. M., editors. *Perspectives on Negation*, Essays in honour of Johan J. de Iongh on his 80th birthday. Tilburg: Tilburg University Press, pp. 177–202.

[19] Weyl, H. (1994). *The Continuum: A Critical Examination of the Foundation of Analysis*. Translated by Stephen Pollard and Thomas Bole. New York: Dover Publications. Originally published in 1918.

DEPARTMENT OF MATHEMATICS
SHARIF UNIVERSITY OF TECHNOLOGY
11365-9415 TEHRAN, IRAN
E-mail: mardeshir@sharif.edu

DEPARTMENT OF ECONOMICS, UNIVERSITY OF LAUSANNE
INTERNEF
1015 LAUSANNE, SWITZERLAND
E-mail: rasoul.ramezanian@unil.ch