3
Lattices of Closed Subspaces

The main objects of this chapter are the lattices formed by closed subspaces of infinite-dimensional complex normed spaces. Our first result is the following analogue of the Fundamental Theorem of Projective Geometry: all isomorphisms of such lattices are induced by linear or conjugate-linear homeomorphisms between the corresponding normed spaces (for the finite-dimensional case this fails). This statement is closely connected to the remarkable Kakutani–Mackey theorem [31] which states that every orthomodular lattice consisting of all closed subspaces of an infinite-dimensional complex Banach space is the orthomodular lattice associated to an infinite-dimensional complex Hilbert space.

At the end, we consider the partially ordered set formed by all closed subspaces of a complex Hilbert space whose dimension and codimension both are infinite. It will be shown that every isomorphism between such partially ordered sets can be uniquely extended to an isomorphism of the lattices of closed subspaces. Using the same arguments, we obtain an analogue of Chow's theorem for connected components of the Grassmamm graphs associated to infinite-dimensional Hilbert spaces (note that for Grassmannians of infinite-dimensional vector spaces there is no result of such kind).

3.1 Linear and Conjugate-Linear Operators

We will work with semilinear maps of complex normed spaces. For this reason, we need some information on endomorphisms of the field of complex numbers. The automorphism group of this field contains the conjugate map $a \to \overline{a}$ and infinitely many other elements.

https://doi.org/10.1017/9781108800327.005 Published online by Cambridge University Press

Example 3.1 Using Zorn's lemma and [34, Chapter V, Theorem 2.8], we can show that every automorphism of a field can be extended to an automorphism of any algebraically closed extension of this field (see, for example, [45, Section 1.1]). The field $\mathbb{Q}(\sqrt{p})$, where p is a prime number, is contained in the algebraically closed field $\mathbb{C}$. Consider the automorphism of $\mathbb{Q}(\sqrt{p})$ sending every $v + w\sqrt{p}$ to $v - w\sqrt{p}$ and extend it to an automorphism of $\mathbb{C}$. Any such extension is not identity on $\mathbb{R}$, which implies that it is different from the conjugate map.

Lemma 3.2 *Every continuous endomorphism of the field $\mathbb{C}$ is identity or the conjugate map.*

Proof If σ is an endomorphism of $\mathbb{C}$, then the restriction of σ to $\mathbb{Q}$ is identity. Therefore, the restriction of σ to $\mathbb{R}$ is identity if σ is continuous. It is clear that $\sigma(\mathbf{i}) = \pm\mathbf{i}$ and we get the claim. □

A *complex normed vector space* is a complex vector space N together with a real-valued norm function $x \to ||x||$ satisfying the following conditions:

- $||x|| \geq 0$ for every $x \in N$ and $||x|| = 0$ only in the case when $x = 0$,
- $||ax|| = |a| \cdot ||x||$ for all $x \in N$ and $a \in \mathbb{C}$,
- $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in N$.

A normed vector space can be considered as a metric space, where the distance between any two vectors x and y is equal to $||x-y||$. In the case when this metric space is complete, the normed space is called a *Banach space*.

Let N and N' be complex normed spaces. A semilinear map $L : N \to N'$ is *bounded* if there is a non-negative real number a such that

$$||L(x)|| \leq a||x||$$

for all vectors $x \in N$. The smallest number a satisfying this condition is called the *norm* of L and denoted by $||L||$.

Proposition 3.3 *For every bounded semilinear map between complex normed spaces the associated endomorphism of the field $\mathbb{C}$ is identity or the conjugate map.*

This statement is a simple consequence of the following lemma (which will be also exploited in the next section).

Lemma 3.4 *If σ is an endomorphism of the field $\mathbb{C}$ such that for every sequence of complex numbers $\{a_n\}_{n\in\mathbb{N}}$ converging to zero the sequence $\{\sigma(a_n)\}_{n\in\mathbb{N}}$ is bounded, then σ is identity or the conjugate map.*

https://doi.org/10.1017/9781108800327.005 Published online by Cambridge University Press

Proof By Lemma 3.2, we need to show that σ is continuous. Since σ is additive, it is sufficient to establish that σ is continuous in zero.

Suppose that a sequence $\{a_n\}_{n\in\mathbb{N}}$ converges to zero and the same fails for the sequence $\{\sigma(a_n)\}_{n\in\mathbb{N}}$. Then $\{a_n\}_{n\in\mathbb{N}}$ contains a subsequence $\{a'_n\}_{n\in\mathbb{N}}$ such that the inequality $|\sigma(a'_n)| > a$ holds for a certain real number $a > 0$ and all natural n. In the sequence $\{a'_n\}_{n\in\mathbb{N}}$, we choose a subsequence $\{a''_n\}_{n\in\mathbb{N}}$ satisfying $na''_n \to 0$. Since σ is an endomorphism of $\mathbb{C}$, we have $\sigma(n) = n$ for every natural n. Then

$$|\sigma(na''_n)| = n|\sigma(a''_n)| > na$$

and the sequence $\{\sigma(na''_n)\}_{n\in\mathbb{N}}$ is unbounded, which contradicts our assumption. □

Remark 3.5 For a semilinear map $L : N \to N'$ the following two conditions are equivalent:

- L is continuous,
- L is bounded.

This is well known if L is linear, but we need some explanations in the general case. If L is continuous, then the associated endomorphism of $\mathbb{C}$ is continuous, i.e. it is identity or the conjugate map, and for every real $\varepsilon > 0$ there is a real $\delta > 0$ such that $||L(x)|| < \varepsilon$ if $||x|| = \delta$; since the field endomorphism associated to L preserves each real number, we have

$$||L(x)|| < \varepsilon\delta^{-1}||x||$$

for every $x \in N$ and L is bounded. Conversely, suppose that L is bounded. Then it sends every vector sequence converging to zero to a bounded sequence. As in the proof of Lemma 3.4, we show that $L(x_n) \to 0$ for every vector sequence $x_n \to 0$, i.e. L is continuous in zero. Then L is continuous by additivity.

Linear maps of normed spaces are called *linear operators*. A semilinear map between complex normed spaces is said to be a *conjugate-linear operator* if the associated endomorphism of $\mathbb{C}$ is the conjugate map.

Example 3.6 Let H be a complex Hilbert space and let $B = \{e_i\}_{i\in I}$ be an orthonormal basis of H. There is a unique conjugate-linear operator C_B which leaves fixed every vector from this basis. If J is a countable or finite subset of I and $x = \sum_{j\in J} a_j e_j$, then

$$C_B(x) = \sum_{j\in J} \overline{a}_j e_j.$$

Since $||C_B(x)|| = ||x||$ for every vector $x \in H$, the operator C_B is bounded.

https://doi.org/10.1017/9781108800327.005 Published online by Cambridge University Press

Denote by $\mathcal{L}(N)$ and $\mathcal{L}(N')$ the sets of all closed subspaces of N and N', respectively. The partially ordered sets $(\mathcal{L}(N), \subset)$ and $(\mathcal{L}(N'), \subset)$ are bounded lattices (such lattices were considered in Section 1.3 for Hilbert spaces). Every linear or conjugate-linear homeomorphism $A : N \to N'$ induces an isomorphism between these lattices and each non-zero scalar multiple of A defines the same lattice isomorphism. In the case when N and N' are Banach spaces, every invertible bounded linear or conjugate linear operator $A : N \to N'$ is a homeomorphism (if A is linear, then this follows easily from the Open Map Theorem, see [55, Corollary 2.12]; readers can check that the Open Map Theorem [55, Theorem 2.11] holds for the conjugate-linear maps).

Let H and H' be complex Hilbert spaces. Recall that for every bounded linear operator $A : H \to H'$ the adjoint linear operator $A^* : H' \to H$ satisfies

$$\langle A(x), y\rangle = \langle x, A^*(y)\rangle$$

for all $x \in H$ and $y \in H'$. Now, we consider the case when $A : H \to H'$ is a bounded conjugate-linear operator. For every vector $y \in H'$ the map

$$x \to \overline{\langle A(x), y\rangle}$$

is a linear functional on H and there is a unique vector $A^*(y) \in H$ such that

$$\overline{\langle A(x), y\rangle} = \langle x, A^*(y)\rangle$$

for all vectors $x \in H$. So, we get a conjugate-linear operator $A^* : H' \to H$ which will be called *adjoint* to A. As in the linear case, the operator A^* is bounded and $||A^*|| = ||A||$.

For every linear or conjugate-linear bounded operator $A : H \to H'$ we have $A^{**} = A$. The kernel of A^* is the orthogonal complement of the image of A. Therefore, A^* is invertible if and only if A is invertible. In this case, the operators $(A^{-1})^*$ and $(A^*)^{-1}$ are coincident. For every scalar $a \in \mathbb{C}$ we have $(aA)^* = \overline{a}A^*$ if A is linear, and $(aA)^* = aA^*$ if it is conjugate-linear.

There is the following relation between adjoint operators and lattice isomorphisms.

Proposition 3.7 *For every invertible bounded linear or conjugate-linear operator $A : H \to H'$ the map of $\mathcal{L}(H)$ to $\mathcal{L}(H')$ defined as*

$$X \to A(X^\perp)^\perp$$

for every $X \in \mathcal{L}(H)$ is the lattice isomorphism induced by the operator $(A^*)^{-1}$.

Proof If $x, y \in H$, then $\langle y, x\rangle = \langle A^{-1}A(y), x\rangle$ is equal to

$$\langle A(y), (A^{-1})^*(x)\rangle \quad \text{or} \quad \overline{\langle A(y), (A^{-1})^*(x)\rangle}.$$

https://doi.org/10.1017/9781108800327.005 Published online by Cambridge University Press

In other words, y is orthogonal to x if and only if $A(y)$ is orthogonal to $(A^{-1})^*(x)$. This implies that

$$A(X^{\perp})^{\perp} = (A^{-1})^*(X) = (A^*)^{-1}(X)$$

for every closed subspace $X \subset H$. □

3.2 Lattice Isomorphisms

We prove the following analogue of the Fundamental Theorem of Projective Geometry for the lattices of closed subspaces of infinite-dimensional complex normed spaces.

Theorem 3.8 (Mackey [35] and Kakutani and Mackey [31]) *Let N and N' be infinite-dimensional complex normed spaces. Then every isomorphism between the lattices $\mathcal{L}(N)$ and $\mathcal{L}(N')$ is induced by a linear or conjugate-linear homeomorphism $A : N \to N'$ and any other operator inducing this lattice isomorphism is a scalar multiple of A.*

Remark 3.9 For finite-dimensional complex normed spaces this statement fails. Indeed, if N is a complex normed space of finite dimension, then $\mathcal{L}(N)$ consists of all subspaces of N (since every finite-dimensional subspace is closed) and some automorphisms of $\mathcal{L}(N)$ are induced by unbounded semilinear automorphisms of N, i.e. semilinear automorphisms associated to non-continuous automorphisms of the field $\mathbb{C}$.

Let f be an isomorphism between the lattices $\mathcal{L}(N)$ and $\mathcal{L}(N')$, where N and N' are complex normed spaces. Then $f(\mathcal{G}_1(N)) = \mathcal{G}_1(N')$ and the restriction of f to $\mathcal{G}_1(N)$ is an isomorphism between the projective spaces Π_N and $\Pi_{N'}$, i.e. this restriction is induced by a semilinear isomorphism $L : N \to N'$. It is easy to see that $f(X) = L(X)$ for every $X \in \mathcal{L}(N)$.

Now, we suppose that our normed spaces are infinite-dimensional and show that L is linear or conjugate-linear. After that we establish that L is a homeomorphism.

Lemma 3.10 *If N is infinite-dimensional, then there exist a linearly independent set of vectors $\{x_n\}_{n\in\mathbb{N}}$ and a sequence of bounded linear functionals $\{v_n\}_{n\in\mathbb{N}}$ on N such that*

$$v_i(x_j) = \delta_{ij}. \tag{3.1}$$

Proof We take any closed hyperplane $H \subset N$ and a vector $x_1 \notin H$. Consider the bounded linear functional v_1 such that $v_1(tx_1 + y) = t$ for every vector

https://doi.org/10.1017/9781108800327.005 Published online by Cambridge University Press

$y \in H$ and every scalar t. Suppose that (3.1) holds for linearly independent vectors $x_1, \ldots, x_n$ and bounded linear functionals $v_1, \ldots, v_n$. Since N is infinite-dimensional, there are a bounded linear functional v_{n+1} whose kernel contains all x_i and a vector x'_{n+1} such that $v_{n+1}(x'_{n+1}) = 1$ [55, Theorem 3.5]. We define

$$x_{n+1} = x'_{n+1} - \sum_{i=1}^{n} v_i(x'_{n+1})x_i.$$

Then $v_{n+1}(x_{n+1}) = 1$ and $v_i(x_{n+1}) = 0$ for all $i \leq n$. □

Lemma 3.11 *If N is infinite-dimensional, then it contains a linearly independent subset $\{x_n\}_{n\in\mathbb{N}}$ satisfying the following condition: for every bounded sequence of scalars $\{a_n\}_{n\in\mathbb{N}}$ there is a bounded linear functional v on N such that*

$$v(x_n) = a_n$$

for every natural n.

Proof Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ be as in the previous lemma. We can assume that

$$\|v_n\| = 1/2^n \qquad \text{for all } \ n \in \mathbb{N}$$

(for every natural n there is a scalar b_n such that $\|b_n v_n\| = 1/2^n$ and we take a scalar multiple x'_n of x_n satisfying $b_n v_n(x'_n) = 1$).

Let $\{a_n\}_{n\in\mathbb{N}}$ be a bounded sequence of scalars and $a = \sup |a_i|$. Denote by X the subspace formed by all linear combinations of vectors from the sequence $\{x_n\}_{n\in\mathbb{N}}$. For every vector

$$x = v_1(x)x_1 + \cdots + v_n(x)x_n \in X$$

we define

$$v(x) := v_1(x)a_1 + \cdots + v_n(x)a_n.$$

Then

$$|v(x)| \leq |a_1| \cdot \|v_1\| \cdot \|x\| + \cdots + |a_n| \cdot \|v_n\| \cdot \|x\|$$

$$\leq a\|x\|(1/2 + \cdots + 1/2^n) < a\|x\|.$$

This means that v is a bounded linear functional on X. By [55, Theorem 3.6], v can be extended to a bounded linear functional of N. □

The following lemma is crucial in our proof.

Lemma 3.12 (Kakutani and Mackey [31]) *Suppose that N and N' are infinite-dimensional. If a semilinear isomorphism $L : N \to N'$ sends closed hyperplanes to closed hyperplanes, then L is linear or conjugate-linear.*

Proof Let σ be the automorphism of $\mathbb{C}$ associated to L. Let also $\{x_n\}_{n\in\mathbb{N}}$ be a subset of N with the property described in the previous lemma. By Lemma 3.4, we need to show that for every sequence of complex numbers $\{a_n\}_{n\in\mathbb{N}}$ converging to zero the sequence $\{\sigma(a_n)\}_{n\in\mathbb{N}}$ is bounded.

If the latter sequence is unbounded, then $\{a_n\}_{n\in\mathbb{N}}$ contains a subsequence $\{b_n\}_{n\in\mathbb{N}}$ such that

$$|\sigma(b_n)| \geq n||L(x_n)|| \tag{3.2}$$

for all natural n. By Lemma 3.11, there is a bounded linear functional v on N satisfying $v(x_n) = b_n$ for every n. We take any vector $z \in N$ such that $v(z) = 1$. Then $x_n = y_n + b_n z$, where $y_n \in \operatorname{Ker} v$. We have

$$L(x_n)/\sigma(b_n) = L(y_n/b_n) + L(z)$$

and (3.2) implies that

$$L(x_n)/\sigma(b_n) \to 0 \ \text{ and } \ L(-y_n/b_n) \to L(z).$$

The latter means that $L(z)$ belongs to the closure of $L(\operatorname{Ker} v)$. On the other hand, $\operatorname{Ker} v$ is a closed hyperplane and the same holds for $L(\operatorname{Ker} v)$ by our assumption. Therefore, $L(z)$ belongs to $L(\operatorname{Ker} v)$. Since L is bijective, we get $z \in \operatorname{Ker} v$, which contradicts $v(z) = 1$. □

Proof of Theorem 3.8 It was noted above that every isomorphism between the lattices $\mathcal{L}(N)$ and $\mathcal{L}(N')$ is induced by a semilinear isomorphism $L : N \to N'$. Then L and L^{-1} send closed hyperplanes to closed hyperplanes and Lemma 3.12 implies that L is linear or conjugate-linear.

Let v be a non-zero bounded linear functional on N'. Then $\operatorname{Ker} v$ is a closed hyperplane of N' and $S = L^{-1}(\operatorname{Ker} v)$ is a closed hyperplane of N. For every closed hyperplane of N there is a bounded linear functional whose kernel coincides with this hyperplane. Consider a bounded linear functional w on N such that $\operatorname{Ker} w = S$ and fix $z \in N$ satisfying $w(z) = 1$. Every vector $x \in N$ can be presented as the sum $x = y + w(x)z$, where $y \in S$. Then we have

$$v(L(x)) = v(L(y)) + v(L(w(x)z)) = v(L(w(x)z))$$

(since $L(y) \in \operatorname{Ker} v$) and

$$v(L(x)) = w(x)v(L(z)) \ \text{ or } \ v(L(x)) = \overline{w(x)}v(L(z)) \tag{3.3}$$

(if L is linear or conjugate-linear, respectively).

https://doi.org/10.1017/9781108800327.005 Published online by Cambridge University Press

Let X be a bounded subset of N. Then $w(X)$ is a bounded subset of $\mathbb{C}$ and (3.3) implies that the same holds for $v(L(X))$. Since v is taken arbitrarily, the set $v(L(X))$ is bounded for every bounded linear functional v on N', i.e. $L(X)$ is weakly bounded in N'. In a normed space, every weakly bounded subset is bounded [55, Theorem 3.18]. So, L transfers bounded subsets to bounded subsets, which means that L is bounded.

Similarly, we show that L^{-1} is bounded. □

Remark 3.13 Theorem 3.8 was first proved by Mackey [35] for the lattices of closed subspaces of infinite-dimensional real normed spaces. Lemma 3.12 was obtained in [31]; it shows that the arguments given in [35] work for the complex case. See also [22].

Corollary 3.14 *If H and H' are infinite-dimensional complex Hilbert spaces, then for every anti-isomorphism f of $\mathcal{L}(H)$ to $\mathcal{L}(H')$ there is an invertible bounded linear or conjugate-linear operator $A : H \to H'$ such that*

$$f(X) = A(X)^{\perp}$$

for every $X \in \mathcal{L}(H)$ and any other operator inducing this anti-isomorphism is a scalar multiple of A.

Proof We apply Theorem 3.8 to the composition of f and the orthocomplementation map. □

The algebra of all bounded linear operators on a normed space N is denoted by $\mathcal{B}(N)$. All ring isomorphisms between the algebras of linear operators on real normed spaces were described by Eidelheit [20]. Following [22] we combine the methods of [20] together with the arguments used to prove Theorem 3.8 and get a simple proof of the complex version of Eidelheit's theorem.

Theorem 3.15 (Arnold [2]) *Let N and N' be infinite-dimensional complex normed spaces. For every ring isomorphism $f : \mathcal{B}(N) \to \mathcal{B}(N')$ there is an invertible bounded linear or conjugate-linear operator $L : N \to N'$ such that*

$$f(A) = LAL^{-1} \tag{3.4}$$

for every $A \in \mathcal{B}(N)$.

Proof Let P be a rank one idempotent of $\mathcal{B}(N)$ (see Section 1.3). Then $f(P)$ is an idempotent of $\mathcal{B}(N')$. We fix non-zero vectors x_0 and x'_0 belonging to the images of P and $f(P)$, respectively. For any vector $x \in N$ there is $A \in \mathcal{B}(N)$ satisfying $x = A(x_0)$ and we set

$$L(x) = f(A)[x'_0].$$

If $B \in \mathcal{B}(N)$ and $B(x_0) = x$, then AP and BP are rank one operators sending x_0 to x. We have $AP = BP$, which implies that

$$f(A)f(P) = f(B)f(P) \quad \text{and} \quad f(A)[x_0'] = f(B)[x_0'].$$

Therefore, the map $L : N \to N'$ is well defined. It is easy to see that this map is additive and bijective.

If $A, B \in \mathcal{B}(N)$ and $B(x_0) = x$, then

$$LA(x) = LAB(x_0) = f(AB)[x_0'] = f(A)f(B)[x_0'] = f(A)L(x),$$

i.e. $LA(x) = f(A)L(x)$ for every $x \in N$, which means that the equality (3.4) holds for all $A \in \mathcal{B}(N)$.

The centres of $\mathcal{B}(N)$ and $\mathcal{B}(N')$ are formed by all scalar multiples of the identity transformations of N and N', respectively. We have

$$f(a\,\mathrm{Id}_N) = \sigma(a)\mathrm{Id}_{N'},$$

where σ is an automorphism of the field $\mathbb{C}$. Since

$$f(aA) = f(a\,\mathrm{Id}_N)f(A) = \sigma(a)f(A)$$

for every $a \in \mathbb{C}$, the map L is σ-linear.

For every closed hyperplane $S \subset N$ there is non-zero $A \in \mathcal{B}(N)$ whose kernel coincides with S. The kernel of $f(A)$ contains $L(S)$. If the hyperplane $L(S)$ is not closed, then its closure coincides with N' and $f(A) = 0$ by continuity. The latter is impossible and $L(S)$ is closed. Similarly, we show that L^{-1} sends closed hyperplanes to closed hyperplanes. Then L is linear or conjugate-linear by Lemma 3.12. As in the proof of Theorem 3.8, we establish that L is bounded. □

3.3 Kakutani–Mackey Theorem

In this section, we consider the lattice $\mathcal{L}(B)$ formed by closed subspaces of an infinite-dimensional complex Banach space B. We show that this lattice is orthomodular only in the case when it is the lattice of closed subspaces of a complex Hilbert space.

Theorem 3.16 (Kakutani and Mackey [31]) *Let B be an infinite-dimensional complex Banach space. Suppose that there is a bijective transformation $X \to X^{\perp}$ of the lattice $\mathcal{L}(B)$ satisfying the following conditions for any $X, Y \in \mathcal{L}(B)$:*

(1) *the inclusion $X \subset Y$ implies that $Y^{\perp} \subset X^{\perp}$,*
(2) *$X^{\perp\perp} = X$,*

https://doi.org/10.1017/9781108800327.005 Published online by Cambridge University Press

(3) $X \cap X^{\perp} = 0$.

Then there is an inner product $B \times B \to \mathbb{C}$ such that the following assertions are fulfilled:

- *The vector space B together with this inner product is a complex Hilbert space.*
- *The identity transformation of B is an invertible bounded linear operator of the Banach space to the Hilbert space, i.e. a subspace of B is closed in the Banach space if and only if it is closed in the Hilbert space*[1].
- *For every $X \in \mathcal{L}(B)$ the subspace $X^{\perp}$ is the orthogonal complement of X with respect to the inner product.*

Proof Consider the Banach space B^* formed by all bounded linear functionals on B [55, Theorem 4.1]. For every closed subspace $X \subset B$ we denote by X^0 the annihilator of $X^{\perp}$ in N^*, i.e. the set of all bounded linear functionals $v : B \to \mathbb{C}$ satisfying $v(X^{\perp}) = 0$. This is a closed subspace of B^*. If P is a 1-dimensional subspace of B, then (1) shows that $P^{\perp}$ is a closed hyperplane of B and P^0 is a 1-dimensional subspace of B^*. Consider the map of $\mathcal{G}_1(B)$ to $\mathcal{G}_1(B^*)$ which sends every 1-dimensional subspace P to P^0. It follows from (1) that this is an isomorphism between the projective spaces Π_B and Π_{B^*}, i.e. there is a semilinear isomorphism $L : B \to B^*$ such that $L(P) = P^0$ for every $P \in \mathcal{G}_1(B)$.

Let H be a closed hyperplane of B. Then $H = P^{\perp}$ for a certain $P \in \mathcal{G}_1(B)$. By (1) and (2), $Q \in \mathcal{G}_1(B)$ is contained in H if and only if $P \subset Q^{\perp}$; in other words, the kernel of every $v \in L(Q)$ contains P. Then $L(H)$ consists of all $v \in B^*$ satisfying $v(P) = 0$ (since $Q \in \mathcal{G}_1(B)$ is contained in H if and only if $L(Q)$ is contained in $L(H)$). Therefore, $L(H)$ is a closed hyperplane of B^*.

So, L sends closed hyperplanes to closed hyperplanes, and by Lemma 3.12, L is linear or conjugate-linear. As in the proof of Theorem 3.8, we show that the operator $L^{-1} : B^* \to B$ is bounded. This means that L is bounded (it was noted above that the Open Map Theorem [55, Theorem 2.11] holds also for conjugate-linear maps).

Suppose that L is linear. Let x and y be linearly independent vectors of B. We set $l = L(x)$ and $s = L(y)$. Then

$$L(x + ay) = l + as.$$

By (3), we have $X \cap X^{\perp} = 0$ for every $X \in \mathcal{L}(B)$. This implies that each of the

[1] We cannot state that the norm related to the inner product coincides with the primordial norm, but these norms define the same topology on B.

scalars

$$l(x), \ s(y), \ (l + as)(x + ay)$$

is non-zero. On the other hand,

$$(l + as)(x + ay) = l(x) + a(l(y) + s(x)) + a^2 s(y)$$

and the equation

$$l(x) + a(l(y) + s(x)) + a^2 s(y) = 0$$

has a solution for a. We get a contradiction. Therefore, L is conjugate-linear.

For all vectors $x, y \in B$ we set

$$\langle x, y\rangle = l(x), \quad \text{where} \ \ l = L(y).$$

Then the product $\langle\cdot,\cdot\rangle$ is linear in the first variable and conjugate-linear in the second. The condition (3) guarantees that $\langle x, x\rangle$ is non-zero for every non-zero vector $x \in B$. Since L can be replaced by any non-zero scalar multiple, we assume that for a certain vector $x_0 \in B$ the scalar $\langle x_0, x_0\rangle$ is a positive real number.

It follows from (1) that for any two vectors $x, y \in B$ we have $\langle x, y\rangle = 0$ if and only if $\langle y, x\rangle = 0$. Suppose that $\langle x, y\rangle$ is non-zero. We choose non-zero scalars $a, b \in \mathbb{C}$ such that

$$a\langle x, x\rangle + \langle x, y\rangle = 0 = b\langle y, y\rangle + \langle x, y\rangle. \tag{3.5}$$

Then $\langle x, \overline{a}x + y\rangle = 0$, which implies that $\langle\overline{a}x + y, x\rangle = 0$ and $\overline{\langle\overline{a}x + y, x\rangle} = 0$, i.e.

$$a\overline{\langle x, x\rangle} + \overline{\langle y, x\rangle} = 0. \tag{3.6}$$

Similarly, we obtain that

$$b\overline{\langle y, y\rangle} + \overline{\langle y, x\rangle} = 0. \tag{3.7}$$

Using (3.5), (3.6) and (3.7), we establish that

$$\overline{\langle x, x\rangle} : \langle x, x\rangle = \overline{\langle y, y\rangle} : \langle y, y\rangle.$$

In other words, for any two vectors $x, y \in B$ satisfying $\langle x, y\rangle \neq 0$ the scalar $\langle x, x\rangle$ is real if and only if $\langle y, y\rangle$ is real. Recall that there is non-zero $x_0 \in B$ such that $\langle x_0, x_0\rangle$ is real. Then $\langle x, x\rangle$ is real if $\langle x_0, x\rangle$ is non-zero. In the case when $\langle x_0, x\rangle = 0$, we take any vector $y \in B$ such that $\langle x_0, y\rangle$ and $\langle x, y\rangle$ both are non-zero (for example, $y = x_0 + x$ is as required). So, for every non-zero vector $x \in B$ the scalar $\langle x, x\rangle$ is a non-zero real number. Then (3.5) and (3.6) imply that

$$\langle x, y\rangle = \overline{\langle y, x\rangle}.$$

https://doi.org/10.1017/9781108800327.005 Published online by Cambridge University Press

Now, we show that $\langle x, x\rangle$ is a positive real number for every non-zero vector $x \in B$. If $x = sx_0$ for a certain scalar s, then $\langle x, x\rangle = |s|^2\langle x_0, x_0\rangle > 0$. In the case when x and x_0 are linearly independent, we consider the real function

$$h(t) = \langle tx + (1-t)x_0, tx + (1-t)x_0\rangle$$

defined on the segment $[0, 1]$. The function is continuous (because L is bounded) and $h(t)$ is non-zero for every $t \in [0, 1]$. Since $h(0) > 0$, we have always $h(t) > 0$.

So, $\langle\cdot,\cdot\rangle$ is an inner product on B. Using the fact that L is bounded, we show that the identity transformation of B is an invertible bounded linear operator of the Banach space to the normed vector space with respect to the inner product $\langle\cdot,\cdot\rangle$. This guarantees that the norm defined by the inner product is complete. □

3.4 Extensions of Isomorphisms

For an infinite-dimensional complex Hilbert space H we denote by $\mathcal{G}_\infty(H)$ the set of all closed subspaces of H whose dimension and codimension both are infinite. Note that the partially ordered set $(\mathcal{G}_\infty(H), \subset)$ is not a lattice.

Theorem 3.17 (Pankov [42, 43]) *Let H and H' be infinite-dimensional complex Hilbert spaces. Then every isomorphism between the partially ordered sets $(\mathcal{G}_\infty(H), \subset)$ and $(\mathcal{G}_\infty(H'), \subset)$ can be uniquely extended to an isomorphism between the lattices $\mathcal{L}(H)$ and $\mathcal{L}(H')$.*

Since the orthocomplementation map sends $\mathcal{G}_\infty(H)$ to itself, Theorem 3.17 implies the following.

Corollary 3.18 *If H and H' are as in Theorem* 3.17*, then every anti-isomorphism of $(\mathcal{G}_\infty(H), \subset)$ to $(\mathcal{G}_\infty(H'), \subset)$ can be uniquely extended to an anti-isomorphism between the lattices $\mathcal{L}(H)$ and $\mathcal{L}(H')$.*

Let f be an isomorphism of the partially ordered set $\mathcal{G}_\infty(H)$ to the partially ordered set $\mathcal{G}_\infty(H')$.

Lemma 3.19 *For every $X \in \mathcal{G}_\infty(H)$ there is an invertible bounded linear or conjugate-linear operator $A_X : X \to f(X)$ such that*

$$f(Y) = A_X(Y)$$

for every $Y \in \mathcal{G}_\infty(H)$ contained in X.

https://doi.org/10.1017/9781108800327.005 Published online by Cambridge University Press

Proof Let $\mathcal{X}$ be the set of all elements of $\mathcal{G}_\infty(H)$ contained in X. Then $f(\mathcal{X})$ consists of all elements of $\mathcal{G}_\infty(H')$ contained in $X' = f(X)$. We consider the closed subspaces X and X' as Hilbert spaces and write $Y^\perp$ and $Y'^\perp$ for the orthogonal complements of $Y \subset X$ and $Y' \subset X'$ in these Hilbert spaces. Let $\mathcal{Y}$ and $\mathcal{Y}'$ be the sets formed by all closed subspaces of infinite codimension in X and X', respectively. Then

$$Y \in \mathcal{Y} \iff Y^\perp \in \mathcal{X} \text{ and } Y' \in \mathcal{Y}' \iff Y'^\perp \in f(\mathcal{X}).$$

Denote by g the bijection sending every $Y \in \mathcal{Y}$ to $f(Y^\perp)^\perp$. This is an isomorphism between the partially ordered sets $(\mathcal{Y}, \subset)$ and $(\mathcal{Y}', \subset)$. The restriction of g to $\mathcal{G}_1(X)$ is an isomorphism of Π_X to $\Pi_{X'}$ and there is a semilinear isomorphism $L : X \to X'$ such that $g(Y) = L(Y)$ for every $Y \in \mathcal{G}_1(X)$. The same holds for all $Y \in \mathcal{Y}$ (since g is an isomorphism of partially ordered sets). Also, for every $Y \in \mathcal{Y}$ the lattice $\mathcal{L}(Y)$ is contained in $\mathcal{Y}$ and the restriction of g to this lattice is an isomorphism to the lattice $\mathcal{L}(g(Y))$. Theorem 3.8 implies that L is bounded on any infinite-dimensional subspace $Y \in \mathcal{Y}$. Show that L is bounded.

We take any orthogonal $Y, Z \in \mathcal{Y}$ such that $X = Y \oplus Z$. If a sequence $\{y_i + z_i\}_{i\in\mathbb{N}}$ converges to $y_0 + z_0$ and $y_i \in Y$, $z_i \in Z$ for all $i = 0, 1, \dots$, then the sequences $\{y_i\}_{i\in\mathbb{N}}$ and $\{z_i\}_{i\in\mathbb{N}}$ converge to y_0 and z_0, respectively. Since L is bounded on Y and Z, the sequences $\{L(y_i)\}_{i\in\mathbb{N}}$ and $\{L(z_i)\}_{i\in\mathbb{N}}$ converge to $L(y_0)$ and $L(z_0)$, respectively. This means that $\{L(y_i + z_i)\}_{i\in\mathbb{N}}$ converges to $L(y_0 + z_0)$. Therefore, L is continuous and, consequently, bounded.

So, $L : X \to X'$ is an invertible bounded linear or conjugate-linear operator. We have

$$f(Y) = L(Y^\perp)^\perp$$

for every $Y \in \mathcal{X}$. Proposition 3.7 shows that the operator $A_X = (L^*)^{-1}$ satisfies the required condition. □

Lemma 3.20 *Let X and Y be elements of $\mathcal{G}_\infty(H)$ satisfying*

$$\dim(X \cap Y) < \infty.$$

Then there exists $Z \in \mathcal{G}_\infty(H)$ such that $X \cap Z$ and $Y \cap Z$ are elements of $\mathcal{G}_\infty(H)$ containing $X \cap Y$.

Proof Let X' and Y' be the orthogonal complements of $X \cap Y$ in X and Y, respectively. The subspaces X' and Y' both are infinite-dimensional and we choose inductively a sequence of mutually orthogonal vectors $\{x_n, x'_n, y_n, y'_n\}_{n\in\mathbb{N}}$ such that

$$x_n, x'_n \in X' \text{ and } y_n, y'_n \in Y'$$

https://doi.org/10.1017/9781108800327.005 Published online by Cambridge University Press

for every $n \in \mathbb{N}$. Denote by Z' the closed subspace spanned by $\{x'_n, y'_n\}_{n\in\mathbb{N}}$. The subspace

$$Z = (X \cap Y) + Z'$$

is as required. □

Proof of Theorem 3.17 For any finite-dimensional subspace $S \subset H$ we take any $X \in \mathcal{G}_\infty(H)$ containing S and set

$$g(S) = A_X(S).$$

We need to show that the definition of $g(S)$ does not depend on the choice of $X \in \mathcal{G}_\infty(H)$.

Suppose that S is contained in $X \in \mathcal{G}_\infty(H)$ and $Y \in \mathcal{G}_\infty(H)$. In the case when $X \cap Y$ is an element of $\mathcal{G}_\infty(H)$, take $X', Y' \in \mathcal{G}_\infty(H)$ contained in $X \cap Y$ and such that $X' \cap Y' = S$. Then

$$A_X(S) = A_X(X') \cap A_X(Y') = f(X') \cap f(Y') = A_Y(X') \cap A_Y(Y') = A_Y(S).$$

If $X \cap Y$ is finite-dimensional, then, by Lemma 3.20, there is $Z \in \mathcal{G}_\infty(H)$ such that $X \cap Z$ and $Y \cap Z$ are elements of $\mathcal{G}_\infty(H)$ containing $X \cap Y$. Applying the above arguments to the pairs X, Z and Y, Z, we establish that

$$A_X(S) = A_Z(S) = A_Y(S).$$

The map g is an isomorphism between the lattices $\mathcal{L}_{\text{fin}}(H)$ and $\mathcal{L}_{\text{fin}}(H')$, hence it is induced by a semilinear isomorphism $A : H \to H'$. For every $X \in \mathcal{G}_\infty(H)$ we have $f(X) = A(X)$ and the restriction of A to X is a scalar multiple of A_X. Since each A_X is bounded and H can be presented as the sum of two orthogonal elements of $\mathcal{G}_\infty(H)$, the operator A is bounded, i.e. it is an invertible bounded linear or conjugate-linear operator. □

Remark 3.21 The above proof is an essential modification of the proof given in [42, 43].

Remark 3.22 Let H be a complex Hilbert space and let $\mathcal{I}(H)$ be the set of all idempotents of the algebra $\mathcal{B}(H)$. The set $\mathcal{I}(H)$ is partially ordered as follows: for $P, Q \in \mathcal{I}(H)$ we have $P \leq Q$ if

$$\text{Im}(P) \subset \text{Im}(Q) \text{ and } \text{Ker}(Q) \subset \text{Ker}(P).$$

Since every closed subspace $X \subset H$ can be identified with the projection P_X, the lattice $\mathcal{L}(H)$ is contained in this partially ordered set. Ovchinikov [41] proved that every automorphism of the partially ordered set $\mathcal{I}(H)$ is of type

$$P \to APA^{-1} \text{ or } P \to AP^*A^{-1},$$

https://doi.org/10.1017/9781108800327.005 Published online by Cambridge University Press

where A is an invertible bounded linear or conjugate-linear operator on H. Now, we suppose that H is infinite-dimensional and consider the partially ordered set $\mathcal{I}_\infty(H)$ formed by all idempotents from $\mathcal{I}(H)$ whose image and kernel both are infinite-dimensional. Plevnik [51] showed that every automorphism of this partially ordered set can be uniquely extended to an automorphism of the partially ordered set $\mathcal{I}(H)$.

For an infinite-dimensional complex Hilbert space H we denote by $\Gamma_\infty(H)$ the graph whose vertex set is $\mathcal{G}_\infty(H)$ and whose edges are pairs of adjacent elements $X, Y \in \mathcal{G}_\infty(H)$, i.e. such that $X \cap Y$ is a hyperplane in both X and Y. This graph is not connected. A connected component containing $X \in \mathcal{G}_\infty(H)$ consists of all $Y \in \mathcal{G}_\infty(H)$ satisfying

$$\dim X/(X \cap Y) = \dim Y/(X \cap Y) < \infty.$$

The restriction of every automorphism or anti-automorphism of the lattice $\mathcal{L}(H)$ to $\mathcal{G}_\infty(H)$ is an automorphism of $\Gamma_\infty(H)$, but there are automorphisms of $\Gamma_\infty(H)$ which cannot be extended to automorphisms or anti-automorphisms of $\mathcal{L}(H)$ (a simple modification of Example 2.18).

Let C be a connected component of $\Gamma_\infty(H)$. As in Chapter 2, we denote by $C_\pm$ the set of all $X \in \mathcal{G}_\infty(H)$ such that X is a subspace of finite codimension in a certain element of C or X contains an element of C as a subspace of finite codimension. Note that $(C_\pm, \subset)$ is an unbounded lattice.

Theorem 3.23 *Let C and C' be connected components of the graph $\Gamma_\infty(H)$. If an automorphism of $\Gamma_\infty(H)$ sends C to C', then the restriction of this automorphism to C can be uniquely extended to an isomorphism or anti-isomorphism of $(C_\pm, \subset)$ to $(C'_\pm, \subset)$. Every automorphism or anti-isomorphism between these lattices can be uniquely extended to an automorphism or, respectively, an anti-automorphism of the lattice $\mathcal{L}(H)$.*

The proof of the first part of Theorem 3.23 is similar to the proof of Theorem 2.19. Using arguments from the proof of Theorem 3.17, we establish that every isomorphism of $(C_\pm, \subset)$ to $(C'_\pm, \subset)$ is uniquely extendable to an automorphism of $\mathcal{L}(H)$. The composition of every anti-isomorphism f of $(C_\pm, \subset)$ to $(C'_\pm, \subset)$ and the orthocomplementation is an isomorphism of $(C_\pm, \subset)$ to $(C''_\pm, \subset)$, where C'' is the connected component of the graph $\Gamma_\infty(H)$ formed by the orthogonal complements of elements from $C'_\pm$. This composition can be uniquely extended to an automorphism of $\mathcal{L}(H)$, which implies that f is uniquely extendable to an anti-automorphism of $\mathcal{L}(H)$.

https://doi.org/10.1017/9781108800327.005 Published online by Cambridge University Press