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On the canonical projection of the third dual of a Banach space onto the first dual

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For a Banach space B let P denote the canonical projection of the third dual space of B onto the embedding of the first dual into the third. It is shown that if $B = l_1$ then $\|I-P\| = 2$. This fact shows to be mistaken a current belief in a statement which is equivalent to the statement that for all Banach spaces B the operator $I - P$ is of norm one.

For a real or complex Banach space $B = B^{(0)}$ let $B^{(n)}$ denote the nth dual space of B , and $Q_n : B^{(n)} \to B^{(n+2)}$ the canonical embedding. Then $P = Q_1 Q_0^* : B^{(3)} \to B^{(3)}$ is a projection of $B^{(3)}$ onto $Q_1 B^{(1)}$ and the kernel of P is the annihilator $\left(Q_0 B\right)^\perp$ of $Q_0 B$ in $B^{(3)}$. The projection P was first considered, though in different phraseology, by Dixmier [2]. Let I denote the identity operator of $B^{(3)}$ and for $x^{(2)} \in B^{(2)}$ (the superscript indicating the space to which the element belongs), let

$$d\left(x^{(2)}, Q_0 B\right) = \inf\left\{\|x^{(2)}-y^{(2)}\| : y^{(2)} \in Q_0 B\right\} .$$

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Some simple properties of the operators P and Q_n will be stated as:

PROPOSITION 1. *(i)* $\|I-P\| = \|Q_2-Q_0^{**}\|$.

(ii) For all $x^{(2)} \in B^{(2)}$,

$$\left\|\left(Q_2-Q_0^{**}\right)x^{(2)}\right\| \geq d\left(x^{(2)}, Q_0B\right) .$$

(iii) For any Banach space B ,

$$\|Q_2-Q_0^{**}\| = 1$$

if and only if, for all $x^{(2)} \in B^{(2)}$,

$$\left\|\left(Q_2-Q_0^{**}\right)\left(x^{(2)}\right)\right\| = d\left(x^{(2)}, Q_0B\right) . \tag{1}$$

Proof. In the interests of accuracy the brief proof will be included. For each $x^{(2)} \in B^{(2)}$ and $y^{(3)} \in B^{(3)}$,

$$\left[\left(Q_2-Q_0^{**}\right)x^{(2)}\right]\left(y^{(3)}\right) = \left((I-P)y^{(3)}\right)\left(x^{(2)}\right). \tag{2}$$

The statement *(i)* follows immediately from (2). It also follows from (2) that

$$\left\|\left(Q_2-Q_0^{**}\right)x^{(2)}\right\| \geq \sup\left\{y^{(3)}\left(x^{(2)}\right) : y^{(3)} \in \left(Q_0B\right)^{\perp}, \|y^{(3)}\| = 1\right\} . \tag{3}$$

The term on the right of (3) equals $d\left(x^{(2)}, Q_0B\right)$. This proves *(ii)*.

The operator $Q_2 - Q_0^{**}$ annihilates Q_0B and it follows that

$$\left\|\left(Q_2-Q_0^{**}\right)x^{(2)}\right\| \leq \|Q_2-Q_0^{**}\| d\left(x^{(2)}, Q_0B\right) .$$

Therefore, by *(ii)*, the first condition of *(iii)* implies the second. The second implies the first, trivially.

There appears in [1] (and in earlier editions of the book) the incorrect statement that the equation (1) holds always, in any Banach space. An attempt to determine whether the statement was true or false was eventually directed by the following.

PROPOSITION 2. *If a Banach space* B' *is either a subspace or a* *quotient space of the space* B *then (with the natural notation)*

$$\|I-P'\| \le \|I-P\| .$$

Proof. If B' is a quotient space of B and $\pi : B \to B'$ is the quotient mapping then the third conjugate mapping $\pi^{(3)} : B'^{(3)} \to B^{(3)}$ is an isometric embedding and

$$(I-P)\pi^{(3)} = \pi^{(3)}(I-P') .$$

The second part of the proposition follows immediately. The proof of the first part is similar.

Every separable Banach space is isometric to a quotient space of the (real or complex) space l_1. In the light of Proposition 2 it is natural to focus attention on the case $B = l_1$.

THEOREM. *If* $B = l_1$ *then* $\|I-P\| = 2$.

Proof. The spaces l_1 and l_∞ can be identified with the duals of c_0 and l_1 respectively. Let $Q : c_0 \to l_\infty$ be the natural embedding. The space c_0 does have the properties of *(iii)* above: more precisely, it can be shown that if $x \in l_1$ and $z \in (Qc_0)^\perp \subseteq (l_1)^{**}$ then

$$\|Q_0x+z\| = \|Q_0x\| + \|z\| . \tag{4}$$

This fact will be used in the proof of the theorem.

Choose $y \in l_\infty$ such that $\|y\| = d(y, Qc_0) = 1$ (for example, y may be the sequence of constant value 1). Then there exists $z \in (Qc_0)^\perp$ such that $\|z\| = z(y) = 1$. Define $F_1 \in (Q_0l_1 \oplus [z])^*$ by

$$F_1(Q_0x) = y(x) \quad \text{for all} \quad x \in l_1 , \tag{5}$$

$$F(z) = -1 .$$

Then, by (4), for all $Q_0x \in Q_0l_1$ and all scalars λ,

$$\|F_1(Q_0x+\lambda z)\| = |y(x)-\lambda| \le \|x\| + |\lambda| = \|Q_0x+\lambda z\| .$$

Therefore $\|F_1\| = 1$. Let $F \in (l_1)^{(3)}$ be of norm one and an extension of

F_1 . Then, by (5), $Q_0^* F = y$. The conclusion of the theorem now follows from the inequality

$$\|I-P\| \geq \left|\left((I-P)F\right)(z)\right| = \left|F(z)-\left(Q_1 Q_0^* F\right)(z)\right| = 2 .$$

References

[1] Mahlon M. Day, *Normed linear spaces*, 3rd ed. (Ergebnisse der Mathematik und ihrer Grenzgebiete, 21. Springer-Verlag, Berlin, Heidelberg, New York, 1973).

[2] J. Dixmier, "Sur un théorème de Banach", *Duke Math. J.* 15 (1948), 1057-1071.

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