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On the Asymptotic Behavior of Complete Kähler Metrics of Positive Ricci Curvature

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Abstract. Let (X, g) be a complete noncompact Kähler manifold, of dimension $n \geq 2$, with positive Ricci curvature and of standard type (see the definition below). N. Mok proved that X can be compactified, *i.e.,* X is biholomorphic to a quasi-projective variety. The aim of this paper is to prove that the L^2 holomorphic sections of the line bundle K_X^{-q} and the volume form of the metric g have no essential singularities near the divisor at infinity. As a consequence we obtain a comparison between the volume forms of the Kähler metric g and of the Fubini–Study metric induced on X. In the case of $\dim_{\mathbb{C}} X = 2$, we establish a relation between the number of components of the divisor D and the dimension of the groups $H^i(\overline{X}, \Omega^1_{\overline{X}}(\log D))$.

1 Introduction

One way to approach noncompact complete Kähler manifolds is to try first to compactify them, *i.e.,* to realize them as Zariski open subsets of some projective varieties. In general, some suitable conditions must be imposed in order to achieve that.

In [4], N. Mok proved that if (X, g) is a complete noncompact Kähler manifold of positive Ricci curvature and satisfying some growth conditions, then K_X^{-1} is *ample*. More precisely, Mok's result says the following: Let (X, g) be a complete noncompact Kähler manifold of dimension $n \geq 2$, with positive Ricci curvature, and of standard type, *i.e.,* satisfying

(i) $\int_X \mathrm{Ric}_g^n < +\infty$,
(ii) $\mathrm{vol}_g(B_g(x_0, r)) \geq c_1 r^{2n}$,
(iii) $|\,\mathrm{sect}(x)| \leq \frac{c_2}{(1+d_g(x_0,x))^2}$,

where c_1 (resp. c_2) is a constant independent of r (resp. x), $\mathrm{vol}_g(B_g(x_0, r))$ is the volume of the geodesic ball $B_g(x_0, r)$ with respect to the metric g, Ric_g is the Ricci $(1, 1)$ form associated with the Kähler metric g, $\mathrm{sect}(x)$ is the sectional curvature of g at the point x, and $d_g(x_0, x)$ is the distance between x_0 and x with respect to the metric g. Then X is biholomorphic to a quasi-projective variety.

He proved the existence of a finite set of sections $\{s_j\}_{j=0}^N \subset H^0(X, K_X^{-q})$ of class L^β for some positive integer q, and for some fixed $\beta > 0$, where K_X^{-1} is the anticanonical line bundle, such that

$$\begin{aligned} \Psi\colon X &\hookrightarrow \overline{X} \subset \mathbb{P}^N \\ x &\longmapsto [s_0(x), \dots, s_N(x)], \end{aligned} \tag{1.1}$$

is an embedding, and $\Psi(X) = \overline{X} \backslash D$, where $\overline{X}$ is a smooth projective variety and $D = \sum_{j=0}^{l} D_j$ is a normal crossing divisor. In [1] the author proved, under the same

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conditions as in Mok's theorem, that X is of logarithmic Kodaira dimension $-\infty$, *i.e.,*

$$H^0(\overline{X}, (K_{\overline{X}} \otimes [D])^{\otimes \nu}) = 0 \text{ for all integers } \nu \geq 1.$$

It was also proved in [1] that $H^0(\overline{X}, \Omega^1_{\overline{X}}) = 0$, that D is connected, and as a consequence of the Castelnuovo rationality criterion and Miyanishi's result on quasi-projective surfaces of logarithmic Kodaira dimension $-\infty$, we deduced that if $\dim_{\mathbb{C}} X = 2$, then $\overline{X}$ is a rational surface and the components D_j of the divisor at infinity D are smooth rational curves. Consequently, X is affine ruled, *i.e.,* X contains a Zariski open set of the form $C \times \mathbb{C}$, where C is an affine curve.

The aim of this paper is to continue the investigation of complete noncompact Kähler manifolds of positive Ricci curvature and of standard type started in [4] and followed by [1]. We prove that the L^2 sections of the line bundle K_X^{-q} and the volume form of the metric g have no essential singularities near the divisor at infinity D.

In what follows, we will identify X with $\Psi(X)$, Ric_g with $(\Psi^{-1})^* \operatorname{Ric}_g$, and so on. We denote by $\omega_{\overline{X}}$ the $(1,1)$ form associated with the metric on $\overline{X}$ induced from the Fubini–Study metric on $\mathbb{P}^N$.

The main results proved in this paper are the following.

Theorem 1.1 *Fix $x \in D_j \setminus \operatorname{sing}(D)$, where D_j is an irreducible component of D and $\operatorname{sing}(D)$ is the singular locus of the divisor D. Let $(U, (z_1, \dots, z_n))$ be a local chart near the point x, with the open set U satisfying*

$$U \cap D = U \cap D_j = \{z \in U \mid z_j = 0\},$$

and let $V = \omega_g^n / |dz_1 \wedge \cdots \wedge dz_n|^2$ be the volume form expressed in terms of the local coordinates $(z_i)_i$, where ω_g is the $(1,1)$ form associated with the Kähler metric g. Then there exist a holomorphic function f and a plurisubharmonic function φ, both defined in U, and real numbers α_j such that

$$V = \frac{|e^f|\, e^{-\varphi}}{|z_j|^{2\alpha_j}} \text{ in } U \setminus D_j.$$

Moreover, $\nu(\varphi, z) = 0$, at a generic point z of $U \cap D_j$, where

$$\nu(\varphi, z) = \liminf_{w \to z} \frac{\varphi(w)}{\log |w - z|}$$

is the Lelong number of the plurisubharmonic function φ at the point z.

Theorem 1.2 *There exist two real numbers a and b and two positive constants μ_1 and μ_2 such that for all $x \in X$,*

$$\mu_1 \delta_D^a(x) \omega_{\overline{X}}^n(x) \leq \omega_g^n(x) \leq \mu_2 \delta_D^b(x) \omega_{\overline{X}}^n(x),$$

where $\delta_D(x) = d_{g_{\overline{X}}}(x, D)$ is the distance from the point $x \in X$ to the divisor D in terms of the induced Fubini–Study metric on $\overline{X}$ under the embedding (1.1) *above.*

Theorem 1.3 *Let X be a complete Kähler surface of positive Ricci curvature and of standard type and let $(\overline{X}, D)$ be the compactification* (1.1) *above. Then*

$$h^0(\overline{X}, \Omega^1_{\overline{X}}(\log D)) + h^1(\overline{X}, \Omega^1_{\overline{X}}) = l + h^1(\overline{X}, \Omega^1_{\overline{X}}(\log D)),$$

where $h^*(\overline{X}, \cdot\,) = \dim_{\mathbb{C}} H^*(\overline{X}, \cdot\,)$, and l is the number of irreducible components of the divisor D.

The paper[1] is organized as follows. In Section 2, we prove that the Ricci $(1,1)$ form Ric_g associated with the Kähler metric g extends across the divisor D as a closed positive $(1,1)$ current denoted by $\widetilde{\mathrm{Ric}_g}$. In Section 3, we establish some estimates for the norms of the L^2 holomorphic sections of K_X^{-q}. In Section 4 we study the behavior of the volume form near the divisor at infinity D and prove Theorem 1.1. In Section 5 we show that it is not possible to get a uniform lower bound for the order of the poles of the volume form. In Section 6 we prove Theorem 1.2. In Section 7 we prove Theorem 1.3.

2 Extension of the Ricci $(1,1)$ Form

Proposition 2.1 *The Ricci $(1,1)$ form Ric_g associated with the Kähler metric g on X extends to $\overline{X}$ as a $(1,1)$ closed positive current, denoted by $\widetilde{\mathrm{Ric}_g}$.*

For a proof of Proposition 2.1 we need the following.

Theorem 2.2 **(Skoda [8])** *Let X, $\overline{X}$, and $\omega_{\overline{X}}$ be as above, T a $(1,1)$ closed positive current defined in X, and suppose that*

$$\int_X T \wedge \omega_{\overline{X}}^{n-1} < +\infty.$$

Then T extends to $\overline{X}$ as a $(1,1)$ closed positive current denoted by $\widetilde{T}$.

Skoda's result is of local nature and is valid for (p,p) closed positive currents, but the weak version we stated above is enough for our purpose.

Proof of Proposition 2.1 By Skoda's result, it is enough to check that

$$\int_X \mathrm{Ric}_g \wedge \omega_{\overline{X}}^{n-1} < +\infty. \tag{2.1}$$

From [1, Lemma 4.2] we know that

$$\omega_{\overline{X}} = \sqrt{-1}\partial\overline{\partial}\log\Big(\sum_{j=0}^{N} \|s_j\|_g^2\Big) + q\,\mathrm{Ric}_g \ \text{ in } X,$$

[1]Most of the results in this paper are contained in the author's Ph.D. thesis, Université Paris-Sud XI, Orsay 1994.

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where the sections $\{s_j\}_{j=0}^{N} \subset H^0(X, K_X^{-q})$ are the sections giving the embedding $\Psi\colon X \hookrightarrow \overline{X}$ defined in the introduction. Therefore, the inequality (2.1) is equivalent to

$$\int_X \mathrm{Ric}_g \wedge \Big(\sqrt{-1}\partial\overline{\partial}\log\Big(\sum_{j=0}^{N}\|s_j\|_g^2\Big) + q\,\mathrm{Ric}_g\Big)^{n-1} < +\infty.$$

For $\varepsilon > 0$, let

$$\sigma_\varepsilon = \log\Big(\sum_{j=0}^{N}\|s_j\|_g^2 + \varepsilon\Big) \text{ and } \eta_\varepsilon = \Big(\sum_{j=0}^{N}\|s_j\|_g^2 + \varepsilon\Big).$$

An easy computation gives

$$\begin{aligned}
(2.2)\quad \sqrt{-1}\partial\overline{\partial}\sigma_\varepsilon = &\sum_{i=0}^{N}\frac{\|s_i\|_g^2}{\eta_\varepsilon}\sqrt{-1}\partial\overline{\partial}\log\|s_i\|_g^2\\
&+\sum_{i=0}^{N}\frac{\|s_i\|_g^2}{\eta_\varepsilon}\sqrt{-1}\partial\log\|s_i\|_g^2\wedge\overline{\partial}\log\|s_i\|_g^2\\
&-\sqrt{-1}\Big(\sum_{i=0}^{N}\frac{\|s_i\|_g^2}{\eta_\varepsilon}\partial\log\|s_i\|_g^2\Big)\wedge\Big(\sum_{i=0}^{N}\frac{\|s_i\|_g^2}{\eta_\varepsilon}\overline{\partial}\log\|s_i\|_g^2\Big).
\end{aligned}$$

It is not hard to prove that

$$\begin{aligned}
(2.3)\quad &\sum_{i=0}^{N}\frac{\|s_i\|_g^2}{\eta_\varepsilon}\sqrt{-1}\partial\log\|s_i\|_g^2\wedge\overline{\partial}\log\|s_i\|_g^2\\
&\geq \tfrac{\sqrt{-1}}{\eta_\varepsilon^2}\Big(\sum_{i=0}^{N}\|s_i\|_g^2\partial\log\|s_i\|_g^2\Big)\wedge\Big(\sum_{i=0}^{N}\|s_i\|_g^2\overline{\partial}\log\|s_i\|_g^2\Big).
\end{aligned}$$

The Poincaré–Lelong equation gives

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\|s_i\|_g^2 = [V_{s_i}] - \frac{q}{2\pi}\mathrm{Ric}_g\,.$$

Therefore, we get

$$(2.4)\qquad \sum_{i=0}^{N}\frac{\|s_i\|_g^2}{\eta_\varepsilon}\sqrt{-1}\partial\overline{\partial}\log\|s_i\|_g^2 \geq -q\,\mathrm{Ric}_g\,.$$

Combining (2.2), (2.3), and (2.4), we get

$$(2.5)\qquad \sqrt{-1}\partial\overline{\partial}\sigma_\varepsilon \geq -q\,\mathrm{Ric}_g\,.$$

Let $\omega_{\overline{X},\varepsilon} = \sqrt{-1}\partial\overline{\partial}\sigma_\varepsilon + q\,\mathrm{Ric}_g$. It follows from (2.5) that $\omega_{\overline{X},\varepsilon}$ is a smooth nonnegative $(1,1)$ form. Suppose that

$$\int_X \mathrm{Ric}_g \wedge \omega_{\overline{X},\varepsilon}^{n-1} \leq C \tag{2.6}$$

for some constant C that is independent of ε. Then by Fatou's lemma, we have

$$\int_X \mathrm{Ric}_g \wedge \omega_{\overline{X}}^{n-1} \leq \liminf_{\varepsilon\to 0} \int_X \mathrm{Ric}_g \wedge \omega_{\overline{X},\varepsilon}^{n-1} \leq C,$$

hence the inequality (2.1). Therefore, the problem is reduced to establishing the inequality (2.6), which will be obtained by induction as follows: Let $r > 0$, let $\alpha_k = \frac{\beta}{2^k}$, $k = 1, 2, \ldots, n-1$, where β is as in the introduction, let $r(x) = d_g(x_0, x)$, and let

$$(\mathrm{A}_k): \int_{B_g(x_0,r)} e^{\alpha_k\sigma_\varepsilon} \wedge \omega_{\overline{X},\varepsilon}^{k-1} \wedge \Big(\frac{\omega_g}{1+r^2(x)}\Big)^{n-k+1} = \varepsilon^{\alpha_{k-1}} O(\log r) + c_\varepsilon.$$

$$(\mathrm{B}_k): \int_{B_g(x_0,r)} \sqrt{-1}\partial e^{\alpha_{k+1}\sigma_\varepsilon} \wedge \overline{\partial} e^{\alpha_{k+1}\sigma_\varepsilon} \wedge \omega_{\overline{X},\varepsilon}^{k-1} \wedge \Big(\frac{\omega_g}{1+r^2(x)}\Big)^{n-k}$$
$$= \varepsilon^{\alpha_{k-1}} O(\log r) + c_\varepsilon.$$

$$(\mathrm{C}_k): \int_{B_g(x_0,r)} \sqrt{-1}\partial\sigma_\varepsilon \wedge \overline{\partial}\sigma_\varepsilon \wedge \omega_{\overline{X},\varepsilon}^{k-1} \wedge \Big(\frac{\omega_g}{1+r^2(x)}\Big)^{n-k} = O(\log r) + c_\varepsilon.$$

$$(\mathrm{D}_k): \int_X \sqrt{-1}\partial\overline{\partial}\sigma_\varepsilon \wedge \omega_{\overline{X},\varepsilon}^{k-1} \wedge \mathrm{Ric}_g^{n-k} < +\infty.$$

The recurrence scheme is the following:

$$(\mathrm{A}_k) \Longrightarrow (\mathrm{B}_k) \Longrightarrow (\mathrm{C}_k) \Longrightarrow (\mathrm{D}_k) \quad \text{and} \quad (\mathrm{A}_k), (\mathrm{B}_k) \Longrightarrow (\mathrm{A}_{k+1}).$$

With obvious modifications, the proof of the induction scheme is included in the two papers [4, 6]. ∎

3 Estimation of the Norms of the Sections of K_X^{-q}

Denote by $H^0_\beta(X, K_X^{-q})$ the set of holomorphic sections of K_X^{-q} which are L^β integrable with respect to the metric g, where β is the fixed positive integer.

Proposition 3.1 *Let $s \in H^0_\beta(X, K_X^{-q})$, $s \neq 0$. Then there exist two positive constants c_1 and c_2 such that*

$$\|s(x)\|_g^2 \leq \frac{c_1}{r(x)^{c_2}} \text{ for all } x \in X\backslash K,$$

where K is a compact subset of X and $r(x)$ is as above.

To prove Proposition 3.1, we need the following.

Lemma 3.2 *Fix* $0 < \delta < 1$ *and consider the following Dirichlet problem*

$$\begin{cases} \Delta_{\omega_g} u_x = \mathrm{Tr}_{\omega_g}(\mathrm{Ric}_g) \ \textit{in}\ B_g(x, \delta r(x)), \\ u_{x|\partial B_g(x,\delta r(x))} = 0. \end{cases} \tag{3.1}$$

Then u_x *exists and is bounded independently of the point* x*.*

Proof The existence, uniqueness and regularity of the solution to the Dirichlet problem (3.1) are very well known. The Riesz representation gives

$$u_x(z) = -\int_{B_g(x,\delta r(x))} G_x^\delta(z, y) Tr_{\omega_g}(\mathrm{Ric}_g)\omega_g^n(y),$$

where $G_x^\delta(\cdot, \cdot)$ is the Green Kernel of the ball $B_g(x, \delta r(x))$ with respect to the metric ω_g.

By assumption

$$\mathrm{Tr}_{\omega_g}(\mathrm{Ric}_g)(x) \leq \frac{c}{(1 + r(x))^2}$$

for some positive constant c which is independent of the point x. Hence

$$|u_x(z)| \leq c \int_{B_g(x,\delta r(x))} \frac{G_x^\delta(z, y)}{(1 + r(y))^2}\omega_g^n(y).$$

But $r(y) \geq (1 - \delta)r(x)$ for all $y \in B_g(x, \delta r(x))$. Therefore

$$|u_x(z)| \leq \frac{c}{(1 + (1 - \delta)r(x))^2} \int_{B_g(x,\delta r(x))} G_x^\delta(z, y)\omega_g^n(y). \tag{3.2}$$

To estimate the right-hand side of the inequality (3.2) we consider the following Dirichlet problem

$$\begin{cases} \Delta_{\omega_g} \mu_x = \chi_{B_g(x,\delta r(x))} \text{ in } B_g(x, \delta r(x)), \\ \mu_{x|\partial B_g(x,\delta r(x))} = 0, \end{cases} \tag{3.3}$$

where

$$\chi_{B_g(x,\delta r(x))}(y) = \begin{cases} 1 & \text{if } y \in B(x, \delta r(x)), \\ 0 & \text{otherwise.} \end{cases}$$

Using the Sobolev inequality and the Nash–Moser iteration process, Mok, Siu, and Yau [5] proved that

$$\sup_{B_g(x,\delta r(x))} |\mu_x| \leq c(\delta r(x))^2,$$

where c is a constant which is independent of $x \in X$ (c depends only on the constant c' appearing in the inequality $\mathrm{vol}(B_g(x, r)) \geq c' r^{2n}$).

From the Riesz representation of the solution of the Dirichlet problem (3.3), we deduce that

$$\int_{B_g(x,\delta r(x))} G_x^\delta(z,y)\omega_g^n(y) = -\mu_x(z) \le cr^2(x). \tag{3.4}$$

Combining (3.2) and (3.4), we deduce that

$$\sup_{B_g(x,\delta r(x))} |u_x| \le c\Big(\frac{r(x)}{1+(1-\delta)r(x)}\Big)^2 \le C,$$

where C is a positive constant independent of $x \in X$. ■

Proof of Proposition 3.1 Let $s \in H_\beta^0(X, K_X^{-q})$, $s \neq 0$. From the convexity of the function $f(t) = -\log t$, $t > 0$, and from the Jensen inequality, we deduce that

$$\begin{aligned}\frac{1}{\mathrm{vol}_g\big(B_g(x,\frac{\delta}{2}r(x))\big)} \int_{B_g(x,\frac{\delta}{2}r(x))} &\log \|s(y)\|_g^\beta \omega_g^n(y) \\ &\le \log\Big(\int_{B_g(x,\frac{\delta}{2}r(x))} \|s(y)\|_g^\beta \frac{\omega_g^n(y)}{\mathrm{vol}_g\big(B_g(x,\frac{\delta}{2}r(x))\big)}\Big).\end{aligned} \tag{3.5}$$

From the fact that the function

$$t \longrightarrow \frac{\mathrm{vol}_g(B(x,t))}{t^{2n}}, \quad t > 0,$$

is decreasing (a consequence of $\mathrm{Ric}_g > 0$) and from the assumption that

$$\mathrm{vol}_g(B_g(x_0,t)) \ge ct^{2n},$$

we deduce the existence of a positive constant c such that for all $x \in X$ and all $t > 0$, we have

$$\mathrm{vol}_g(B_g(x,t)) \ge ct^{2n}. \tag{3.6}$$

(for details, see [3]). Combining (3.5), (3.6), and using the fact that $s \in H_\beta^0(X, K_X^{-q})$, we get

$$\frac{1}{\mathrm{vol}_g\big(B_g(x,\frac{\delta}{2}r(x))\big)} \int_{B_g(x,\frac{\delta}{2}r(x))} \log \|s(y)\|_g^\beta \omega_g^n(y) \le \log\Big(\frac{c}{r^{2n}(x)}\Big). \tag{3.7}$$

The Poincaré–Lelong equation gives

$$\sqrt{-1}\partial\overline{\partial} \log \|s\|_g^2 = 2\pi[V_s] - q\,\mathrm{Ric}_g \ \text{ in } X, \tag{3.8}$$

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where $[V_s]$ is the current of integration on the divisor $s^{-1}(0)$, and the equality is in the sense of currents. By taking the trace of the Poincaré–Lelong equation (3.8) with respect to the Kähler $(1,1)$ form ω_g we get

$$\Delta_{\omega_g}(\log \|s\|_g^2 + qu_x) = 2\pi \operatorname{Tr}_{\omega_g}[V_s] \text{ in } B(x, \delta r(x)),$$

where Δ_{ω_g} is the Laplacian with respect to the Kähler metric g. Hence $\log \|s\|_g^2 + qu_x$ is subharmonic in $B(x, \delta r(x))$. Then

$$\log \|s(x)\|_g^2 + qu_x(x) \leq \frac{1}{\operatorname{vol}_g\big(B_g(x, \frac{\delta}{2} r(x))\big)} \int_{B_g(x, \frac{\delta}{2} r(x))} \Big(\frac{2}{\beta} \log \|s(y)\|_g^\beta + qu_x(y)\Big)\, \omega_g^n(y). \tag{3.9}$$

Since u_x is bounded by a constant that is independent of x, the inequalities (3.7) and (3.9) imply

$$\|s(x)\|_g^2 \leq \frac{c_1}{r(x)^{c_2}} \text{ for all } x \in X \backslash K,$$

where c_1 and c_2 are positive constants independent of x. ■

4 Asymptotic Behavior of the Volume Form

Let $\operatorname{sing}(D)$ be the singular locus of D. Since $\operatorname{codim}_{\mathbb{C}}(\operatorname{sing}(D)) \geq 2$, each holomorphic function defined in $U \backslash D$, where U is a neighborhood of a point $x \in D$, extends holomorphically across $\operatorname{sing}(D)$. Therefore the behavior of the holomorphic sections of K_X^{-q} near the divisor at infinity D can be reduced to their behavior in a neighborhood of $D \backslash \operatorname{sing}(D)$.

Let $x \in D_j \backslash \operatorname{sing}(D)$, where D_j is an irreducible component of D, and let $(U, (z_1, \dots, z_n))$ be a local chart near the point x, with the open set U satisfying

$$U \cap D = U \cap D_j = \{z \in U \mid z_j = 0\}.$$

The aim of this section is to give a description of the volume form near the divisor at infinity D, and hence a proof of Theorem 1.1. Some preparatory lemmas are needed.

Lemma 4.1 *Let*

$$V = \frac{\omega_g^n}{(\sqrt{-1})^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n} \text{ in } U \backslash D_j.$$

Then there exist a holomorphic function f in $U \backslash D_j$, a plurisubharmonic function φ in U, and a real number α_j such that

$$V = \frac{|e^f| e^{-\varphi}}{|z_j|^{2\alpha_j}} \text{ in } U \backslash D_j.$$

Moreover, $\nu(\varphi, z) = 0$, at a generic point z of $U \cap D_j$, where

$$\nu(\varphi, z) = \liminf_{w \longrightarrow z} \frac{\varphi(w)}{\log |w - z|}.$$

Proof We already proved that the Ricci $(1,1)$ form Ric_g extends trivially to $\overline{X}$ as a closed positive $(1,1)$ current, denoted by $\widetilde{\mathrm{Ric}}_g$. Consequently, there exists a plurisubharmonic function $\widetilde{\varphi}$ such that

$$\widetilde{\mathrm{Ric}}_{g_{|U}} = \sqrt{-1}\partial\overline{\partial}\widetilde{\varphi} \text{ in } U. \tag{4.1}$$

Moreover,

$$\mathrm{Ric}_g = -\sqrt{-1}\partial\overline{\partial}\log V \quad \text{in } U\backslash D_j. \tag{4.2}$$

From equations (4.1) and (4.2) we deduce that $\widetilde{\varphi} + \log V$ is pluriharmonic in $U\backslash D_j$. Therefore, $\partial(\widetilde{\varphi} + \log V)$ is a $(1,0)$ d-closed holomorphic form in $U\backslash D_j$. For an appropriate choice of a constant c, the $(1,0)$ form $\partial(\widetilde{\varphi} + \log V) - c(dz_j/z_j)$ is exact in $U\backslash D_j$. Fix a point z_0 in $U\backslash D_j$. Then for z in $U\backslash D_j$, we define a function f_1 as follows

$$f_1(z) = 2\int_{z_0}^{z} \partial(\widetilde{\varphi} + \log V) - 2c\log z_j,$$

where c is a complex number to be determined in such a way that the function f_1 is well defined in $U\backslash D_j$. Moreover, from

$$\frac{1}{2}df_1(z) = \partial(\widetilde{\varphi} + \log V) - d(c\log z_j),$$

we deduce that $\widetilde{\varphi} + \log V = \mathrm{Re}(f_1) + 2\,\mathrm{Re}(c\log z_j) + c_1$, where c_1 is a real number, and Re denotes the real part. Note first that since

$$\mathrm{Re}(c\log z_j) = \mathrm{Re}(c)\ \log|z_j| - \mathrm{Im}(c)\ \arg(z_j),$$

where Im denotes the imaginary part and arg denotes the argument, and since

$$\mathrm{Re}(c\log z_j) = \frac{1}{2}(\widetilde{\varphi} + \log\ V - \mathrm{Re}(f_1) - c_1)$$

is well defined (univalent), we deduce that $\mathrm{Im}\, c = 0$, *i.e.,* c is real. Hence

$$\widetilde{\varphi} + \log V = \mathrm{Re}(f) + 2c\log|z_j|,$$

where $f = f_1 + c_1$. In other words $V = |e^f|\, e^{-\widetilde{\varphi}}|z_j|^{2c}$.

Put $\varphi = \widetilde{\varphi} - \beta_j \log|z_j|$, where β_j is the generic Lelong number of $\widetilde{\varphi}$ along D_j, *i.e.,* $\nu(\widetilde{\varphi}, x) = \beta_j$ for a generic point x in $U \cap D_j$. Since the function φ is plurisubharmonic in $U\backslash D_j$ (a sum of a plurisubharmonic and a pluriharmonic), and since, after shrinking U if necessary, $\widetilde{\varphi} \leq \beta_j \log|z_j| + O(1)$, we deduce that φ is bounded from above. Hence φ extends to a plurisubharmonic function in U, which will be denoted also by φ. The lemma follows if we put $\alpha_j = (\beta_j/2) - c$. ■

Remark 4.2 Let $\mathbb{I}_{D_j}$ be the characteristic function of the component D_j, $[D_j]$ the current of integration over the smooth part of D_j, and let $\widetilde{\mathrm{Ric}}_g$ and $\{\beta_j\}_{j=1}^{l}$ be as above. Then it can be shown that

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(i) $\mathbb{I}_{D_j}\widetilde{\mathrm{Ric}_g}$ is a closed positive current,

(ii) $\mathbb{I}_{D_j}\widetilde{\mathrm{Ric}_g} = \beta_j[D_j]$ in $\overline{X}$,

(iii) $\widetilde{\mathrm{Ric}_g} - \beta_j[D_j]$ is a positive closed current.

Lemma 4.3 *The holomorphic function f, introduced in Lemma 4.1, extends to a holomorphic function in U. Moreover, every holomorphic section s of K_X^{-q} of class L^2 extends to a meromorphic section $\widetilde{s}$ of $K_{\overline{X}}^{-q}$, admitting poles of finite order along D.*

Proof Let $s \neq 0$ be a holomorphic section of K_X^{-q} of class L^2. From Proposition 3.1, we have

$$\|s(x)\|_g^2 \leq \frac{c_1}{r^{c_2}(x)} \text{for all } x \in X \backslash K,$$

where K is a compact subset of X and c_1 and c_2 are two positive constants independent of x. Hence, there exists an integer $p \geq 2$ such that

$$\int_{X\backslash K} \|s(x)\|^{2p} \omega_g^n(x) < +\infty. \tag{4.3}$$

In $U\backslash D_j$, we write $\omega_g^n(z) = V\,d\lambda(z)$, where $d\lambda(z) = (\sqrt{-1})^n dz_1\wedge d\bar{z}_1\wedge\cdots\wedge dz_n\wedge d\bar{z}_n$, and

$$s|_{U\backslash D_j} = h\,e^{\tau} \otimes \Big(\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n}\Big)^{\otimes q},$$

where h and τ are holomorphic functions in $U\backslash D_j$.

The assumption that s is square integrable implies

$$\int_{U\backslash D_j} \|s\|_g^2 \omega_g^n = \int_{U\backslash D_j} |h|^2 |e^{2\tau}| V^{q+1}\, d\lambda(z) < +\infty.$$

By Lemma 4.1, $V = |e^f| e^{-\varphi}/|z_j|^{2\alpha_j}$. Suppose that $\alpha_j \geq 0$. Then from the fact that φ is bounded from above in U, we get

$$\int_{U\backslash D_j} |h^2 e^{2\tau+(q+1)f}|\, d\lambda(z) < +\infty. \tag{4.4}$$

From (4.3) we deduce that

$$\int_{U\backslash D_j} \|s\|_g^{2p} \omega_g^n = \int_{U\backslash D_j} |h|^{2p} |e^{2p\tau}| V^{pq+1}\, d\lambda(z) < +\infty.$$

Hence,

$$\int_{U\backslash D_j} |h^{2p} e^{2p\tau+(pq+1)f}|\, d\lambda(z) < +\infty, \tag{4.5}$$

Let $z \in U\backslash D_j$ and let $B_{ec}(z,t) = \{w \in C^n \mid \|w - z\| < t\}$ be the ball in the euclidean norm in $\mathbb{C}^n$ with center the point z and radius t. Since we are interested in what happens in a neighborhood of the divisor D_j, we can assume that the euclidean ball $B_{\mathrm{ec}}(z, |z_j|/2)$ is relatively compact in $U\backslash D_j$.

From the plurisubharmonicity of the function $|h^2 e^{2\tau+(q+1)f}|$ and (4.4) we deduce that

$$(4.6)\qquad \begin{aligned} |h^2(z)e^{2\tau(z)+(q+1)f(z)}| &\le \frac{c}{|z_j|^{2n}} \int_{B_{\mathrm{ec}}(z,|z_j|/2)} |h^2(w)e^{2\tau(w)+(q+1)f(w)}|\, d\lambda(w) \\ &\le \frac{C}{|z_j|^{2n}}, \end{aligned}$$

where the positive constant C is independent of z.

From (4.6), we deduce that the function $z_j^{2n}h^2(z)\, e^{2\,\tau(z)+(q+1)f(z)}$ is bounded in $U\backslash D_j$, and therefore extends to a holomorphic function in U. Also, from the plurisubharmonicity of the function $|h^{2p}e^{2p\tau+(pq+1)f}|$ and (4.5) we deduce that the function $z_j^{2n}h^{2p}e^{2p\tau+(pq+1)f}$ extends to a holomorphic function in U. Consequently the functions $2\tau+(q+1)f$ and $2p\tau+(p\,q+1)f$ extend to holomorphic functions in U. Hence, f extends to a holomorphic function in U, and therefore the function τ also extends across the divisor D.

If $\alpha_j < 0$, then an argument similar to the one above implies that for some integer $\nu \ge -\alpha_j$, the holomorphic functions

$$z_j^{2n+\nu}h^2(z)\, e^{2\,\tau(z)+(q+1)f(z)} \quad \text{and} \quad z_j^{2n+\nu}h^{2p}e^{2p\tau+(pq+1)f}$$

extend to holomorphic functions in U. Hence f and τ extend to holomorphic functions in U.

As a consequence, we deduce that the function h extends to a meromorphic function in U with poles of finite order.

Therefore the L^2 holomorphic sections of the line bundle K_X^{-q} and the volume form, have no essential singularities along the divisor D. ■

Remark 4.4 Lemma 4.3 is true for holomorphic sections of K_X^{-q} of class L^γ for any $\gamma > 0$.

Proposition 4.5 *The real number α_j is independent of the chart (U, z_i), i.e., α_j depends only on the component D_j of the divisor D.*

Proof Let (U_1, z) and (U_2, w) be two open charts such that $U_1 \cap U_2 \ne \varnothing$,

$$U_1 \cap D = U_1 \cap D_j = \{z \in U_1 \mid z_j = 0\},$$
$$U_2 \cap D = U_2 \cap D_j = \{w \in U_2 \mid w_j = 0\}.$$

Let

$$\begin{aligned} \vartheta_{1,2}\colon U_1 \cap U_2 &\longrightarrow U_1 \cap U_2 \\ z &\longmapsto w = \vartheta_{1,2}(z) = (\vartheta_{1,2}^1(z), \dots, \vartheta_{1,2}^n(z)) \end{aligned}$$

be the transition biholomorphism. Then from

$$V = \frac{|e^{f_1(z)}|e^{-\varphi_1(z)}}{|z_j|^{2\alpha_{1,j}}} \quad \text{in } U_1\backslash D_j$$
$$= \frac{|e^{f_2(w)}|e^{-\varphi_2(w)}}{|w_j|^{2\alpha_{2,j}}} \quad \text{in } U_2\backslash D_j$$

we get

$$\frac{|e^{f_1(z)}|e^{-\varphi_1(z)}}{|z_j|^{2\alpha_{1,j}}} = \frac{|e^{f_2(\vartheta_{1,2}(z))}|e^{-\varphi_2(\vartheta_{1,2}(z))}}{|\vartheta^j_{1,2}(z)|^{2\alpha_{2,j}}} \quad \text{in } (U_1 \cap U_2)\backslash D_j.$$

Therefore,

$$\frac{|\vartheta^j_{1,2}(z)|^{2\alpha_{2,j}}}{|z_j|^{2\alpha_{1,j}}} = e^{\varphi_1(z)-\varphi_2(\vartheta_{1,2}(z))}|e^{f_2(\vartheta_{1,2}(z))-f_1(z)}|.$$

Since $\varphi_1(z) - \varphi_2(\vartheta_{1,2}(z))$ is pluriharmonic, hence bounded (after shrinking U_1 and U_2 if necessary) and since f_1 and f_2 are bounded in $U_1 \cap U_2$, we deduce that

$$C_1 \leq \frac{|\vartheta^j_{1,2}(z)|^{2\alpha_{2,j}}}{|z_j|^{2\alpha_{1,j}}} \leq C_2,$$

where C_1 and C_2 are positive constants that are independent of z. But the last inequality is possible only if $\alpha_{1,j} = \alpha_{2,j}$. ∎

Definition 4.6 The real number α_j which appears in Proposition 4.5 will be called the order of the poles of the volume form along the component D_j of the divisor D.

Remark 4.7 Theorem 1.1 follows from Lemmas 4.1 and 4.3.

5 Volume Forms with Poles of Arbitrary Order

In this section we will construct a family of complete Kähler metrics of positive Ricci curvature and of standard type on $\mathbb{C}^2$ such that the corresponding volume forms have poles along the divisor at infinity D of arbitrary negative orders. The idea is to start with a given complete Kähler metric of positive Ricci curvature and of standard type and then deform it using the automorphism group of $\mathbb{C}^2$ to obtain new complete Kähler metrics of positive Ricci curvature and of standard type with the desired properties.

Using the continuity method for the Monge–Ampère equation, Yeung [9] proved that if $\overline{X}$ is a smooth projective variety with $\dim_{\mathbb{C}} \overline{X} \geq 2$ and D a smooth hypersurface such that the associated line bundles $[D]$ and $(K_{\overline{X}} \otimes [D])^{-1}$ are positive, then the affine variety $\overline{X}\backslash D$ admits a complete Kähler metric of positive Ricci curvature and of standard type.

Example 1 Let $\overline{X}=\mathbb{P}^n$, $n \geq 2$ and let D be a smooth hypersurface in $\mathbb{P}^n$ of degree n, where $1 \leq d \leq n$. Then $[D] = \mathcal{O}_{\mathbb{P}^n}(d)$ and $(K_{\mathbb{P}^n} \otimes [D])^{-1} = \mathcal{O}_{\mathbb{P}^n}(n+1-d)$ are positive. Then $X = \mathbb{P}^n\backslash D$ admits a complete Kähler metric of positive Ricci curvature and of standard type.

For a fixed β, $0 < \beta < 1$, Yeung's method [9] guarantees the existence of a complete Kähler metric with positive Ricci curvature and of standard type in $\mathbb{C}^2$ which is equivalent to ω_β near the hyperplane at infinity, where ω_β is the closed $(1,1)$ form

$$\omega_\beta = \sqrt{-1}\partial\overline{\partial}(1 + \|z\|^2)^\beta.$$

It can be shown that ω_β is associated with a complete Kähler metric in $\mathbb{C}^2$. An easy computation gives

$$\omega_\beta = \beta(1 + \|z\|^2)^{\beta-1}\sqrt{-1}\partial\overline{\partial}\|z\|^2 + \beta(\beta - 1)(1 + \|z\|^2)^{\beta-2}\sqrt{-1}\partial\|z\|^2 \wedge \overline{\partial}\|z\|^2.$$

Hence

$$\omega_\beta^2 = \beta^2(1 + \|z\|^2)^{2\beta-3}(1 + \beta\|z\|^2)\left(\sqrt{-1}\partial\overline{\partial}\|z\|^2\right)^2.$$

In a neighborhood of the hyperplane at infinity we have the following estimate:

$$\omega_\beta^2 \sim \|z\|^{2(2\beta-2)}\left(\sqrt{-1}\partial\overline{\partial}\|z\|^2\right)^2.$$

Let $\xi = 1/z_1$ and $\eta = z_2/z_1$ be local coordinates at infinity *i.e.*, the hyperplane at infinity is defined by $\xi = 0$. The Jacobian of the transformation is given by $-1/z_1^3$. Hence

$$\omega_\beta^2 \sim \frac{1}{|\xi|^{2[(2\beta-2)+3]}} d\lambda(\xi,\eta),$$

where $d\,\lambda(\xi,\eta) = (\sqrt{-1})^2 d\,\xi \wedge d\,\bar{\xi} \wedge d\,\eta \wedge d\,\bar{\eta}$. In this case, the order of the poles of the volume form is $\alpha = 2\beta + 1$. Let p be an integer ≥ 1 and let

$$\Phi_p \colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$
$$(z_1, z_2) \longmapsto (z_1 + z_2^p, z_2).$$

Then Φ_p defines an automorphism of $\mathbb{C}^2$ with Jacobian equal to one. Hence

$$\Phi_p^*(\omega_\beta^2) = \beta^2(1 + \|\Phi_p(z)\|^2)^{2\beta-3}(1 + \beta\|\Phi_p(z)\|^2)\left(\sqrt{-1}\partial\overline{\partial}\|z\|^2\right)^2.$$

For $|z_1|$ sufficiently large and $z_1/z_2 =$ constant, we obtain

$$\Phi_p^*(\omega_\beta^2) \sim |z_1|^{2p\,(2\beta-2)}\left(\sqrt{-1}\partial\overline{\partial}\|z\|^2\right)^2 \sim \frac{1}{|\xi|^{2[p(2\beta-2)+3]}} d\,\lambda(\xi,\eta).$$

In this case, the order of the poles of the volume form $\Phi_p^*(\omega_\beta^2)$ is $\alpha_p = p\,(2\beta - 2) + 3$. Therefore, α_p is arbitrarily negative (depends only on the choice of the integer p which is arbitrary).

Note that the method developed by Yeung [9] gives a complete Kähler metric of positive Ricci curvature and of standard type in $\mathbb{C}^2$ (resp. in the blow up of $\mathbb{C}^2$ at a point) such that the order of the poles of the volume form satisfies $1 < \alpha < 3$ (resp. $1 < \alpha < 2$) with α sufficiently close to 1.

6 The Kähler Metric Versus the Fubini–Study Metric

Let $g_{\overline{X}}$ be the induced Fubini–Study metric on $\overline{X}$ under the embedding $\overline{X} \hookrightarrow \mathbb{P}^N$, described in (1.1), and let $\omega_{\overline{X}}$ be the $(1,1)$ form associated with $g_{\overline{X}}$. Denote by $\delta_D(x) = d_{g_{FS}}(x, D)$ the distance from the point $x \in X$ to the divisor D in terms of the metric $g_{\overline{X}}$. As a consequence of Proposition 3.1 and Theorem 1.1, we get the following.

Corollary 6.1 *There exist two real numbers a and b and two positive constants μ_1 and μ_2 such that for all $x \in X$,*

$$\mu_1 \delta_D^a(x)\omega_{\overline{X}}^n(x) \le \omega_g^n(x) \le \mu_2 \delta_D^b(x)\omega_{\overline{X}}^n(x).$$

Proof Let $(U, (z_1, \dots, z_n))$ be a chart defined as in the beginning of Section 4 and let

$$V = \frac{|e^f|e^{-\varphi}}{|z_j|^{2\alpha_j}}$$

be the expression of the volume form in $U\backslash D_j$ obtained in Theorem 1.1. After shrinking U, if necessary, we can assume that the function $|e^f|$ is bounded and φ is bounded from above. Then

$$V \ge \frac{c}{|z_j|^{2\alpha_j}}.$$

By a compactness argument, we deduce that there exist a real number a and a positive constant c_1 such that

$$c_1 \delta_D^a(x)\omega_{\overline{X}}^n(x) \le \omega_g^n(x).$$

Let $s \in H^0_\beta(X, K_X^{-q})$, $s \neq 0$. The same argument as in Lemma 4.3 shows that s extends to a meromorphic section of $K_{\overline{X}}^{-q}$ with poles of finite order. In $U\backslash D_j$ we have

$$s = s_0 \otimes \Big(\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n} \Big)^q,$$

where s_0 is a holomorphic function in $U\backslash D_j$ which extends to a meromorphic function in U admitting poles of finite order along the divisor D_j.

Proposition 3.1 guarantees the existence of a compact subset K of X and two positive constants c_1 and c_2 such that for all $x \in X\backslash K$

$$\|s(x)\|^2 \le \frac{c_1}{r^{c_2}(x)},$$

Consequently

$$\|s(x)\|^2_{|U\backslash D_j} = V^q |s_0|^2 \le c.$$

Then

$$V \le \Big(\frac{c}{|s_0|^2} \Big)^{1/q} \le \frac{C}{|z_j|^\gamma}.$$

By a compactness argument we deduce that there exist a real number b and a positive constant c_2 such that

$$\omega_g^n(x) \le c_2 \delta_D^b(x) \omega_{\overline{X}}^n(x). \qquad \blacksquare$$

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7 Logarithmic One-Forms on the Compactification

The aim of this section is to give a proof of Theorem 1.3. Before doing that let us mention the following.

Remark 7.1 Let $(\overline{X}, D)$ and $(\overline{X}', D')$ be two compactifications of X, *i.e.,* $\overline{X}$ and $\overline{X}'$ are smooth projective varieties and D and D' are divisors with normal crossing singularities satisfying

$$X \overset{\text{biholomorphic}}{\cong} \overline{X}\backslash D \overset{\text{biholomorphic}}{\cong} \overline{X}'\backslash D'.$$

It can be shown that if the biholomorphism $\overline{X}\backslash D \cong \overline{X}'\backslash D'$ extends to a birational map $\overline{X} \dashrightarrow \overline{X}'$, then

$$H^0\big(\overline{X}, \Omega^1_{\overline{X}}(\log D)\big) \cong H^0\big(\overline{X}', \Omega^1_{\overline{X}'}(\log D')\big).$$

Hence the cohomology group $H^0(\overline{X}, \Omega^1_{\overline{X}}(\log))$ reflects properties of X and does not depend on birational compactifications.

Proof of Theorem 1.3 Consider the following exact sequence

$$0 \longrightarrow \Omega^1_{\overline{X}} \longrightarrow \Omega^1_{\overline{X}}(\log D) \xrightarrow{\sigma} \bigoplus_{j=1}^{l} O_{D_j} \longrightarrow 0. \tag{7.1}$$

Let ϑ be an element of $H^0(\overline{X}, \Omega^1_{\overline{X}}(\log D))$. In a local chart (U, z) such that

$$D \cap U = \{z \in U \mid z_1 \cdots z_s = 0\},$$

we have

$$\vartheta|_{U\backslash D} = \sum_{j=1}^{s} a_j(z)\frac{dz_j}{z_j} + \sum_{i=s+1}^{l} a_i(z)dz_i,$$

where the functions a_i, $i = 1, \dots, n$, are holomorphic functions in U. A local description of σ is given by

$$\sigma\Big(\sum_{j=1}^{s} a_j(z)\frac{dz_j}{z_j} + \sum_{i=s+1}^{l} a_i(z)dz_i\Big) = \bigoplus_{j=1}^{s} a_j|_{D_j}.$$

The exact sequence (7.1) induces an exact sequence at the level of cohomology groups, *i.e.,* the following sequence

$$\begin{aligned} &H^0(\overline{X}, \Omega^1_{\overline{X}}) \longrightarrow H^0(\overline{X}, \Omega^1_{\overline{X}}(\log D)) \longrightarrow \bigoplus_{j=1}^{l} H^0(D_j, \mathcal{O}_{D_j}) \\ &\qquad \xrightarrow{\varrho} H^1(\overline{X}, \Omega^1_{\overline{X}}) \longrightarrow H^1(\overline{X}, \Omega^1_{\overline{X}}(\log D)) \longrightarrow \bigoplus_{j=1}^{l} H^1(D_j, \mathcal{O}_{D_j}) \end{aligned} \tag{7.2}$$

is exact.

If $\dim_{\mathbb{C}} X = 2$, then by combining Proposition 7.1, Corollary 8.1 from [1], and the exact sequence (7.2) above, we get

$$0 \longrightarrow H^0(\overline{X}, \Omega^1_{\overline{X}}(\log D)) \longrightarrow \mathbb{C}^l \xrightarrow{\rho} H^1(\overline{X}, \Omega^1_{\overline{X}}) \longrightarrow H^1(\overline{X}, \Omega^1_{\overline{X}}(\log D)) \longrightarrow 0.$$

Therefore

$$\dim_{\mathbb{C}} H^0(\overline{X}, \Omega^1_{\overline{X}}(\log D)) - l + \dim_{\mathbb{C}} H^1(\overline{X}, \Omega^1_{\overline{X}}) - \dim_{\mathbb{C}} H^1(\overline{X}, \Omega^1_{\overline{X}}(\log D)) = 0. \qquad \blacksquare$$

Corollary 7.2 *Let X and $(\overline{X}, D)$ be as in Theorem 1.3. Then*

$$l = h^1(X, \mathbb{C}) - h^2(X, \mathbb{C}) + h^1(\overline{X}, \Omega^1_{\overline{X}}) + \chi(\mathcal{O}_{\overline{X}}) - 1,$$

where l is the number of irreducible components of the divisor D and $\chi(\mathcal{O}_{\overline{X}})$ is the Euler characteristic of the structure sheaf $\mathcal{O}_{\overline{X}}$.

Proof This follows easily from Theorem 1.3, Deligne's isomorphism [2]

$$H^k(X, \mathbb{C}) \cong H^k(\overline{X} \backslash D, \mathbb{C}) \cong \bigoplus_{p+q=k} H^p(\overline{X}, \Omega^q_{\overline{X}}(\log D)),$$

and the fact (proved in [1]) that $H^0(X, K_{\overline{X}} \otimes [D]) = 0$. $\blacksquare$

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