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ON POLYGONAL PRODUCTS OF FINITELY GENERATED ABELIAN GROUPS

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We prove that a polygonal product of polycyclic-by-finite groups amalgamating subgroups, with trivial intersections, is cyclic subgroup separable (hence, it is residually finite) if the amalgamated subgroups are contained in the centres of the vertex groups containing them. Hence a polygonal product of finitely generated abelian groups, amalgamating any subgroups with trivial intersections, is cyclic subgroup separable. Unlike this result, most polygonal products of four finitely generated abelian groups, with trivial intersections, are not subgroup separable ($\mathcal{LERF}$). We find necessary and sufficient conditions for certain polygonal products of four groups to be subgroup separable.

1. INTRODUCTION

Polygonal products of groups were introduced by Karrass, Pietrowski and Solitar [**6**]. Using their result, Brunner, Frame, Lee and Wielenberg [**5**] determined all torsion-free subgroups of finite index in the Picard group $PSL(2, Z[i])$. In [**3**], Allenby and Tang proved that polygonal products of four finitely generated free abelian groups, amalgamating cyclic subgroups with trivial intersections, is residually finite. Kim and Tang [**9**] showed that certain polygonal products of four nilpotent groups, amalgamating cyclic subgroups with trivial intersections, are residually finite. In this paper, we prove that polygonal products of more than four polycyclic-by-finite groups amalgamating any subgroups, contained in the centres of their vertex groups, with trivial intersections are π_c (Theorem 2.11), hence they are residually finite. Thus, polygonal products of more than four finitely generated abelian groups, amalgamating any subgroups with trivial intersections, are π_c. It was relatively easy to prove the same result for those polygonal products with four vertex groups and cyclic subgroups amalgamated [**8**]. Note that polygonal products of four polycyclic-by-finite groups amalgamating cyclic subgroups, contained in the centres of their vertex groups, with trivial intersections is conjugacy separable [**7**]. Unlike the case for residual finiteness or for conjugacy separability, most

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polygonal products of four finitely generated abelian groups amalgamating cyclic subgroups with trivial intersections are not subgroup separable (Theorem 3.3). We also find necessary and sufficient conditions for certain polygonal products of four groups to be subgroup separable (Theorem 3.2).

Briefly polygonal products of groups can be described as follows [**3**]: Let P be a polygon. Assign a group G_v to each vertex v and a group G_e to each edge e of P. Let α_e and β_e be monomorphisms which embed G_e as a subgroup of the two vertex groups at the ends of the edge e. Then the *polygonal product* G is defined to be the group generated by the generators and relations of the vertex groups together with the extra relations obtained by identifying $g_e\alpha_e$ and $g_e\beta_e$ for each $g_e \in G_e$.

By abuse of language, we say that G is the polygonal product of the (vertex) groups $G_0, G_1, \ldots, G_n$, amalgamating the (edge) subgroups $H_0, H_1, \ldots, H_n$ with *trivial intersections*, if $G_i \cap G_{i+1} = H_i$ and $H_i \cap H_{i+1} = 1$, where $0 \leqslant i \leqslant n$ and the subscripts i are taken modulo $n+1$.

Finally, we note that a polygonal product can appear as a subgroup of a group, and then the residual properties of the polygonal product determine the residual properties of the whole group, as in the following example.

EXAMPLE 1.1: Let $G = \langle a, b; [a, b^{-1}ab], a^m, b^n\rangle$, where $n \neq 0$. Clearly G is a finite cyclic extension of $\langle a\rangle^G$. We note that $\langle a\rangle^G = \langle a_0, a_1, \ldots, a_{n-1}; a_i^m, [a_i, a_{i+1}]\rangle$, where $a_i = b^{-i}ab^i$ and the subscripts i are considered modulo n. For $n \geqslant 3$, we may consider $\langle a\rangle^G$ as the polygonal product of the abelian subgroups $\langle a_i, a_{i+1}\rangle$ amalgamating the subgroups $\langle a_{i+1}\rangle$, where the subscripts are taken modulo n. By Theorem 2.12, we can see that $\langle a\rangle^G$ is π_c for $n \geqslant 4$. If $n = 1, 2, 3$ then $\langle a\rangle^G$ is finite abelian. Therefore, G is π_c for all $n \neq 0$.

We shall adopt the following notation and terminology:

We use $N \lhd_f G$ to denote that the normal subgroup N of G has finite index in G and "f.g." means "finitely generated". We denote by $A *_H B$ the generalised free product of A and B with the subgroup H amalgamated. If $G = A *_H B$ and $x \in G$, then $\|x\|$ denotes the free product length of x in G. On the other hand, we use $|x|$ to denote the order of x. If $\overline{G}$ is a homomorphic image of G, then we use $\overline{x}$ to denote the image of $x \in G$ in $\overline{G}$.

Let H be a subgroup of a group G. Then G is said to be *H-separable* if, for each $x \in G \setminus H$, there exists $N \lhd_f G$ such that $x \notin NH$. A group G is *locally extended residually finite* ($\mathcal{LERF}$ or *subgroup separable*) if G is H-separable for all f.g. subgroups H of G.

A group G is *extended residually finite* ($\mathcal{ERF}$) if G is H-separable for all subgroups H of G.

A group G is cyclic subgroup separable (π_c) if G is $\langle x\rangle$-separable for all

$x \in G$.

A group G is *residually finite* ($\mathcal{RF}$) if G is $\langle 1\rangle$-separable.

Clearly, every $\mathcal{LERF}$ group is π_c, and every π_c group is $\mathcal{RF}$.

We shall use the following results:

Theorem 1.2. [4], [1] *If A and B are $\mathcal{RF}$ (π_c, $\mathcal{LERF}$) and U is finite, then $A *_U B$ is $\mathcal{RF}$ (π_c, $\mathcal{LERF}$, respectively).*

Theorem 1.3. [1] *Let $G = N \cdot H$ be a split extension of the normal f.g. subgroup N by H. If N is $\mathcal{ERF}$ and H is $(\mathcal{L})\mathcal{ERF}$, then G is $(\mathcal{L})\mathcal{ERF}$. If N and H are both π_c groups, then G is a π_c group.*

Theorem 1.4. [8] *Let $G = E *_H F$. Suppose that*

(a) *E and F are π_c and H-separable,*

(b) *for each $N \lhd_f H$ there exist $N_E \lhd_f E$ and $N_F \lhd_f F$ such that $N_E \cap H = N_F \cap H \subset N$.*

Then G is π_c.

A group G is polycyclic-by-finite if it has a normal subgroup N such that N is polycyclic and G/N is finite.

2. Cyclic subgroup separability (π_c)

A group G is polycyclic-by-finite if it has a normal subgroup N such that N is polycyclic and G/N is finite.

In this section we shall prove that a polygonal product of polycyclic-by-finite groups $A_0, A_1, \ldots, A_n$ $(n \geqslant 3)$, amalgamating any subgroups $H_0, H_1, \ldots, H_n$ with trivial intersections, is π_c if H_{i-1}, $H_i \subset Z(A_i)$ for all i. To prove this result, we have to study some properties of the group $E_m = A_1 *_{H_1} A_2 *_{H_2} \cdots *_{H_{m-1}} A_m$, where $H_j = A_j \cap A_{j+1}$ for $1 \leqslant j \leqslant m-1$, and each A_i is a polycyclic-by-finite group containing subgroups H_{i-1} and H_i such that $H_{i-1} \cap H_i = 1$, where $H_{i-1}, H_i \subset Z(A_i)$ for $1 \leqslant i \leqslant m$. Throughout this section E_m denotes the above group.

Lemma 2.1. *For given subgroups $U \lhd_f H_0$ and $V \lhd_f H_m$, there exists $N \lhd_f E_m$ such that $N \cap H_0 = U$, $N \cap H_m = V$ and $NH_0 \cap NH_m = N$.*

Proof: Case 1: $m > 1$. There exists a homomorphism $\pi \colon E_m \to (A_1/H_1U) * (A_m/H_{m-1}V)$, since H_{i-1}, $H_i \subset Z(A_i)$. Now A_1/H_1U and $A_m/H_{m-1}V$ are polycyclic-by-finite, hence, $\overline{E}_m = E_m\pi$ is $\mathcal{RF}$. Thus, since $\overline{H}_0 = H_0H_1U/H_1U \cong H_0/U$ and $\overline{H}_m \cong H_m/V$ are finite, there exists $\overline{N} \lhd_f \overline{E}_m$ such that $1 = \overline{N} \cap \overline{H_mH_0}$. Let N be the preimage of $\overline{N}$ in E_m.

Case 2. $m = 1$. Let $U \lhd_f H_0$ and $V \lhd_f H_1$ be given. Since $U, V \subset Z(A_1)$, we can consider $\overline{A}_1 = A_1/UV$. Then the subgroups $\overline{H}_0$ and $\overline{H}_1$ of $\overline{A}_1$ are finite. Since $\overline{A}_1$ is

$\mathcal{RF}$, there exists $\overline{N} \lhd_f \overline{A}_1$ such that $1 = \overline{N} \cap \overline{H_1 H_0}$. Let N be the preimage of $\overline{N}$ in A_1.

It is not difficult to see that N satisfies our requirements. ∎

Lemma 2.2. *For any given $M \lhd_f E_{n-1}$ and $N \lhd_f A_n$, there exist $P \lhd_f E_{n-1}$ and $Q \lhd_f A_n$ such that $P \subset M$, $Q \subset N$, $P \cap H_{n-1} = Q \cap H_{n-1}$, $PH_0 \cap PH_{n-1} = P$ and $QH_{n-1} \cap QH_n = Q$.*

Proof: By Lemma 2.1, there exists $M_1 \lhd_f E_{n-1}$ such that $M_1 \cap H_0 = M \cap H_0$, $M_1 \cap H_{n-1} = M \cap N$, and $M_1 H_0 \cap M_1 H_{n-1} = M_1$. Similarly, there exists $N_1 \lhd_f A_n$ such that $N_1 \cap H_{n-1} = M \cap N$, $N_1 \cap H_n = N \cap H_n$, and $N_1 H_{n-1} \cap N_1 H_n = N_1$. Let $P = M \cap M_1$ and $Q = N \cap N_1$. Then it is easy to see that P and Q satisfy the required conditions. ∎

Theorem 2.3. *Let $G = E *_H F$, where E, F are H-separable. Let S be a subgroup of E and suppose that E is S-separable. Suppose, further, that*

(W) *for each $N \lhd_f H$ there exist $N_E \lhd_f E$ and $N_F \lhd_f F$ such that $N_E \cap H = N_F \cap H \subset N$.*

Then G is S-separable.

Proof: Let $g \in G$ be such that $g \notin S$.

Case 1. $g \in E$. Since E is S-separable, there exists $P \lhd_f E$ such that $g \notin PS$. Now, by (W), there exist $P_1 \lhd_f E$ and $Q_1 \lhd_f F$ such that $P_1 \cap H = Q_1 \cap H \subset P \cap H$. Let $N_E = P \cap P_1$ and $N_F = Q_1$. Then $N_E \lhd_f E$, $N_F \lhd_f F$, and $N_E \cap H = P_1 \cap H = Q_1 \cap H = N_F \cap H$. Thus, we have a homomorphism $\pi\colon E *_H F \to E/N_E *_{\overline{H}} F/N_F$, where $\overline{H} = HN_E/N_E = HN_F/N_F$. It is clear that $\overline{g} \notin \overline{S}$, where $\overline{G} = G\pi$. Since $\overline{G}$ is $\mathcal{LERF}$ by Theorem 1.2, there exists $\overline{M} \lhd_f \overline{G}$ such that $\overline{g} \notin \overline{MS}$. Let M be the preimage of $\overline{M}$ in G. Then, clearly, $M \lhd_f G$ and $g \notin MS$.

Case 2. $g \in F \setminus H$. Since F is H-separable, we can find $Q \lhd_f F$ such that $g \notin QH$. By (W), there exist $P_1 \lhd_f E$ and $Q_1 \lhd_f F$ such that $P_1 \cap H = Q_1 \cap H \subset Q \cap H$. Let $N_E = P_1$ and $N_F = Q \cap Q_1$. Then $N_E \lhd_f E$, $N_F \lhd_f F$, and $N_E \cap H = P_1 \cap H = Q_1 \cap H = N_F \cap H$. Now we consider $\pi\colon G \to E/N_E *_{\overline{H}} F/N_F$ as in Case 1. Then $\overline{g} \in \overline{F} \setminus \overline{H}$ and $\overline{S} \subset \overline{E}$. It follows that $\overline{g} \notin \overline{S}$. As in Case 1, we can find $M \lhd_f G$ such that $g \notin MS$.

Case 3. $\|g\| \geqslant 2$. Assume that $g = e_1 f_1 \cdots e_n f_n$, where $e_i \in E \setminus H$ and $f_i \in F \setminus H$ (the other cases are similar). Since E and F are H-separable, there exist $P \lhd_f E$ and $Q \lhd_f F$ such that $e_i \notin PH$ and $f_i \notin QH$ for all i. Considering $P \cap Q \lhd_f H$, by assumption (W), we can find $P_1 \lhd_f E$ and $Q_1 \lhd_f F$ such that $P_1 \cap H = Q_1 \cap H \subset P \cap Q$. Let $N_E = P \cap P_1$ and $N_F = Q \cap Q_1$. Then $N_E \lhd_f E$, $N_F \lhd_f F$ and $N_E \cap H = N_F \cap H$. Thus we have a homomorphism $\pi\colon G \to E/N_E *_{\overline{H}} F/N_F$ as in Case 1. Then we have

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$\|\overline{g}\| = \|g\| \geqslant 2$, where $\overline{G} = G\pi$. It follows that $\overline{g} \notin \overline{S}$. Now, as in Case 1, we can find $M \triangleleft_f G$ such that $g \notin MS$. This proves the theorem. ∎

If $S = \langle 1 \rangle$, then Theorem 2.3 proves that $G = E *_H F$ is $\mathcal{RF}$. Thus Theorem 2.3 is a generalised version of [**11**, Criterion] or [**2**, Lemma 3.1]. Furthermore, using the above theorem, we can prove the following two results which are unavoidable for our main result in this section.

Corollary 2.4. *The group E_n is H_0-separable and H_n-separable.*

Proof: Note that E_1 and E_2 are $\mathcal{LERF}$ by [**1**, Theorem 5]. Hence the lemma holds for $n = 1, 2$. Inductively, we assume that E_{n-1} is H_0-separable and H_{n-1}-separable. Note that the A_i are polycyclic-by-finite, hence A_n is also H_{n-1}-separable. By Lemma 2.1, for each $N_H \triangleleft_f H_{n-1}$, there exist $N \triangleleft_f E_{n-1}$ and $M \triangleleft_f A_n$ such that $N \cap H_{n-1} = N_H = M \cap H_{n-1}$. This proves (W) in Theorem 2.3 for $E_n = E_{n-1} *_{H_{n-1}} A_n$. Thus E_n is H_0-separable by Theorem 2.3. By symmetry, E_n is H_n-separable. ∎

Corollary 2.5. *The group E_n is π_c.*

Proof: Write $E_n = E_{n-1} *_{H_{n-1}} A_n$. Then E_{n-1} and A_n are H_{n-1}-separable (Corollary 2.4) and satisfy (W) in Theorem 2.3 as in the previous proof. It follows, by induction and Theorem 1.4, that E_n is π_c. ∎

Lemma 2.6. *For given $x \in E_m$ such that $x \notin H_0 H_m$, there exists $N \triangleleft_f E_m$ such that $x \notin N H_0 H_m$.*

Proof: For $m = 1$, the lemma is trivial, since $E_1 = A_1$ is polycyclic-by-finite (hence, it is $\mathcal{LERF}$) and since $H_0 H_1$ is a f.g. subgroup. For an induction, we assume that the lemma holds for E_{m-1}; that is, for given $e \in E_{m-1}$ such that $e \notin H_0 H_{m-1}$, there exists $P \triangleleft_f E_{m-1}$ such that $e \notin P H_0 H_{m-1}$. We consider $E_m = E_{m-1} *_{H_{m-1}} A_m$.

Case 1. Suppose that $x \notin H_0 H_m$ is implied by the syllable length of x; that is,

(1) $\|x\| \geqslant 3$; or,
(2) $\|x\| = 2$ and $x \in A_m E_{m-1}$.

Consider the case $x = e_1 a_1 \cdots e_n a_n$, where $e_i \in E_{m-1} \backslash H_{m-1}$ and $a_i \in A_m \backslash H_{m-1}$ (the other cases are similar). Since E_{m-1}, A_m are H_{m-1}-separable by Corollary 2.4, there exist $P_1 \triangleleft_f E_{m-1}$ and $Q_1 \triangleleft_f A_m$ such that $e_i \notin P_1 H_{m-1}$ and $a_i \notin Q_1 H_{m-1}$, for all i. Now, by Lemma 2.2, there exist $P \triangleleft_f E_{m-1}$ and $Q \triangleleft_f A_m$ such that $P \subset P_1$, $Q \subset Q_1$, $P \cap H_{m-1} = Q \cap H_{m-1}$, $PH_0 \cap PH_{m-1} = P$, and $QH_{m-1} \cap QH_m = Q$. Hence, considering the natural homomorphism $\pi\colon E_m \to (E_{m-1}/P) *_{\overline{H}_{m-1}} (A_m/Q)$, where $\overline{H}_{m-1} = H_{m-1}P/P = H_{m-1}Q/Q$, we have $\|\overline{x}\| = \|x\|$ and $\overline{H}_0 \cap \overline{H}_{m-1} = \langle 1 \rangle = \overline{H}_{m-1} \cap \overline{H}_m$. Then clearly $\overline{x} \notin \overline{H_0 H_m}$.

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Case 2. $\|x\| = 2$ and $x \notin A_m E_{m-1}$; that is, $x = ea$ where $e \in E_{m-1} \setminus H_{m-1}$ and $a \in A_m \setminus H_{m-1}$. Thus, by Corollary 2.4, there exist $P_1 \lhd_f E_{m-1}$ and $Q_1 \lhd_f A_m$ such that $e \notin P_1 H_{m-1}$ and $a \notin Q_1 H_{m-1}$. We note that $x = ea \notin H_0 H_m$ if, and only if, one of the following is true:

(1) $e \notin H_0 H_{m-1}$; or
(2) $e = h_1 r$ and $ra \notin H_m$, where $h_1 \in H_0$ and $r \in H_{m-1}$.

If (1) is true then, by the induction hypothesis, there exists $P_2 \lhd_f E_{m-1}$ such that $e \notin P_2 H_0 H_{m-1}$. Then, by Lemma 2.2, there exist $P \lhd_f E_{m-1}$ and $Q \lhd_f A_m$ such that $P \subset P_1 \cap P_2$, $Q \subset Q_1$, $P \cap H_{m-1} = Q \cap H_{m-1}$, $PH_0 \cap PH_{m-1} = P$, and $QH_{m-1} \cap QH_m = Q$. Consider the homomorphism $\pi\colon E_m \to E_{m-1}/P *_{\overline{H}_{m-1}} A_m/Q$, as above. Note that $\overline{H}_0 \cap \overline{H}_{m-1} = 1 = \overline{H}_{m-1} \cap \overline{H}_m$, $\overline{x} = \overline{e}\overline{a}$ and $\overline{e} \notin \overline{H_0 H}_{m-1}$. It follows that $\overline{x} \notin \overline{H_0 H_m}$.

If (2) is true then, since A_m is $\mathcal{LERF}$, we can find $Q_2 \lhd_f A_m$ such that $ra \notin Q_2 H_m$. As before, we can find $P \lhd_f E_{m-1}$ and $Q \lhd_f A_m$ such that $P \subset P_1$, $Q \subset Q_1 \cap Q_2$, $P \cap H_{m-1} = Q \cap H_{m-1}$, $PH_0 \cap PH_{m-1} = P$, and $QH_{m-1} \cap QH_m = Q$. Then, as before, we have $\overline{x} = \overline{ea} \notin \overline{H_0 H_m}$, where $\overline{E_m} = E_m \pi$.

Case 3. $\|x\| = 1$. Consider the case $x \in E_{m-1} \setminus H_{m-1}$ (the other case being similar). Since $x \notin H_0$, there exists $P_1 \lhd_f E_{m-1}$ such that $x \notin P_1 H_0 \cup P_1 H_{m-1}$. Now, by Lemma 2.2, there exist $P \lhd_f E_{m-1}$ and $Q \lhd_f A_m$ such that $P \subset P_1$, $P \cap H_{m-1} = Q \cap H_{m-1}$, $PH_0 \cap PH_{m-1} = P$, and $QH_{m-1} \cap QH_m = Q$. Then, we can easily see that $\overline{x} \notin \overline{H_0 H_m}$, where $\overline{E}_m = E_m \pi$ as above.

Case 4. $\|x\| = 0$. In this case we have $x \in H_{m-1}$ and $x \neq 1$. Since E_{m-1} is $\mathcal{RF}$ (Corollary 2.5), there exists $P_1 \lhd_f E_{m-1}$ such that $x \notin P_1$. As in Case 3, we can find $P \lhd_f E_{m-1}$ and $Q \lhd_f A_m$ such that $\overline{x} \notin \overline{H_0 H_m}$, where $\overline{E}_m = E_m \pi$.

Consequently, we have found $P \lhd_f E_{m-1}$ and $Q \lhd_f A_m$ such that $\overline{x} \notin \overline{H_0 H_m}$, where $\overline{E}_m = E_m \pi = E_{m-1}/P *_{\overline{H}_{m-1}} A_m/Q$. Since $|\overline{H}_0|$ and $|\overline{H}_m|$ are finite, and since $\overline{E}_m$ is $\mathcal{RF}$, it is not difficult to find $\overline{N} \lhd_f \overline{E}_m$ such that $\overline{x} \notin \overline{N H_0 H_m}$. Let N be the preimage of $\overline{N}$ in E_m. Then clearly, $N \lhd_f E_m$ and $x \notin N H_0 H_m$ as required. ∎

Definition 2.7: [9] Let $G = G_1 *_H G_2$. Let X, Y be subgroups of G_1, G_2 respectively. Let $\mathcal{N} = \{(N_i, M_i); i \in I\}$ be a collection of pairs of normal subgroups of G_1 and G_2 satisfying the following:

(1) $N_i \lhd G_1$, $M_i \lhd G_2$, and $N_i \cap H = M_i \cap H$, for all $i \in I$.
(2) $N_i \cap XH = (N_i \cap X)(N_i \cap H)$ and $M_i \cap YH = (M_i \cap Y)(M_i \cap H)$, for all $i \in I$,
(3) $\left(\bigcap_{j=1}^{n} N_{\alpha_j}, \bigcap_{j=1}^{n} M_{\alpha_j}\right) \in \mathcal{N}$ for all $\alpha_1, \ldots, \alpha_n \in I$, where n is finite,

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(4) $\bigcap_{i\in I} N_iX = X$, $\bigcap_{i\in I} N_iH = H$, $\bigcap_{i\in I} M_iY = Y$, and $\bigcap_{i\in I} M_iH = H$,

(5) $\bigcap_{i\in I} N_iXH = XH$ and $\bigcap_{i\in I} M_iYH = YH$.

Then $\mathcal{N}$ is called a *compatible filter* of G with respect to the subgroups X and Y.

Lemma 2.8. [9] *Let $G = G_1 *_H G_2$. Let X, Y be subgroups of G_1, G_2 respectively, such that $X \cap H = Y \cap H = 1$. Let $\mathcal{N}$ be a compatible filter of G with respect to X and Y. Then, for each $g \in G \setminus (X * Y)$ with $\|g\| \geqslant 1$, there exists $(N, M) \in \mathcal{N}$ such that $\|g\pi\| = \|g\|$ and $g\pi \notin X\pi * Y\pi$, where π is the canonical homomorphism of G onto $\overline{G} = \overline{G}_1 *_{\overline{H}} \overline{G}_2$, and where $\overline{G}_1 = G_1/N$, $\overline{G}_2 = G_2/M$ and $\overline{H} = HN/N = HM/M$.*

For example, we can see, by Corollary 2.4 and Lemma 2.6 together with Lemma 2.2, that $\mathcal{N} = \{(P, Q) : P \lhd_f E_{m-1}, Q \lhd_f A_m, P \cap H_{m-1} = Q \cap H_{m-1}, PH_0 \cap PH_{m-1} = P, QH_{m-1} \cap QH_m = Q\}$ is a compatible filter of $E_m = E_{m-1} *_{H_{m-1}} A_m$ with respect to H_0 and H_m.

Theorem 2.9. *Let $G = G_1 *_H G_2$, and let $X < G_1$, $Y < G_2$ be such that $X \cap H = 1 = Y \cap H$. Suppose that G has a compatible filter $\mathcal{N} = \{(N_i, M_i) : i \in I\}$ of G with respect to X and Y, where $N_i \lhd_f G_1$ and $M_i \lhd_f G_2$, for all i, and suppose further that*

(W$'$) *for each $N_H \lhd_f H$ there exists $(N_j, M_j) \in \mathcal{N}$ such that $M_j \cap H = N_j \cap H \subset N_H$ for some $j \in I$.*

Then G is $X * Y$-separable whenever H is $\mathcal{RF}$.

Proof: Let $g \in G \setminus (X * Y)$.

Case 1. $g \in H$. Since $g \neq 1$ and H is $\mathcal{RF}$, there exists $N_H \lhd_f H$ such that $g \notin N_H$. By (W$'$), there exists $(N_j, M_j) \in \mathcal{N}$ such that $N_j \cap H = M_j \cap H \subset N_H$ for some $j \in I$. Then, $\overline{g} \notin \overline{X} * \overline{Y}$ where $\overline{G} = G_1/N_j *_{\overline{H}} G_2/M_j$ and $\overline{H} = N_jH/N_j = M_jH/M_j$. Now, $\overline{G}$ is $\mathcal{LERF}$ by Theorem 1.2. Hence, there exists $\overline{N} \lhd_f \overline{G}$ such that $\overline{g} \notin \overline{N}(\overline{X} * \overline{Y})$. Let N be the preimage of $\overline{N}$ in G. Then $N \lhd_f G$ and $g \notin N(X * Y)$.

Case 2. $g \notin H$. By Lemma 2.8, there exists $(N_j, M_j) \in \mathcal{N}$ such that $\overline{g} \notin \overline{X} * \overline{Y}$, where $\overline{G} = G_1/N_j *_{\overline{H}} G_2/M_j$. Then, as before, we can find $N \lhd_f G$ such that $g \notin N(X * Y)$. ∎

Lemma 2.10. *For each $N \lhd_f H_0 * H_m (m \geqslant 2)$, there exists $N_{E_m} \lhd_f E_m$ such that $N_{E_m} \cap (H_0 * H_m) = N$.*

Proof: There exists a natural homomorphism $\pi \colon E_m \to A_1/H_1 * A_m/H_{m-1}$, obtained by defining $z\pi = 1$, for all $z \in A_2 \cup \cdots \cup A_{m-1}$, if $m \geqslant 3$; or $z\pi = 1$, for all $z \in H_1$, if $m = 2$. Let $\overline{E}_m = E_m\pi = \overline{A}_1 * \overline{A}_m$, where $\overline{A}_1 = A_1/H_1$ and

$\overline{A}_m = A_m/H_{m-1}$. We note that $\overline{H}_0 \cong H_0$, $\overline{H}_m \cong H_m$ and $N \cong \overline{N} \lhd_f \overline{H}_0 * \overline{H}_m$. Now, considering $\overline{A}_1 * \overline{A}_m = \overline{A}_1 *_{\overline{H}_0} (\overline{H}_0 * \overline{H}_m) *_{\overline{H}_m} \overline{A}_m$, we have a homomorphism

$$\phi : \overline{E}_m \to (\overline{A}_1/\overline{N} \cap \overline{H}_0) *_{\widetilde{H}_0} (\overline{H}_0 * \overline{H}_m/\overline{N}) *_{\widetilde{H}_m} (\overline{A}_m/\overline{N} \cap \overline{H}_m),$$

where $\widetilde{H}_0 = \overline{H}_0/\overline{N} \cap \overline{H}_0 = \overline{H_0 N}/\overline{N}$ and $\widetilde{H}_m = \overline{H_m N}/\overline{N} = \overline{H}_m/\overline{N} \cap \overline{H}_m$. Since $\widetilde{H}_0$ and $\widetilde{H}_m$ are finite, therefore, $\overline{E}_m\phi$ is $\mathcal{RF}$. Note that $(\overline{H}_0 * \overline{H}_m)/\overline{N}$ is finite. It follows that there exists $\widetilde{M} \lhd_f \overline{E}_m\phi$ such that $\widetilde{M} \cap ((\overline{H}_0 * \overline{H}_m)/\overline{N}) = 1$. Now, let N_{E_m} be the preimage of $\widetilde{M}$ in E_m under the homomorphism $\pi \circ \phi$. Then $N_{E_m} \lhd_f E_m$ and $N_{E_m} \cap (H_0 * H_m) = N$ as required. ∎

Now we are ready to show our main result of this section.

Theorem 2.11. *Let G be the polygonal product of the polycyclic-by-finite groups $A_0, A_1, \dots, A_n$ $(n \geqslant 3)$, amalgamating any subgroups $H_0, H_1, \dots, H_n$, with trivial intersections, where $H_i \subset Z(A_i) \cap Z(A_{i+1})$ for all i, and where subscripts are taken modulo $n+1$. Then G is π_c.*

Proof: We write $G = E *_H F$, where $E = A_1 *_{H_1} \cdots *_{H_{n-2}} A_{n-1}$, $F = A_0 *_{H_n} A_n$, and $H = H_0 * H_{n-1}$. With G in this form, we can apply Theorem 1.4. For condition (a) in the theorem, Corollary 2.5 proves that E and F are π_c. Theorem 2.9, using Lemma 2.1, proves that E and F are H-separable. Also, Lemma 2.10 proves condition (b) in the theorem. Hence, by Theorem 1.4, G is π_c. ∎

We immediately have the following result:

Theorem 2.12. *Let G be the polygonal product of the f.g. abelian groups $A_0, A_1, \dots, A_n$ $(n \geqslant 3)$, amalgamating any subgroups $H_0, H_1, \dots, H_n$, with trivial intersections. Then G is π_c.*

We note that the above two results are generalisations of Theorem 3.4 in [**3**].

3. Subgroup separability ($\mathcal{LERF}$)

Now we consider the subgroup separability of polygonal products of f.g. abelian groups. Throughout this section we assume that the amalgamated subgroups of polygonal products are not trivial.

Lemma 3.1. [**1**] *If a group G contains a subgroup $F_2 \times F_2$, where F_2 is a free group of rank 2, then G is not $\mathcal{LERF}$.*

Theorem 3.2. *Let P be the polygonal product of the four $\mathcal{LERF}$ groups A, B, C, D, amalgamating the finite subgroups H_2, H_3, H_4, H_1, with trivial intersections, where the H_i are contained in the centres of the vertex groups containing them. Then P is $\mathcal{LERF}$ if, and only if, either $|H_1| = |H_3| = 2$ or $|H_2| = |H_4| = 2$.*

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PROOF: Let P_0 be the polygonal product of $A_0 = \langle H_1, H_2\rangle$, $B_0 = \langle H_2, H_3\rangle$, $C_0 = \langle H_3, H_4\rangle$, $D_0 = \langle H_4, H_1\rangle$ amalgamating H_2, H_3, H_4, H_1. Let $E = A_0 *_{H_2} B_0$, $F = D_0 *_{H_4} C_0$ and $H = H_1 * H_3$. Then $E = H_2 \times H$, $F = H_4 \times H$ and $P_0 = E *_H F = (H_2 * H_4) \times (H_1 * H_3)$.

($\Longrightarrow$) Note that $H_2 * H_4$ contains a free subgroup of rank 2 unless $|H_2| = 2 = |H_4|$ [**10**, p.195]. Similarly, $H_1 * H_3$ contains a free subgroup of rank 2 unless $|H_1| = 2 = |H_3|$. Now if P is $\mathcal{LERF}$ then P_0 is $\mathcal{LERF}$. It follows from Lemma 3.1 that $|H_2| = 2 = |H_4|$ or $|H_1| = 2 = |H_3|$.

($\Longleftarrow$) Assume that $|H_1| = 2 = |H_3|$. Then every subgroup of $H_1 * H_3$ is f.g., hence, $H_1 * H_3$ is $\mathcal{ERF}$. It follows from Theorem 1.3 that P_0 is $\mathcal{LERF}$. Note that $P = (((P_0 *_{A_0} A) *_{B_0} B) *_{C_0} C) *_{D_0} D$. Since A_0, B_0, C_0, D_0 are finite, by Theorem 1.2, P is $\mathcal{LERF}$. ∎

THEOREM 3.3. *Let P be the polygonal product of the four groups A, B, C, D, amalgamating the subgroups H_2, H_3, H_4, H_1, with trivial intersections, where the H_i are contained in the centres of the vertex groups containing them. If P is $\mathcal{LERF}$, then either $|H_1| = |H_3| = 2$ or $|H_2| = |H_4| = 2$.*

PROOF: If P is $\mathcal{LERF}$, then the subgroup P_0 constructed in the proof of Theorem 3.2 is also $\mathcal{LERF}$. It follows that $|H_1| = |H_3| = 2$, or $|H_2| = |H_4| = 2$, as in the proof of Theorem 3.2. ∎

COROLLARY 3.4. *Let G be the polygonal product of the f.g. abelian groups A, B, C, D, amalgamating the subgroups $\langle b\rangle$, $\langle c\rangle$, $\langle d\rangle$, $\langle a\rangle$, with trivial intersections. If G is $\mathcal{LERF}$, then $|a| = 2 = |c|$ or $|b| = 2 = |d|$. In particular, the polygonal product of the four free abelian groups amalgamating the cyclic subgroups, with trivial intersections, is not $\mathcal{LERF}$.*

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