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PRODUCTS OF LOCALLY FINITE GROUPS WITH min-p

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Abstract

The paper is devoted to showing that if the factorized group $G = AB$ is almost solvable, if A and B are π-subgroups with min-p for some prime p in π, and also if the hypercenter factor group $A/H(A)$ or $B/H(B)$ has min-p for every prime p, then G is a π-group with min-p for the prime p.

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Introduction

Let Y be a group theoretical property. Then a group is *almost-Y* if it contains a Y-subgroup of finite index. A group satisfies min-p for the prime p, if each of its p-subgroups satisfies the minimum condition on subgroups. A group is a *Černikov group* if it is almost abelian and satisfies the minimum condition on subgroups.

The main theorem of this note is the following.

THEOREM. *Let the almost soluble group $G = AB$ be the product of two π-subgroups A and B with* min-p *for some prime p in the set of primes π. If the hypercenter factor group $A/H(A)$ or $B/H(B)$ has* min-p *for every prime $p \in \pi$, then G is a π-group with* min-p *for the prime p.*

It seems likely that the additional requirement for the subgroups A or B that $A/H(A)$ or $B/H(B)$ satisfy min-p for every p can be omitted. One notes that this condition is in particular satisfied if A or B is hypercentral.

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Since by N. S. Černikov [5] every product of two almost central groups is almost soluble, the above theorem has the following consequence.

COROLLARY. *If the group $G = AB$ is the product of two almost central π-subgroups A and B with* min-p *for the prime p in the set of primes π, then G is an almost soluble π-group with* min-p.

In addition, a bound for the p^∞-rank of G in this situation is given in Section 6; see Corollary 6.3.

Notation

π = an arbitrary set of primes
$O_\pi(G)$ = the maximal normal π-subgroup of the group G
$T(G)$ = the torsion subgroup of G
$H(G)$ = the hypercenter of G
$Z(G)$ = the center of G
$J(G)$ = the intersection of all subgroups of finite index in G
$R(G)$ = the Hirsch-Plotkin radical of G
$N(S)$ = the normalizer of the subgroup S in G

2. Preparatory results

If the group $G = AB$ is the product of two subgroups A and B, then G is also called *factorized* by its subgroups A and B. A subset S of a factorized group $G = AB$ is called *factorized* (with respect to the factorization $G = AB$) if $ab \in S$ with $a \in A$ and $b \in B$ implies that $a \in S$. If N is a normal subgroup of $G = AB$, then the *factorizer* $X(N)$ of N is defined to be the smallest factorized subgroup of G containing N. It is easy to show that

$$X(N) = AN \cap BN = N(A \cap BN) = N(B \cap AN) = (A \cap BN)(B \cap AN);$$

see for instance [1, Theorem 1.7].

The following lemma of Wielandt is crucial for many arguments concerning factorized groups.

LEMMA 2.1. *Let the group $G = AB$ be the product of two subgroups A and B, and let A_0 and B_0 be normal subgroups of A and B, respectively. If $S = \langle A_0, B_0 \rangle$, then the normalizer $N_G(S)$ is factorized if at least one of the following conditions holds*

(i) *S is finite*;
(ii) *A or B is periodic.*

This may easily be proved directly. It also follows from [6, Lemma 1.2].

Let G be a group and π a set of primes. The maximal ith central normal π-subgroup of G, for $i = 1, 2, \ldots$, is denoted by $Z_{i\pi}(G)$. In particular, if $\pi = \tau$ is the set of all primes, then $Z_{i\tau}(G)$ denotes the maximal ith central torsion normal subgroup of G. If $\pi = \varnothing$ is the empty set of primes, then $Z_{i\pi}(G) = Z_i(G)$, for $i = 1, 2, \ldots$, is the ith term of the upper central series of G.

It is well known that the proof of a factorization theorem for almost soluble groups usually reduces quickly to a triple factorization problem. To deal with this situation the following lemma is needed.

LEMMA 2.2. *Let the group $G = AB = AK = BK$ be the product of two subgroups A and B and an abelian normal subgroup K of G such that $A \cap K = B \cap K = 1$. Then, for $i = 1, 2, \ldots$, we have*

$$Z_{i\pi}(A)K = Z_{i\pi}(B)K.$$

In particular, $Z_{i\tau}(A)K = Z_{i\tau}(B)K$ and $Z_i(A)K = Z_i(B)K$.

PROOF. By the assumption we observe that

$$A \cong A/(A \cap K) \cong AK/K = BK/K \cong B/(B \cap K) \cong B.$$

Therefore, if the π-part of any term of the upper central series of A or B is trivial, then the other is also trivial and the assertions follow.

Since $G = AK$, it follows that $Z_{i\pi}(A)K/K \subseteq Z_{i\pi}(G/K)$ for $i = 1, 2, \ldots$. Let Z be the uniquely determined normal subgroup of G such that $K \subseteq Z$ and $Z/K = Z_{i\pi}(G/K)$. Clearly, $Z = Z \cap AK = (Z \cap A)K$, and $Z/K = (Z \cap A)K/K \cong (Z \cap A)/K \cap (Z \cap A) \cong (Z \cap A)$. Now the properties of the upper central series give the following:

$$\begin{aligned} 1 &= [Z/K, G/K, \ldots, G/K] = [(Z \cap A)K/K, AK/K, \ldots, AK/K] \quad (i\text{-times}) \\ &= [(Z \cap A), A, \ldots, A]K/K \quad (i\text{-times}). \end{aligned}$$

Thus $[Z \cap A, A, \ldots, A] \subseteq A \cap K = 1$, and this implies that $(Z \cap A)K \subseteq Z_{i\pi}(A)K = Z$ for $i = 1, 2, \ldots$.

Similarly, one obtains that $Z_{i\pi}(B)K = Z$ and hence $Z_{i\pi}(A)K = Z_{i\pi}(B)K$ for $i = 1, 2, \ldots$. This proves the lemma.

For the hypercenter $H(A)$ and its π-component $H(A)_\pi$ we obtain the following corollary.

COROLLARY 2.3. *If the group $G = AB = AK = BK$ is the product of two subgroups A and B and an abelian normal subgroup K of G such that $A \cap K = B \cap K = 1$, then for every set of primes π, we have $H(A)_\pi K = H(B)_\pi K$; in particular, $H(A)_\tau K = H(B)_\tau K$ and $H(A)K = H(B)K$.*

The next lemma deals again with a triple factorization.

LEMMA 2.4. *Let the group $G = AB = AK = BK$ be the product of two subgroups A and B and an abelian normal subgroup K of G with $A \cap K = B \cap K = 1$, and for some prime q let the maximal normal q-subgroups $O_q(A)$ and $O_q(B)$ be Černikov groups. Then $O_q(A) = O_q(B)$ if at least one of the following conditions holds:*
(i) *K is a p-group for some prime $p \neq q$;*
(ii) *K is torsionfree and A or B is periodic.*

PROOF. Clearly $A \cong B$. By [2, Lemma 1.2], $J(O_q(A))K = J(O_q(B))K$, and this group is metabelian. If $1 \neq a \in J(O_q(A))$, then $a = bx$ with $b \in J(O_q(B))$ and $x \in K$. Then a is contained in a finite normal q-subgroup A_1 of A. Also b is contained in a finite normal q-subgroup B_1 of B. Clearly $S = \langle A_1, B_1 \rangle$ is soluble, and in case (i) it is also finite. In both cases (i) and (ii) the normalizer $N(S)$ is factorized by Lemma 2.1. Application of [1, Proposition 5.3] yields that S is a finite q-subgroup. Then $x = b^{-1}a$ is a p-element and a q-element, so that $x = 1$. It follows that $a = b \in J(O_q(B))$. This shows that $J(O_q(A)) \subseteq J(O_q(B))$. Since the other inclusion follows similarly, we have $J(O_q(A)) = J(O_q(B)) = D$, say.

In view of the identities $G/D = (A/D)(B/D) = (A/D)(KD/D) = (B/D)(KD/D)$, we may assume that $D = 1$. Hence $O_q(A)$ and $O_q(B)$ are finite. By [2, Lemma 1.2], $O_q(A)K = O_q(B)K$. In the same way as above, it follows that $O_q(A) = O_q(B)$. The lemma is proved.

3. Products of π-groups

The following theorem gives a condition for the product of two π-groups to be a π-group.

THEOREM 3.1. *If the almost soluble group $G = AB$ is the product of two π-subgroups A and B with* min-p *for every prime p in the set of primes π, then G is a π-group.*

PROOF. Assume that the theorem is false and let $G = AB$ be a counterexample. Then G is infinite and $O_\pi(G) \subsetneqq G$. Since

$$G/O_\pi(G) = \left(AO_\pi(G)/O_\pi(G)\right)\left(BO_\pi(G)/O_\pi(G)\right)$$

is also a counterexample, it may be assumed that $O_\pi(G) = 1$.

Since G is infinite and almost soluble, there exists an abelian normal subgroup $K \neq 1$ of G. Then $O_\pi(K) = 1$. The factorizer of K has the form $X = X(K) = A^*B^* = A^*K = B^*K$, where $A^* = A \cap BK$ and $B^* = B \cap AK$. Since X is also a

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counterexample, it may be assumed that $G = AB = AK = BK$ with a nontrivial abelian normal subgroup K of G. As above, we may assume that $O_\pi(G) = 1$. Clearly $A \cap K$ is normal in $AK = G$, and $B \cap K$ is normal in $BK = G$. Since $O_\pi(G) = 1$, it follows that $A \cap K = B \cap K = 1$. Now we distinguish two cases: (i) $T(K) \neq 1$, so that it may be assumed that $K = T(K)$ is a p-group for some prime $p \notin \pi$; (ii) K is a torsionfree abelian group. Since A and B are periodic almost soluble groups, $O_q(A) \neq 1 \neq O_q(B)$ for some prime $q \in \pi$. By hypothesis $O_q(A)$ and $O_q(B)$ are Černikov groups. Now in both cases, Lemma 2.4 yields that $O_q(A) = O_q(B)$ is normal in G. This contradicts the fact that $O_\pi(G) = 1$. The theorem is proved.

The next theorem gives another condition for an almost soluble product of two π-groups to be a π-group.

THEOREM 3.2. *Let the almost soluble group $G = AB$ be the product of two subgroups A and B such that $A/H(A)$ or $B/H(B)$ is almost locally normal. If A and B are π-groups for the set of primes π, then G is a π-group. In particular, if A and B are periodic, then G is periodic.*

PROOF. Assume that G is not a π-group. Then G is infinite and $O_\pi(G) \subsetneqq G$. Since $G/O_\pi(G)$ is also a counterexample, it may be assumed that $O_\pi(G) = 1$.

Since G is infinite and almost soluble, there exists an abelian normal subgroup $K \neq 1$ of G. The factorizer of K is

$$X(K) = (A \cap BK)(B \cap AK) = K(A \cap BK) = K(B \cap AK).$$

Since this is also a counterexample, it may be assumed that $G = AB = AK = BK$, where K is a nontrivial abelian normal subgroup of G. As above, we may assume that $O_\pi(G) = 1$. Clearly $A \cap K$ is normal in $AK = G$, and $B \cap K$ is normal in $BK = G$. Since $A \cap K$ and $B \cap K$ are normal π-subgroups of G, it follows that $A \cap K = B \cap K = 1$. This implies that $A \cong B$, so that $A/H(A)$ and $B/H(B)$ are almost locally normal.

If $H(A) = H(B) = 1$, then A and B are almost locally normal, and G is a π-group by [1, Theorem 5.4]. This is a contradiction, so that $H(A) \neq 1 \neq H(B)$, and hence also $Z(A) \neq 1 \neq Z(B)$.

Assume that K is torsionfree. Application of [4, Lemma 2.2] yields that $Z_\pi(A) = Z_\pi(B)$ is normal in $AB = G$. Since $O_\pi(G) = 1$, it follows $Z(A) = Z(B) = 1$. This contradiction shows that $T = T(K) \neq 1$.

Since the factorizer $X(T)$ is also a counterexample, it may be assumed that $G = AB = AK = BK$, where K is a normal π'-subgroup of G. As above, it follows that $A \cap K = B \cap K = 1$, that $O_\pi(G) = 1$, and that $Z(A) \neq 1 \neq Z(B)$.

By Lemma 2.2, $Z(A)K = Z(B)K$. If $1 \neq a \in Z(A)$, then $a = bx$ with $b \in Z(B)$ and $x \in K$. Clearly $\langle a \rangle$ is normal in a, $\langle b \rangle$ is normal in B, and $S = \langle a, b \rangle$ is soluble. By Lemma 2.1, $N(S)$ is factorized. By [1, Proposition 5.3]

S is a finite π-group. Therefore the π'-element $x = b^{-1}a \in S$ is also a π-element, so that $x = 1$. This gives $a = b \in Z(B)$, and hence $Z(A) \subseteq Z(B)$. Similarly, one sees that $Z(B) \subseteq Z(A)$. Hence $Z(A) = Z(B)$ is a normal π-subgroup of G, so that $Z(A) = Z(B) = 1$. This contradiction proves the theorem.

4. The condition 'min-p' for every prime p

If G is an almost soluble group with min-p for every prime p, then the finite residual $J(G)$ is a radicable torsion group with artinian primary components, and $G/J(G)$ is residually finite with finite maximal p-subgroups for every prime p; see [7, Corollary 3.18, page 95]. The following theorem generalizes some of the results in [3].

THEOREM 4.1. *If the almost soluble group $G = AB$ is the product of two π-subgroups A and B with* min-p *for every prime p, then G is a π-group with* min-p *for every prime p.*

PROOF. By Theorem 3.1, G is a π-group. Assume now that G does not satisfy min-p for some prime p. Then there exists an infinite elementary abelian normal p-factor of G. Considering a suitable epimorphic image of G, we may assume that G contains an infinite elementary abelian normal p-subgroup K. The factorizer of K has the form

$$X = X(K) = (A \cap BK)(B \cap AK) = K(A \cap BK) = K(B \cap AK).$$

Therefore we may assume that $G = AB = AK = BK$ is a counterexample, where K is an elementary abelian normal p-subgroup of G. Factoring out $O_{p'}(G)$, we may also assume that $O_{p'}(G) = 1$. Clearly $A \cap K$ and $B \cap K$ are finite normal p-subgroups of G, and so $C = (A \cap K)(B \cap K)$ is a finite normal p-subgroup of G. Factoring out C, we obtain

$$G/C = (AC/C)(BC/C) = (AC/C)(K/C) = (BC/C)(K/C),$$

where $(AC/C) \cap (K/C) = (A \cap K)C/C = 1$, and where $(BC/C) \cap (K/C) = 1$. Now, using [7, Lemma 3.16, page 94], we may interchange the finite factor with the p'-factor to obtain that $G/O_{p'}(G)$ is finite. Therefore we may assume that $G = AB = AK = BK$, where $A \cap K = B \cap K = 1$, where A and B are subgroups with min-p for every prime p, where K is an elementary abelian normal p-subgroup, and where $O_{p'}(G)$ is finite.

Since G is locally finite, it follows that for every prime q, $O_q(G) = R_q(G)$ is the q-component of the Hirsch-Plotkin radical $R(G)$ of G. By Lemma 2.4, $O_q(A) = O_q(B) \subseteq O_q(G)$. Therefore

$$R_{p'}(A) = \prod_{q \neq p} O_q(A) \subseteq R_{p'}(G) \subseteq O_{p'}(G),$$

so that $R_{p'}(A)$ is finite. Since $R_p(A) = O_p(A)$ is a Černikov group, $R(A) = R_{p'}(A) \times R_p(A)$ is also a Černikov group. By assumption, A contains a soluble normal subgroup N of finite index. Since the characteristic subgroup $R(N)$ of N is normal in A, we have $R(N) \subseteq R(A)$, so that $R(N)$ is also a Černikov group. Since N is soluble, we have $C_N(R(N)) \subseteq R(N)$. As a periodic group of automorphisms of a Černikov group, $N/C_N(R(N))$ is also a Černikov group; see [7, Theorem 1.F.3]. Hence $N/R(N)$ is also a Černikov group. This implies that N, and hence also A, is a Černikov group. Since B is isomorphic to A, it, too, is a Černikov group. Hence by [1, Theorem 5.5], $G = AB$ is also a Černikov group. But then K must be finite. This contradiction proves the theorem.

5. The condition 'min-p' for some prime p

We are now ready to prove the following theorem, which is identical to the theorem in the introduction.

THEOREM 5.1. *Let the almost soluble group $G = AB$ be the product of two π-subgroups A and B with* min-p *for some prime p in the set of primes π. If $A/H(A)$ or $B/H(B)$ has* min-p *for every $p \in \pi$, then G is a π-group with* min-p *for the prime p.*

PROOF. By Theorem 3.1, G is a π-group. Assume that the theorem is false and let $G = AB$ be a counterexample, so that G does not satisfy min-p for the prime p. Since G is infinite and almost soluble, there exists an abelian normal subgroup K of G. Factoring out $O_{p'}(G)$, we may assume that K is a p-group.

The socle $M = K[p]$ is an elementary abelian p-group which is infinite, since K does not satisfy min-p. The factorizer of M has the form

$$X = X(M) = (A \cap BM)(B \cap AM) = M(A \cap BM) = M(B \cap AM).$$

Clearly $A \cap M$ and $B \cap M$ are finite normal p-subgroups of X. Hence $M/(A \cap M)(B \cap M)$ is an infinite elementary abelian normal p-subgroup of $X/(A \cap M)(B \cap M)$. Therefore we may assume that $G = AB = AM = BM$, where $A \cap M = B \cap M = 1$, and where M is an infinite elementary abelian normal p-subgroup of G.

Since $A \cong G/M \cong B$, it follows that $A/H(A)$ and $B/H(B)$ both satisfy min-p for every prime $p \in \pi$. Recall that the hypercenter of a group is nontrivial if and only if its center is nontrivial. Therefore, by Lemma 2.2, $Z(A)_{p'}M = Z(B)_{p'}M$, which is a metabelian group. If $Z(A)_{p'} \neq 1$, then there exists $1 \neq a \in Z(A)_{p'}$ such that $a = bx$, where $b \in Z(B)_{p'}$ and $x \in M$. Clearly $\langle a \rangle$ is normal in A,

$\langle b \rangle$ is normal in B, and $S = \langle a, b \rangle$ is a finite soluble group. By Lemma 2.1, $N(S)$ is factorized, and by [1, Proposition 5.3], S is a p'-group. Hence $x = b^{-1}a$ is both a p-element and a p'-element, so that $x = 1$. This shows that $Z(A)_{p'} \subseteq Z(B)_{p'}$. Similarly $Z(B)_{p'} \subseteq Z(A)_{p'}$, and hence $Q = Z(A)_{p'} = Z(B)_{p'} \subseteq Z(G)$. Factoring out this normal subgroup, we obtain

$$G/Q = (AQ/Q)(BQ/Q) = (A/Q)(MQ/Q) = (B/Q)(MQ/Q),$$

where $A \cap MQ = Q(A \cap M)$ and $B \cap AM = Q(B \cap M) = Q$. Thus, without loss of generality, we may assume that $Z(A)_{p'} = Z(B)_{p'} = 1$. Hence $Z(A)$ and $Z(B)$ are p-groups.

By [8, Theorem 2.25, p. 53], the hypercenters $H(A)$ and $H(B)$ are also p-groups, and, since they satisfy min-p, they are Černikov groups. It follows that A and B satisfy min-p for every prime p. Hence, by Theorem 4.1, G satisfies min-p for every prime p. This implies that M is finite. This contradiction proves the theorem.

6. The p^∞-rank

If G is a locally finite group with min-p for the prime p, then by [7, Section 3.A], the so-called Sylow-p-subgroups of G are isomorphic. Thus we can define the *p^∞-rank* $m_p(G)$ of G to be the number of Prüfer p-subgroups (C_{p^∞}-factors) in a Sylow-p-subgroup of G.

LEMMA 6.1. *If G is a locally finite group with* min-p *for the prime p, then* $m_p(G) = m_p(G/O_{p'}(G))$.

PROOF. By [7, Theorem 3.17], $G/O_{p'}(G)$ is an almost p-group. Since G has min-p, $G/O_{p'}(G)$ is a Černikov group. Therefore $P/O_{p'}(G) = J(G/O_{p'}(G))$ is a countable p-group. By [7, Lemma 1.D.4], there exists a p-subgroup S of G such that $G = O_{p'}(G)S$, so that $S \cong G/O_{p'}(G)$. Hence $m_p(G) = m_p(S) = m_p(G/O_{p'}(G))$.

LEMMA 6.2. *Let the almost locally-(finite and soluble) group $G = AB$ with* min-p *for the prime p be the product of two subgroups A and B. Then $m_p(G) \leqslant m_p(A) + m_p(B) - m_p(A \cap B)$.*

PROOF. Let $O = O_{p'}(G)$. By [7, Theorem 3.17], G/O is an almost-p Černikov group. It follows from [1, Lemma 5.6] that

$$J(G/O) = J(AO/O)J(BO/O).$$

Using Lemma 6.1, we obtain

$$m_p(G) = m_p(G/O) = m_p(AO/O) + m_p(BO/O) - m_p(AO \cap BO/O)$$
$$\leqslant m_p(A) + m_p(B) - m_p(A \cap B).$$

Since every product of two almost central groups is almost soluble, by Theorem 5.1 we have the following.

COROLLARY 6.3. *If the group* $G = AB$ *is the product of two almost central* π*-subgroups* A *and* B *with* min-p *for the prime* p, *then* G *is an almost soluble* π*-group with* min-p *and*

$$m_p(G) \leqslant m_p(A) + m_p(B) - m_p(A \cap B).$$

Added in proof

The results of Section 3 are a special case of a theorem of Y. P. Sysak, Preprint 82.53, Kiev (1982).

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