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LEFT ANNIHILATORS CHARACTERIZED BY DIFFERENTIAL IDENTITIES

TSIU-KWEN LEE AND CHING-YUEH PAN

Department of Mathematics, National Taiwan University, Taipei 106, Taiwan (tklee@math.ntu.edu.tw; cypan@math.ntu.edu.tw)

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Abstract Let R be a semiprime ring with U_{s} its maximal symmetric ring of quotients and let ρ_1 and ρ_2 be two right ideals of R. We show that $\ell_R(\rho_1) = \ell_R(\rho_2)$ if and only if ρ_1 and ρ_2 satisfy the same differential identities with coefficients in U_{s}, where $\ell_R(\rho_i)$ denotes the left annihilator of ρ_i in R. This gives a generalization of several previous results in this area.

Keywords: semiprime ring; maximal ring of quotients; generalized polynomial identities (GPIs); differential identity

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1. Introduction

Throughout, R always denotes a semiprime ring. A right ideal ρ of R is said to be *dense* if ρ_R is a dense submodule of R_R. That is, given $x, y \in R$ with $y \neq 0$, there exists $r \in R$ such that $xr \in \rho$ and $yr \neq 0$. The maximal right ring of quotients (or the right Utumi quotient ring used in [**8**]) of R can be characterized as a ring U satisfying the following axioms.

(1) R is a subring of U.

(2) For each $a \in U$, there exists a dense right ideal ρ of R such that $a\rho \subseteq R$.

(3) If $a \in U$ and $a\rho = 0$ for some dense right ideal ρ of R, then $a = 0$.

(4) For any dense right ideal ρ of R and for any right R-module homomorphism $\phi : \rho_R \to R_R$, there exists $a \in U$ such that $\phi(x) = ax$ for all $x \in \rho$.

The maximal symmetric ring of quotients U_{s} of R is then defined as

$$U_{\mathrm{s}} = \{x \in U \mid \lambda x \subseteq R \text{ for some dense left ideal } \lambda \text{ of } R\}.$$

Then U and U_{s} are still semiprime rings and have the same centre, denoted by C, which is called the extended centroid of R. For these basic properties we refer to [**2**]. The lifting properties of generalized polynomial identities (or GPIs for brevity) have been studied by Beidar [**1**] and by Chuang [**3**]. We mention the two main results here.

Theorem 1.1 (Beidar). *Let R be a semiprime ring. Then R and U satisfy the same GPIs with coefficients in U.*

Theorem 1.2 (Chuang). *Let R be a prime ring. Then R and each dense right ideal of R satisfy the same GPIs with coefficients in U.*

In [**8**], the first-named author gave another viewpoint on the two theorems above. We characterize the GPIs satisfied by two right ideals ρ_1 and ρ_2 of a semiprime ring R by checking left annihilators of ρ_1 and ρ_2. For a subset A of R we denote by $\ell_R(A)$ the left annihilator of A in R, that is, $\ell_R(A) = \{x \in R \mid xA = 0\}$. Similarly, we define the right annihilator $r_R(A)$ of A in R. Applying [**8**, Main Theorem] and Theorem 1.1 we have the following immediate consequence.

Theorem 1.3. *Let R be a semiprime ring with two right ideals ρ_1 and ρ_2. Then $\ell_R(\rho_1) = \ell_R(\rho_2)$ if and only if ρ_1 and ρ_2 satisfy the same GPIs with coefficients in U_{s}.*

This theorem says that to test whether the two right ideals ρ_1 and ρ_2 satisfy the same GPIs with coefficients in U_{s}, it suffices to check only the simplest type aX with $a \in U_{\mathrm{s}}$. In further study along this line, the first-named author [**9**] considered the lifting properties of differential identities (DIs) on right ideals with zero left annihilators. The structure theory of differential identities has been established by Kharchenko in [**6**, **7**]. To state these results precisely, let us recall some notation. An additive map $d : A \to U$, where A is a subring of U, is called a *derivation* if $(xy)^d = x^d y + xy^d$ for all $x, y \in A$. We denote by $\mathrm{Der}(U)$ the set of all derivations of U. For $d \in \mathrm{Der}(U)$ and $x \in U$, we define $x^{d\beta} = x^d\beta$ for $\beta \in C$. It follows that $\mathrm{Der}(U)$ forms a right C-module. Let $\boldsymbol{D}$ be the C-submodule of $\mathrm{Der}(U)$ defined by

$$\boldsymbol{D} = \{\delta \in \mathrm{Der}(U) \mid I^{\delta} \subseteq R \text{ for some dense ideal } I, \text{ depending on } \delta, \text{ of } R\}.$$

In fact, every derivation $d : I \to R$, where I is a dense ideal of R, can be uniquely extended to a derivation of U such that $U_{\mathrm{s}}^d \subseteq U_{\mathrm{s}}$. Thus $d \in \boldsymbol{D}$ in this case. By a *derivation word* we mean an additive map Δ from U into itself assuming the form $\Delta = \delta_1\delta_2\cdots\delta_t$, where each $\delta_i \in \boldsymbol{D}$. If Δ is empty, we define $x^{\Delta} = x$ for $x \in U$. A *differential polynomial* means a generalized polynomial with coefficients in U and with non-commuting variables X_i which are acted on by derivation words. Thus every differential polynomial can be written in the form $\phi(X_i^{\Delta_j})$, where $\phi(Z_{ij})$ is a generalized polynomial over U in distinct indeterminates Z_{ij}, and the Δ_j are derivation words. A differential polynomial $\phi(X_i^{\Delta_j})$ is called a *differential identity* (DI) for a subset T of U if $\phi(X_i^{\Delta_j})$ assumes 0 for any assignment of values from T to its indeterminates X_i. The first-named author proved the following theorem (see [**9**, Theorem 6]).

Theorem 1.4. *Let R be a semiprime ring and let ρ be a right ideal of R with zero left annihilator. Then R and ρ satisfy the same DIs with coefficients in U_{s}.*

The first-named author also raised a natural generalization of Theorem 1.4 (see [**9**, p. 807]). Let ρ_1 and ρ_2 be two right ideals of a semiprime ring R with $\ell_R(\rho_1) = \ell_R(\rho_2)$. Do ρ_1 and ρ_2 satisfy the same DIs with coefficients in U_{s}? The purpose of this paper is to give an answer in the affirmative. Our main theorem is as follows.

Theorem 1.5. *Let R be a semiprime ring with two right ideals ρ_1 and ρ_2. Then $\ell_R(\rho_1) = \ell_R(\rho_2)$ if and only if ρ_1 and ρ_2 satisfy the same DIs with coefficients in U_{s}.*

Theorem 1.5 implies that, to test whether the two right ideals ρ_1 and ρ_2 satisfy the same DIs with coefficients in U_{s}, it suffices to check only the simplest type aX with $a \in U_{\mathrm{s}}$. Also, Theorem 1.4 is an immediate consequence of Theorem 1.5. Indeed, if ρ is a right ideal of R with $\ell_R(\rho) = 0$, then $\ell_R(\rho) = 0 = \ell_R(R)$. Thus, by Theorem 1.5, R and ρ satisfy the same DIs with coefficients in U_{s}, as asserted.

2. The prime case

The aim of this section is to prove the prime case of Theorem 1.5. We denote by Q the symmetric Martindale quotient ring of R, that is

$$Q = \{x \in U \mid Ix + xI \subseteq R \text{ for some dense ideal } I \text{ of } R\}.$$

We note that $U_{\mathrm{s}} = Q$ if R is a prime ring with non-zero socle $\mathrm{soc}(R)$. A derivation $d \in \boldsymbol{D}$ is called X-inner if d is an inner derivation induced by an element of Q. Otherwise, it is called X-outer. We set D_{int} to be the C-submodule of $\mathrm{Der}(U)$ consisting of all X-inner derivations. Then the following hold.

(1) For $\delta, d \in \boldsymbol{D}$ we have that $[\delta, d] \in \boldsymbol{D}$ and $\delta^p \in \boldsymbol{D}$ if $\mathrm{char}\, R = p$ is a prime integer.

(2) $D_{\mathrm{int}} \subseteq \boldsymbol{D} \subseteq \mathrm{Der}(Q)$.

(3) If $\delta \in \boldsymbol{D}$ is U-inner, then δ must be X-inner.

Recall the following basic identities due to Kharchenko [**6**, p. 155].

(B1) $(XY)^\delta = X^\delta Y + XY^\delta$ for $\delta \in \boldsymbol{D}$.

(B2) $(X + Y)^\delta = X^\delta + Y^\delta$ for $\delta \in \boldsymbol{D}$.

(B3) $X^\delta = Xa - aX$ if δ is the inner derivation induced by $a \in Q$.

(B4) $X^{[d,\delta]} = (X^d)^\delta - (X^\delta)^d$ for $d, \delta \in \boldsymbol{D}$.

(B5) $X^{\delta^p} = (\cdots((X^\delta)^\delta)\cdots)^\delta$ (p-times) for $\delta \in \boldsymbol{D}$ and $\mathrm{char}\, R = p > 0$. If $\mathrm{char}\, R = 0$, then this identity assumes the form $X = X$.

(B6) $X^{d\alpha+\delta\beta} = \alpha X^d + \beta X^\delta$ for $d, \delta \in \boldsymbol{D}$ and $\alpha, \beta \in C$.

Let R always be a prime ring from now on. Then C is a field. We choose a fixed basis M_0 for D_{int} and augment it to a basis M for $\boldsymbol{D}$ over C. Fix a total order '$>$' in the set M such that $\mu_0 > \mu$ for $\mu_0 \in M_0$ and $\mu \in M \setminus M_0$, and then extend this order to the set of all derivation words by assuming that a longer word is greater than a shorter one and that words of the same length are ordered lexicographically. A *regular word* means a derivation word of the form $\Delta = \delta_1^{s_1}\delta_2^{s_2}\cdots\delta_m^{s_m}$ possessing the following properties:

(W1) $\delta_i \in M \setminus M_0$ for $1 \leqslant i \leqslant m$;

(W2) $\delta_1 < \delta_2 < \cdots < \delta_m$; and

(W3) $s_i < p$ for $1 \leqslant i \leqslant m$, if $\operatorname{char} R = p > 0$.

As pointed out in [**6**, **7**], each differential identity can be transformed, via the basic identities (B1)–(B6), into a form $\phi(X_i^{\Delta_j})$ such that

(R1) $\phi(Z_{ij})$ is a generalized polynomial with coefficients in U in non-commuting indeterminates Z_{ij}; and

(R2) the Δ_j are distinct regular words.

A differential polynomial is called *reduced* if it assumes the form $\phi(X_i^{\Delta_j})$ satisfying (R1) and (R2). Kharchenko actually proved the following powerful result [**7**, Theorem 2].

Theorem 2.1 (Kharchenko's Theorem). *Let R be a prime ring. If $\phi(X_i^{\Delta_j})$ is a reduced DI (with coefficients in U) for a non-zero ideal of R, then $\phi(Z_{ij})$ is a GPI for R.*

Since every differential polynomial can be transformed into a reduced differential polynomial via (B1)–(B6) and $\rho R \subseteq \rho \subseteq \rho C \subseteq \rho U$, applying Theorems 2.1, 1.1 and 1.2 we have the following corollary.

Corollary 2.2. *Let R be a prime ring with extended centroid C and let ρ be a right ideal of R. Then ρU, ρC and ρ satisfy the same DIs with coefficients in U.*

Corollary 2.3. *Let R be a prime ring with I a non-zero ideal of R. Then I and U satisfy the same DIs with coefficients in U.*

Let B be a set of C-independent elements of U_{s} and let $\Delta_1, \Delta_2, \ldots, \Delta_t$ be distinct regular words. A B-monomial in X^{Δ_i} means a monomial of the form $u_0Y_1u_1Y_2\cdots Y_nu_n$, where $u_i \in B$ and $Y_i \in \{X^{\Delta_1}, X^{\Delta_2}, \ldots, X^{\Delta_t}\}$ for each i. Here, each u_iY_{i+1} is called a submonomial appeared in this monomial. Thus for each non-zero $\phi \in U_{\mathrm{s}} *_C C\{X^{\Delta_1}, X^{\Delta_2}, \ldots, X^{\Delta_t}\}$, the free product of the C-algebra U_{s} and the free C-algebra $C\{X^{\Delta_1}, X^{\Delta_2}, \ldots, X^{\Delta_t}\}$, there exists a B such that ϕ is a C-linear combination of B-monomials in X^{Δ_i}. Also, B is said to be C-independent modulo $\ell_{U_{\mathrm{s}}}(\rho)$, where ρ is a right ideal of R, if B satisfies the following condition: if $\beta_1, \ldots, \beta_\ell \in C$ satisfy $(\beta_1 b_1 + \cdots + \beta_\ell b_\ell)\rho = 0$, where these b_i are distinct elements in B, then $\beta_i = 0$ for all i. We begin our proof with the following result [**8**, Lemma 3].

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Lemma 2.4. *Let ρ be a non-zero right ideal of R. Suppose that $a_1, a_2, \dots, a_t \in U_{\mathrm{s}}$ are C-independent modulo $\ell_{U_{\mathrm{s}}}(\rho)$. Then there exists an element $u \in \rho$ such that $a_1 u, \dots, a_t u$ are C-independent unless R is a PI-ring.*

We write $U_{\mathrm{s}} = \ell_{U_{\mathrm{s}}}(\rho) \oplus W_\rho$ as C-spaces and fix a basis B_ρ for W_ρ. Thus B_ρ is C-independent modulo $\ell_{U_{\mathrm{s}}}(\rho)$.

Lemma 2.5. *Let ρ be a right ideal of R and let Δ be a non-empty regular word. If $a \in \ell_{U_{\mathrm{s}}}(\rho)$, then there exist finitely many regular words $G_n < G_{n-1} < \dots < G_1 < \Delta$ such that*

$$ax^{\Delta} = \sum_{i=1}^{n} \biggl(\sum_{j=1}^{m_i} \beta_{ij} b_{ij} \biggr) x^{G_i} \tag{2.1}$$

for all $x \in r_R(\ell_R(\rho))$, where, for each i, the set $\{b_{ij} \mid j = 1, \dots, m_i\}$ is C-independent in B_ρ, and $\beta_{ij} \in C$.

Proof. Since $a \in \ell_{U_{\mathrm{s}}}(\rho)$, we have $a\rho = 0$ and hence $ax = 0$ for all $x \in r_R(\ell_R(\rho))$. Let $x \in r_R(\ell_R(\rho))$. Then

$$0 = (ax)^{\Delta} = ax^{\Delta} + \sum_{i>1} a^{E_i} x^{F_i},$$

where these (E_i, F_i) run over all pairs of subwords of Δ with $E_1 = \emptyset$, $F_1 = \Delta$. Note that $a^{E_i} \in U_{\mathrm{s}}$ since ${U_{\mathrm{s}}}^{E_i} \subseteq U_{\mathrm{s}}$. By the fact that $U_{\mathrm{s}} = \ell_{U_{\mathrm{s}}}(\rho) \oplus B_\rho C$, we can write

$$ax^{\Delta} = \sum_{i>1} \sum_{j} \mu_{ij} c_{ij} x^{F_i} + \sum_{i>1} c_i x^{F_i},$$

where $c_{ij} \in B_\rho$, $c_i \in \ell_{U_{\mathrm{s}}}(\rho)$ and $\mu_{ij} \in C$. Since $c_i \in \ell_{U_{\mathrm{s}}}(\rho)$ and $F_i < \Delta$ for $i > 1$, we can repeat the same argument on $c_i x^{F_i}$ and get our conclusion by the inductive hypothesis. This proves the lemma. $\square$

We are now in a position to prove the following key lemma, which reduces Theorem 1.5 to the prime GPI case.

Lemma 2.6. *Let $\phi(X^{\Delta_i})$ be a reduced DI for a right ideal ρ of R. Then either R is a GPI-ring or $\phi(X^{\Delta_i})$ is a reduced DI for $r_R(\ell_R(\rho))$.*

Proof. Choose B'_ρ to be a C-basis for $\ell_{U_{\mathrm{s}}}(\rho)$. Since $U_{\mathrm{s}} = \ell_{U_{\mathrm{s}}}(\rho) \oplus B_\rho C = B'_\rho C \oplus B_\rho C$, we can write

$$\phi(X^{\Delta_i}) = \phi_0(X^{\Delta_i}) + \phi_1(X^{\Delta_i}),$$

where $\phi_0(X^{\Delta_i})$ is a C-linear combination of B_ρ-monomials in X^{Δ_i} and where each monomial of $\phi_1(X^{\Delta_i})$ has coefficients in $B_\rho \cup B'_\rho$ and has at least a coefficient in B'_ρ. Applying (2.1) to all possible submonomials aX^{Δ_i}, where $a \in \ell_{U_{\mathrm{s}}}(\rho)$, of $\phi_1(X^{\Delta_i})$, we can transform $\phi(X^{\Delta_i})$ to a differential polynomial $\psi(X^{\Gamma_j})$, where $\psi(X^{\Gamma_j})$ is a C-linear combination of B_ρ-monomials in X^{Γ_j} or B_ρ-monomials in X^{Γ_j} with their last right coefficients in B'_ρ, where these Γ_j are distinct regular words. That is,

$$\psi(X^{\Gamma_j}) = \psi_0(X^{\Gamma_j}) + \psi_1(X^{\Gamma_j}),$$

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where $\psi_0(X^{\Gamma_j})$ is a C-linear combination of B_ρ-monomials in X^{Γ_j} and each monomial of $\psi_1(X^{\Gamma_j})$ has the form $u_0Y_1u_1Y_2\cdots Y_nu_n$, where $u_i\in B_\rho$ for each $1\leqslant i\leqslant n-1$ and $u_n\in B'_\rho$ and $Y_i\in\{X^{\Gamma_1},X^{\Gamma_2},\dots\}$ for each i. We may assume that $\Gamma_j\neq\emptyset$ for some j. Otherwise, we are done by Theorem 1.3.

Case 1. Suppose that $\psi(Z_j)$ is zero as a generalized polynomial in indeterminates Z_j. Then, in particular, $\psi(x^{\Gamma_j})=0$ for all $x\in r_R(\ell_R(\rho))$. Applying (2.1) to reverse the process from $\psi(X^{\Gamma_j})$ to $\phi(X^{\Delta_i})$, we conclude that $\phi(X^{\Delta_i})$ is a DI for $r_R(\ell_R(\rho))$, as asserted.

Case 2. Suppose that $\psi(Z_j)$ is not zero as a generalized polynomial in indeterminates Z_j. We claim that R is a prime GPI-ring. Suppose not. In particular, R is not a PI-ring. We list all coefficients appearing in $\psi(Z_j)$ as $b_1,b_2,\dots,b_m\in B_\rho$ and $b'_1,b'_2,\dots,b'_{m'}\in B'_\rho$. Then these elements b_s are C-independent modulo $\ell_{U_{\mathrm{s}}}(\rho)$. By Lemma 2.4, there exists an element $u\in\rho$ such that $b_1u,\dots,b_mu$ are C-independent. Since $\psi(X^{\Gamma_j})$ is a DI for ρ, $\psi((uX)^{\Gamma_j})$ is a DI for R. Note that

$$b_s(uX)^{\Gamma_j}=(b_su)X^{\Gamma_j}+\sum_l b_su^{A_{jl}}X^{B_{jl}}, \tag{2.2}$$

where the (A_{jl},B_{jl}) are pairs of subwords of Γ_j with $B_{jl}<\Gamma_j$. Thus, by (2.2), we can write

$$\psi((uX)^{\Gamma_j})=g(X^{\Gamma_j})+h(X^{\Gamma_j},X^{D_k}), \tag{2.3}$$

where $g(X^{\Gamma_j})$ is the differential polynomial obtained from $\psi(X^{\Gamma_j})$ by replacing all coefficients b_s by b_su and where each D_k is a subword of some Γ_j and does not appear in $\{\Gamma_1,\Gamma_2,\dots\}$. Applying Kharchenko's Theorem to (2.3), $g(X_j)+h(X_j,Z_k)$ is a GPI for R. This GPI is indeed non-trivial since the largest monomial (ordered by considering these Γ_j appearing in this monomial plus their weights and the C-independence of these b_su and $b'_{s'}$) of $g(X^{\Gamma_j})$ cannot be cancelled by the terms in $h(X^{\Gamma_j},X^{D_k})$, a contradiction. Thus R is a prime GPI-ring, proving the lemma. □

With Lemma 2.6 in hand we turn our attention to the prime GPI case. The key to this case is to study the continuity of derivations in certain endomorphism rings under the finite topology. We need to recall some notation from [**5**, p. 27]. Let Γ be a ring and let ${}_\Gamma M_1$ and ${}_\Gamma M_2$ be left Γ-modules. For $S\subseteq\mathrm{Hom}_\Gamma(M_1,M_2)$, we let

$$S^\perp=\{m\in M_1\mid ms=0\ \forall s\in S\},$$

and for $N\subseteq M_1$, we let

$$N^\perp=\{g\in\mathrm{Hom}_\Gamma(M_1,M_2)\mid ng=0\ \forall n\in N\}.$$

First we quote the following result [**5**, Theorem 1].

Theorem 2.7. *Let M_1 and M_2 be left vector spaces over a division ring Γ, let A be a subring of $\mathrm{End}({}_\Gamma M_2)$ and let $(B,+,0)$ be an additive subgroup of $\mathrm{Hom}_\Gamma(M_1,M_2)$ such that $BA\subseteq B$. Suppose that (i) $(M_2)_A$ is irreducible and (ii) $\Gamma=\mathrm{End}((M_2)_A)$. Then $(B^\perp)^\perp=\mathrm{cl}(B)$, the closure of B in the finite topology of $\mathrm{Hom}_\Gamma(M_1,M_2)$.*

We now apply the theorem to our case. Let R be a prime ring with non-zero socle $\mathrm{soc}(R)$. We choose a minimal idempotent $e \in R$ and let $D = eRe$ and $V = eR$. By the Density Theorem [**4**, Theorem 2.1.2], and considering right multiplication, we can regard R as a dense subring in $\mathrm{End}({}_DV)$ under its finite topology $\mathcal{T}_R$. For a finite-dimensional D-subspace W of ${}_DV$, we define $W^\perp = \{f \in \mathrm{End}({}_DV) \mid wf = 0\ \forall w \in W\}$. We recall that $\mathcal{T}_R$ has these $W^\perp + g$, where $g \in \mathrm{End}({}_DV)$, as its subbasis. Since $\mathrm{soc}(R) = ReR$, $eRq \subseteq eR$ for $q \in Q$. Thus V forms a right Q-module and $D = \mathrm{End}(V_Q)$. So Q is also embedded in $\mathrm{End}({}_DV)$. We will make these assumptions in Theorems 2.8 and 2.9

Theorem 2.8. *Let ρ be a right ideal of R. Then $\mathrm{cl}(\rho) \cap R = r_R(\ell_R(\rho))$.*

Proof. Since $\rho R \subseteq \rho$ and $D = \mathrm{End}(V_R)$, by Theorem 2.7 we have $(\rho^\perp)^\perp = \mathrm{cl}(\rho)$ in $\mathrm{End}({}_DV)$. We first notice that

$$\rho^\perp = \{er \in V \mid er\rho = 0\} = eR \cap \ell_R(\rho).$$

Also, $y \in \mathrm{cl}(\rho) \cap R = (\rho^\perp)^\perp \cap R$ if and only if $(eR \cap \ell_R(\rho))y = 0$. Thus it suffices to prove that if $y \in R$, then $(eR \cap \ell_R(\rho))y = 0$ if and only if $y \in r_R(\ell_R(\rho))$. The 'if' part is trivial. Let $y \in R$ be such that $(eR \cap \ell_R(\rho))y = 0$. Then, for $u \in \ell_R(\rho)$, $eRu \in eR \cap \ell_R(\rho)$ and so $eRuy = 0$, implying that $uy = 0$ by the primeness of R. So $\ell_R(\rho)y = 0$ and hence $y \in r_R(\ell_R(\rho))$, proving the theorem. □

Theorem 2.9. *Suppose that $d : A \to Q$ is a derivation, where A is a subring of R containing the socle of R. Then d is a continuous map.*

Proof. Let ${}_DL$ be a finite-dimensional D-subspace of ${}_DV$. It suffices to find a finite-dimensional D-subspace W of ${}_DV$ such that $(W^\perp \cap A)^d \subseteq L^\perp$. We write $L = Du_1 \oplus \cdots \oplus Du_s$, where $u_i \in V$, and let $W = L + \sum_{i=1}^{s} Deu_i^d$.

Let $a \in W^\perp \cap A$, then $u_ia = 0 = eu_i^da$. Since $u_i \in eR \subseteq \mathrm{soc}(R) \subseteq A$, we have $(u_ia)^d = 0$ and so $u_i^da + u_ia^d = 0$. Thus $u_ia^d = -u_i^da$ and so $u_ia^d = eu_ia^d = -eu_i^da = 0$. This implies that $a^d \in L^\perp$. The theorem is thus proved. □

As an immediate consequence of Theorem 2.9, we have the following corollary.

Corollary 2.10. *Suppose that $d : I \to Q$ is a derivation, where I is an ideal of R. Then d is a continuous map.*

We are now ready to prove the main result in this section.

Theorem 2.11. *Let R be a prime ring with two right ideals ρ_1 and ρ_2. Then $\ell_R(\rho_1) = \ell_R(\rho_2)$ if and only if ρ_1 and ρ_2 satisfy the same DIs with coefficients in U_{s}.*

Proof. The 'if' part is trivial. We prove the 'only if' part. Suppose that $\phi(X_i^{\Delta_j})$ is a DI for a right ideal ρ_1 of R. Since $\ell_R(\rho_1) = \ell_R(\rho_2)$, we have $r_R(\ell_R(\rho_1)) = r_R(\ell_R(\rho_2))$ and $\rho_2 \subseteq r_R(\ell_R(\rho_2))$. Thus it suffices to prove that $\phi(X_i^{\Delta_j})$ is a DI for $r_R(\ell_R(\rho_1))$. For simplicity of notation, we set $\rho = \rho_1$. Since every DI can be transformed into a reduced DI via (B1)–(B6), we may assume that $\phi(X_i^{\Delta_j})$ is a reduced DI for ρ. Moreover, by assigning $X_2, X_3, \ldots$ to fixed elements in ρ we may assume that ϕ only involves one indeterminate X with coefficients in U_{s}. Write $\phi = \phi(X^{\Delta_j})$, where the $\Delta_1, \ldots, \Delta_t$ are all distinct regular words occurring in ϕ.

In view of Lemma 2.6, either R is a GPI-ring or $\phi(X^{\Delta_j})$ is a reduced DI for $r_R(\ell_R(\rho))$. Thus it suffices to consider the case in which R is a prime GPI-ring. By Martindale's Theorem [**10**, Theorem 3], RC is a primitive ring with a minimal right ideal eRC, where e is a minimal idempotent in RC. We let $D = eRCe$ and $V = eRC$, a left vector space over the division ring D. Denote by H the socle of RC. By the Density Theorem [**4**, Theorem 2.1.2], RC is canonically embedded in $\mathrm{End}(_D V)$ as a dense subring. Let $d \in \boldsymbol{D}$. Then, by definition, there exists a non-zero ideal I of R such that $I^d \subseteq R$. It is clear that $H \subseteq IC$ and $H = H^2$. Thus we always have $H^d \subseteq H$. Also, U_{s} is canonically embedded in $\mathrm{End}(_D V)$. Theorem 2.9 says that $d : H \to H$ is a continuous map. This means that the map $x \in H \mapsto \phi(x^{\Delta_j}) \in \mathrm{End}(_D V)$ defines a continuous map from H into $\mathrm{End}(_D V)$. Since $\rho H \subseteq \rho RC \subseteq \rho C$, applying Corollary 2.2 we have $\phi(x^{\Delta_j}) = 0$ for all $x \in \rho H$. By the continuity of the map $x \in H \mapsto \phi(x^{\Delta_j}) \in \mathrm{End}(_D V)$ and Theorem 2.8, we have $\phi(x^{\Delta_j}) = 0$ for all $x \in r_H(\ell_H(\rho H))$.

Let $a \in r_R(\ell_R(\rho))$. Then $\ell_R(\rho)a = 0$ and so $\ell_{RC}(\rho H)a = 0$. In particular, $\ell_H(\rho H)a = 0$. Thus $aH \subseteq r_H(\ell_H(\rho H))$. Hence, $r_R(\ell_R(\rho))H \subseteq r_H(\ell_H(\rho H))$ follows. Thus $\phi(x^{\Delta_j}) = 0$ for all $x \in r_R(\ell_R(\rho))H$. Since H and U satisfy the same DIs by Corollary 2.3, so do $r_R(\ell_R(\rho))H$ and $r_R(\ell_R(\rho))U$. In particular, $\phi(x^{\Delta_j}) = 0$ for all $x \in r_R(\ell_R(\rho))$, proving the theorem. □

3. Proof of the Main Theorem

Let R be a semiprime ring. Recall that a subset $T \subseteq U$ is called orthogonally complete if $0 \in T$ and, given any set of orthogonal idempotents $\{e_\omega \mid \omega \in \Omega\} \subseteq C$ and any subset $\{x_\omega \mid \omega \in \Omega\} \subseteq T$, there exists $x \in T$ such that $e_\omega x = e_\omega x_\omega$ for all $\omega \in \Omega$. For any subset $K \subseteq U$, denote by $\hat{K}$ the orthogonal completion of K in U which is defined as the intersection of all orthogonally complete subsets of U containing K. Note that $\hat{K}$ itself is an orthogonally complete subset of U. We now come to the proof of Theorem 1.5. Since the method of extending Theorem 2.11 to the semiprime case is almost routine by applying the theory of orthogonal completions for semiprime rings [**2**, Chapter 3], we only sketch its proof.

Proof of Theorem 1.5. The 'if' part is trivial. We prove the 'only if' part. Suppose that $\ell_R(\rho_1) = \ell_R(\rho_2)$, where ρ_1 and ρ_2 are right ideals of R. Suppose that $\phi(X_i^{\Delta_j})$ (with coefficients in U_{s}) is a DI for ρ_1. Then, applying the same argument as that of [**8**, Lemma 6(i)] (with DIs instead of GPIs), we see that $\phi(X_i^{\Delta_j})$ is still a DI for $\hat{\rho}_1$. Let P be a maximal ideal of C. Then the following hold:

(i) PU is a prime ideal of U;

(ii) $(U_{\mathrm{s}} + PU)/PU$ is contained in the maximal symmetric ring of quotients of the prime ring $(\hat{R} + PU)/PU$;

(iii) $\ell_{(\hat{R}+PU)/PU}((\hat{\rho}_1 + PU)/PU) = \ell_{(\hat{R}+PU)/PU}((\hat{\rho}_2 + PU)/PU)$; and

(iv) each $\delta \in \boldsymbol{D}$ naturally induces a derivation $\bar{\delta}$ of U/PU such that $\delta(\bar{I}) \subseteq (\hat{R} + PU)/PU$ for some non-zero ideal $\bar{I}$ of $(\hat{R} + PU)/PU$.

We remark that (i) is referred to in [**2**, Theorem 3.2.7]. For (ii), see [**2**, Theorem 3.2.15], and fact (iii) can be derived from the fact that $\ell_R(\rho_1) = \ell_R(\rho_2)$, Finally, fact (iv) is clear. Using these facts we can reduce the theorem to the prime case and hence $\phi(x_i^{\Delta_j}) \in PU$ for all $x_i \in \hat{\rho}_2$. Applying the fact that $\bigcap_P PU = 0$, where the P run over all maximal ideals of C, we see that $\phi(X_i^{\Delta_j})$ is a DI for $\hat{\rho}_2$ and, therefore, for ρ_2. This proves Theorem 1.5. $\square$

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