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An algebraic approach to a quartic analogue of the Kontsevich model

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Abstract

We consider an analogue of Kontsevich's matrix Airy function where the cubic potential $\mathrm{Tr}(\Phi^3)$ is replaced by a quartic term $\mathrm{Tr}\big(\Phi^4\big)$. Cumulants of the resulting measure are known to decompose into cycle types for which a recursive system of equations can be established. We develop a new, purely algebraic geometrical solution strategy for the two initial equations of the recursion, based on properties of Cauchy matrices. These structures led in subsequent work to the discovery that the quartic analogue of the Kontsevich model obeys blobbed topological recursion.

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1. *Introduction*

Guided by uniqueness of quantum gravity in two dimensions, Witten conjectured in [33] that the generating function of intersection numbers of tautological characteristic classes on the moduli space of stable complex curves has to satisfy the PDE of the Korteweg–de Vries hierarchy. The conjecture was proved a few months later by Kontsevich in his seminal paper [20]. Kontsevich understood that critical graphs of the canonical Strebel differential [31] on a punctured curve give a cell-decomposition of the moduli space of punctured curves, which can be organised into a novel type of matrix model (the 'matrix Airy function') with covariance

$$\langle \Phi(e_{jk})\Phi(e_{lm})\rangle_c = \frac{\delta_{kl}\delta_{jm}}{\lambda_j+\lambda_k}$$

(where (e_{jk}) denotes the standard matrix basis and δ_{kl} the Kronecker symbol) and tri-valent vertices. The λ_j are Laplace transform parameters[1] of the lengths L_j of critical trajectories of the Strebel differentials, and the generically simple zeros of the Strebel differential correspond to tri-valent vertices. Kontsevich went on to establish that the logarithm of the partition function of his matrix model is the τ-function for the KdV-hierarchy, thereby proving that his matrix model is integrable.

The same covariance (up to normalisation)

$$\langle \Phi(e_{jk})\Phi(e_{lm})\rangle_c = \frac{\delta_{kl}\delta_{jm}}{N\big(E_j+E_k\big)}$$

[1] These λ_j will be denoted by E_j in this paper.

arises in quantum field theory models on noncommutative geometries [13], where the E_k are the spectral values ('energy levels') of a Laplace-type operator. These are models for scalar fields with cubic self-interaction. From a quantum field theoretical point of view one would be more interested in a quartic self-interaction, which e.g. is characteristic to the Higgs field. Such quartic models have been understood in [15] at the level of formal power series. Later in [16, 17] exact equations between correlation functions in the quartic (matrix) model were derived. These equations share many aspects with a universal structure called topological recursion [10].

Such recursions typically rely on the initial solution of a non-linear problem (for the Kontsevich model achieved in [25]). For the quartic model, the corresponding equation (for the planar two-point function of cycle type (0, 1)) is given in (3·2) below. Its solution succeeded in [12], via a larger detour. It was assumed that (3·2) converges for $N \to \infty$ to an integral equation with Hölder-continuous measure. The special case of constant measure was solved in [26] with help from computer algebra. Its structure suggested a conjecture for the general case which was proved in [12] by residue theorem and Lagrange-Bürmann resummation.

This paper provides a novel algebraic geometrical solution strategy for the non-linear equation (3·2) and the affine equation (6·2) (which determines the planar two-point function of cycle type (2, 0)). We (re)prove that these cumulants are compositions of rational functions with a preferred inverse of another rational function

$$R(z) = z - \frac{\lambda}{N} \sum_{k=1}^{d} \frac{\varrho_k}{z + \varepsilon_k} .$$

Building on these results it was understood in [3] that derivatives of the partially summed two-point function with respect to the spectral values E_k extend to meromorphic differentials $\omega_{g,n}$ labelled by genus g and number n of marked points of a complex curve. The $\omega_{g,n}$ are supplemented by two families of auxiliary functions and satisfy a coupled system of equations. The solution of this system for small $-\chi = 2g + n - 2$ in [3] gave strong support for the conjecture that the $\omega_{g,n}$ obey blobbed topological recursion [7] for the spectral curve $\big(x : \hat{\mathbb{C}} \to \hat{\mathbb{C}}, \omega_{0,1} = x dy, \omega_{0,2}\big)$ given by

$$x(z) = R(z) , \qquad y(z) = -R(-z) , \qquad \omega_{0,2}(w, z) = \frac{dw\, dz}{(w-z)^2} + \frac{dw\, dz}{(w+z)^2} .$$

The proof of this conjecture for $g = 0$ was achieved in [18]. As shown in [7], blobbed topological recursion generates intersection numbers on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable complex curves. In view of the deep rôle played by the global involution $z \mapsto -z$ [18] we expect that this very natural involution will find a counterpart in the intersection theory encoded in the quartic analogue of the Kontsevich model. Working out the details is a fascinating programme left for the future.

2. *Matrix integrals*

Let H_N be the real vector space of self-adjoint $N \times N$-matrices and $(E_1, \ldots, E_N)$ be not necessarily distinct positive real numbers. By the Bochner–Minlos theorem [6], combined with the Schur product theorem [27, section 4], there is a unique probability measure $d\mu_0(\Phi)$ on the dual space H_N' with

$$\exp\left(-\frac{1}{2N}\sum_{k,l=1}^{N}\frac{M_{kl}M_{lk}}{E_k+E_l}\right)=\int_{H'_N} d\mu_0(\Phi)\, e^{\mathrm{i}\Phi(M)} \tag{2·1}$$

for any $M=M^*=\sum_{k,l=1}^{N} M_{kl}e_{kl}\in H_N$, where (e_{kl}) is the standard matrix basis. The linear forms extend via $\Phi(M_1+\mathrm{i}M_2):=\Phi(M_1)+\mathrm{i}\Phi(M_2)$ to arbitrary complex $N\times N$-matrices. This allows us to evaluate $\Phi(e_{jk})$ and to identify the covariance

$$\int_{H'_N} d\mu_0(\Phi)\,\Phi(e_{jk})\Phi(e_{lm})=\frac{\delta_{kl}\delta_{jm}}{N\big(E_j+E_k\big)}\,.$$

We are going to deform the Gaußian measure (2·1) by a quartic potential,

$$d\mu_\lambda(\Phi):=\frac{d\mu_0(\Phi)\,\mathcal{P}_4(\Phi,\lambda)}{\int_{H'_N} d\mu_0(\Phi)\,\mathcal{P}_4(\Phi,\lambda)}\,, \tag{2·2}$$

$$\mathcal{P}_4(\Phi,\lambda)=\exp\left(-\frac{\lambda N}{4}\mathrm{Tr}(\Phi^4)\right):=\exp\left(-\frac{\lambda N}{4}\sum_{j,k,l,m=1}^{N}\Phi(e_{jk})\Phi(e_{kl})\Phi(e_{lm})\Phi\big(e_{mj}\big)\right),$$

for some $\lambda>0$. This matrix measure is the quartic analogue of the Kontsevich model [20] in which the deformation is given by the cubic term

$$\mathcal{P}_3(\Phi,\lambda)=\exp\left(-\frac{\lambda N}{3}\mathrm{Tr}\big(\Phi^3\big)\right):=\exp\left(-\frac{\lambda N}{3}\sum_{k,l,m=1}^{N}\Phi(e_{kl})\Phi(e_{lm})\Phi(e_{mk})\right).$$

The cubic measure was designed to prove Witten's conjecture [33] that intersection numbers of tautological characteristic classes on the moduli space of stable complex curves are related to the KdV hierarchy. Kontsevich proved that $\log\int_{H'_N} d\mu_0(\Phi)\,\mathcal{P}_3(\Phi,\mathrm{i}/2)$, viewed as function of $t_k=-(2k-1)!!(1/N)\sum_{j=1}^{N}E_j^{-(2k+1)}$, is the generating function of these intersection numbers.

We are interested in moments of the measure (2·2),

$$\langle e_{k_1l_1}\ldots e_{k_nl_n}\rangle:=\int_{H'_N} d\mu_\lambda(\Phi)\,\Phi\big(e_{k_1l_1}\big)\ldots\Phi\big(e_{k_nl_n}\big)=\frac{1}{\mathrm{i}^n}\frac{\partial^n\mathcal{Z}(M)}{\partial M_{k_1l_1}\ldots\partial M_{k_nl_n}}\bigg|_{M=0},$$

$$\mathcal{Z}(M)=\int_{H'_N} d\mu_\lambda(\Phi)\,e^{\mathrm{i}\Phi(M)}\,. \tag{2·3}$$

As explained in Appendix A (see also [23, 30]), the moments (2·3) decompose into cumulants

$$\left\langle\prod_{i=1}^{n}e_{k_il_i}\right\rangle=\sum_{\substack{\text{partitions}\\ \pi\text{ of }\{1,\ldots,n\}}}\ \prod_{\text{blocks }\beta\in\pi}\left\langle\prod_{i\in\beta}e_{k_il_i}\right\rangle_c\,. \tag{2·4}$$

For a quartic potential (2·2), moments and cumulants are only non-zero if n is even and every block β is of even length. The structure of the Gaußian measure (2·1) (together with the invariance of a trace under cyclic permutations) implies that $\big\langle e_{k_1l_1}\ldots e_{k_nl_n}\big\rangle_c$ is only non-zero if $(l_1,\ldots,l_n)=\big(k_{\sigma(1)},\ldots,k_{\sigma(n)}\big)$ is a permutation of $(k_1,\ldots,k_n)$, and in this case the

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cumulant only depends on the *cycle type* of this permutation σ in the symmetric group $\mathcal{S}_n$ (see Appendix A, with $b \geqslant 1$ the number of cycles of length $n_i > 0$, $n_1 + \ldots + n_b = n$):

$$N^{n_1+\cdots+n_b}\Big\langle\Big(e_{k_1^1k_2^1}e_{k_2^1k_3^1}\ldots e_{k_{n_1}^1k_1^1}\Big)\ldots\Big(e_{k_1^bk_2^b}e_{k_2^bk_3^b}\ldots e_{k_{n_b}^bk_1^b}\Big)\Big\rangle_c =: N^{2-b}G_{|k_1^1\ldots k_{n_1}^1|\ldots|k_1^b\ldots k_{n_b}^b|}\,. \tag{2·5}$$

To correctly identify the cycles of the permutation it is necessary that all k_j^i are pairwise different in (2·5). These N-rescaled cumulants (2·5) are further expanded as formal power series $G_{\ldots} = \sum_{g=0}^{\infty} N^{-2g} G_{\ldots}^{(g)}$ in N^{-2} so that

$$N^{n_1+\cdots+n_b}\Big\langle\Big(e_{k_1^1k_2^1}e_{k_2^1k_3^1}\ldots e_{k_{n_1}^1k_1^1}\Big)\ldots\Big(e_{k_1^bk_2^b}e_{k_2^bk_3^b}\ldots e_{k_{n_b}^bk_1^b}\Big)\Big\rangle_c = \sum_{g=0}^{\infty} N^{2-b-2g}\cdot G^{(g)}_{|k_1^1\ldots k_{n_1}^1|\ldots|k_1^b\ldots k_{n_b}^b|}\,. \tag{2·6}$$

It turns out that this grading (g, b) of $G^{(g)}_{|k_1^1\ldots k_{n_1}^1|\ldots|k_1^b\ldots k_{n_b}^b|}$ fits with the combinatorics of *ribbon graphs* (with 4-valent vertices) on a *connected* oriented compact topological surface of genus $g \geqslant 0$ with $b \geqslant 1$ boundary components (and n_i labels on the ith boundary component) and *Euler characteristic* $\chi = 2 - 2g - b$ (see e.g. [**14**, section 3] for the particular case of 4-valent vertices, and compare also with [**20**] or [**11**, sections 2 and 6]). Note that the moments are related to ribbon graphs on possibly *non-connected* oriented compact topological surfaces (see e.g. [**22**, section 3, proposition 3·8·3]).

Starting point for the investigation of cumulants are equations of motion for $\mathcal{Z}(M)$:

LEMMA 2·1. *The Fourier transform $\mathcal{Z}(M)$ of the measure (2·2) satisfies*

$$\frac{1}{i}\frac{\partial \mathcal{Z}(M)}{\partial M_{ab}} = \frac{iM_{ba}\mathcal{Z}(M)}{N(E_a+E_b)} - \frac{\lambda}{i^3(E_a+E_b)}\sum_{k,l=1}^{N}\frac{\partial^3\mathcal{Z}(M)}{\partial M_{ak}\partial M_{kl}\partial M_{lb}}\,. \tag{2·7}$$

Proof. This follows from basic properties of the Gaußian measure (2·1). The derivative $(1/\mathrm{i})(\partial/\partial M_{ab})$ applied to $\mathcal{Z}(M)$ produces a factor $\Phi(e_{ab})$ under the integral. Moments of $d\mu_0(\Phi)$ are by (2·2) a sum over pairings. This means that $\Phi(e_{ab})$ is paired in all possible ways with a $\Phi(e_{cd})$ contained in $\exp(\mathrm{i}\Phi(M))$ or in $\mathcal{P}_4(\Phi,\lambda)$. Every such pair contributes a factor $\delta_{ad}\delta_{bc}/\big(N\big(E_a+E_b\big)\big)$, and summing over all pairings is the same as taking the derivative, thus producing a term

$$\frac{1}{N(E_a+E_b)}\left(\mathrm{i}M_{ba} - \lambda N\sum_{k,l=1}^{N}\Phi(e_{ak})\Phi(e_{kl})\Phi(e_{lb})\right)$$

under the integral. The triple product of $\Phi(e_{..})$ is written as a third derivative with respect to the corresponding entries of M.

The Kontsevich model [**20**] with cubic deformation $\mathcal{P}_3(\Phi,\lambda)$ is governed by the equation of motion

$$\frac{1}{\mathrm{i}}\frac{\partial \mathcal{Z}(M)}{\partial M_{ab}} = \frac{\mathrm{i}M_{ba}\mathcal{Z}(M)}{N(E_a+E_b)} - \frac{\lambda}{\mathrm{i}^2(E_a+E_b)}\sum_{k=1}^{N}\frac{\partial^2\mathcal{Z}(M)}{\partial M_{ak}\partial M_{kb}}\,.$$

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For $N=1$ this is essentially the ODE

$$f''(x)+2cf'(x)=xf(x)$$

solved by the Airy function $f(x)=e^{-cx}\mathrm{Ai}(x+c^2)$, hence the title of [20]. Its quartic analogue is the matrix version of the ODE

$$f'''(x)+3cf'(x)=xf(x)\,,$$

which does not seem to have a name. The Airy function is the case $p=2$ of a larger class

$$\mathrm{Ai}_p(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\;\exp\left(\mathrm{i}\left(\frac{t^{p+1}}{p+1}+xt\right)\right)$$

of higher Airy functions. As remarked in [20, section 4·3], they also give rise to *higher matrix Airy functions*. In particular, there is also a 'quartic analogue' $p=3$ in this class, which was studied in [19, 21]. This matrix model does not seem to be related to our 'quartic analogue' of the Kontsevich model.

Another equation of motion will be necessary for the subsequent work in [3]:

LEMMA 2·2. *The Fourier transform $\mathcal{Z}(M)$ of the measure (2·2) satisfies*

$$\frac{1}{N}\frac{\partial\mathcal{Z}(M)}{\partial E_a}=\left(\sum_{k=1}^{N}\frac{\partial^2}{\partial M_{ak}\partial M_{ka}}+\frac{1}{N}\sum_{k=1}^{N}G_{|ak|}+\frac{1}{N^2}G_{|a|a|}\right)\mathcal{Z}(M)\,. \tag{2·8}$$

Proof. Application of $(1/N)(\partial/\partial E_a)-\sum_{k=1}^{N}\partial^2/\big(\partial M_{ak}\partial M_{ka}\big)-(1/N)\sum_{k=1}^{N}1/\big(E_a+E_k\big)$ to the left-hand side of (2·1) yields zero so that it gives

$$\frac{1}{N}\frac{\partial}{\partial E_a}\big(d\mu_0(\Phi)\big)=d\mu_0(\Phi)\sum_{k=1}^{N}\left(\frac{1}{N(E_a+E_k)}-\Phi(e_{ka})\Phi(e_{ak})\right)$$

when applying it to the right-hand side. Apply this identity to (2·2) to get

$$\begin{aligned}\frac{1}{N}\frac{\partial}{\partial E_a}\big(d\mu_\lambda(\Phi)\big)=d\mu_\lambda(\Phi)\sum_{k=1}^{N}\left(\frac{1}{N(E_a+E_k)}-\Phi(e_{ka})\Phi(e_{ak})\right)\\ -d\mu_\lambda(\Phi)\int_{H_N'}d\mu_\lambda(\Phi)\sum_{k=1}^{N}\left(\frac{1}{N(E_a+E_k)}-\Phi(e_{ka})\Phi(e_{ak})\right).\end{aligned}$$

Multiplying with $e^{\mathrm{i}\Phi(M)}$ and integrating over H_N' gives with (A3) the assertion.

The equations of motion (2·7) and (2·8) induce identities between cumulants. Some of them are derived in Appendix B, for others see [17]. Taking also the grading by (g,b) into account, one can establish a partial order in the homogeneous building blocks $G^{(g)}_{\dots}$. The least element is the planar two-point function $G^{(0)}_{|ab|}$, which is the dominant part (at large N) of the cumulant of length 2 and cycle type (0, 1) (i.e. one cycle ab of length 2). It satisfies a closed non-linear equation for it alone, given in (3·1) below. Any other homogeneous building block of (2·6) satisfies an affine equation with inhomogeneity that depends only on functions of strictly larger topological *Euler characteristic* $\chi=2-2g-b$, which are

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known by induction. Similar recursive systems have been identified in many areas of mathematics. Their common universal structure has been axiomatised under the name *topological recursion* [10], since the recursion is by the topological Euler characteristic. Starting from a few initial data called the *spectral curve*, topological recursion constructs a hierarchy of differential forms and understands them as spectral invariants of the curve. A prominent example is the Kontsevich model [20] whose topological recursion is described e.g. in [11, section 6]. Other classes of examples are the one- and two-matrix models [8], Mirzakhani's recursions [24] for the volume of moduli spaces of Riemann surfaces, and recursions in Hurwitz theory [5] and Gromov–Witten theory [4].

3. *The planar two-point function*

The two-point function $G_{|ab|}$ is the cumulant of length 2 and cycle type (0, 1) (i.e. one cycle ab of length 2), see Appendix A. We reprove in Appendix B that the planar two-point function $G^{(0)}_{|ab|}$ (of degree or genus $g=0$) satisfies

$$\Big(E_a+E_b+\frac{\lambda}{N}\sum_{k=1}^{N}G^{(0)}_{|ak|}\Big)G^{(0)}_{|ab|}=1+\frac{\lambda}{N}\sum_{k=1}^{N}\frac{G^{(0)}_{|kb|}-G^{(0)}_{|ab|}}{E_k-E_a}\,. \tag{3·1}$$

This equation was first established in [16]; equation (B7) which involves all $G^{(g)}_{|ab|}$ was obtained in [17].

To give a meaning to the term $k=a$ in (3·1) we make the decisive assumption that $\Big\{G^{(0)}_{|ab|}\Big\}_{a,b=1,\dots N}$ arise by evaluation of a holomorphic function in two complex variables. Let $E_1,\dots,E_d$ be the distinct entries in the tuple (E_k), which occur with multiplicities $r_1,\dots,r_d$, with $N=r_1+\dots+r_d$. We assume that for some neighbourhoods $\mathcal{U}_k\subset\mathbb{C}$ of E_k there is a holomorphic function $G^{(0)}:\bigcup_{k,l=1}^{d}(\mathcal{U}_k\times\mathcal{U}_l)\to\mathbb{C}$ which interpolates $G^{(0)}_{|ab|}=G^{(0)}(E_a,E_b)$ and satisfies the natural (but by no means unique) holomorphic extension

$$\Big(\zeta+\eta+\frac{\lambda}{N}\sum_{k=1}^{d}r_kG^{(0)}(\zeta,E_k)\Big)G^{(0)}(\zeta,\eta)=1+\frac{\lambda}{N}\sum_{k=1}^{d}r_k\frac{G^{(0)}(E_k,\eta)-G^{(0)}(\zeta,\eta)}{E_k-\zeta} \tag{3·2}$$

of (3·1), for $(\zeta,\eta)\in\bigcup_{k,l=1}^{d}(\mathcal{U}_k\times\mathcal{U}_l)$. Equation (3·1) is understood as the limit $\zeta\to E_a$ and $\eta\to E_b$ of (3·2) when taking multiplicities into account. It is not possible to deduce (3·2) from (3·1) alone. Justification of (3·2) comes from the fact that it gives rise to interesting mathematical structures:

THEOREM 3·1. *Construct 2d functions* $\{\varepsilon_k(\lambda),\varrho_k(\lambda)\}_{k=1,\dots,d}$, *with* $\lim_{\lambda\to0}\big(\varepsilon_k,\varrho_k\big)=\big(E_k,r_k\big)$, *as implicitly defined solution of the system*

$$E_k=R(\varepsilon_k)\,,\qquad r_k=\varrho_kR'(\varepsilon_k)\,,\qquad \textit{where}\quad R(z)=z-\frac{\lambda}{N}\sum_{j=1}^{d}\frac{\varrho_j}{\varepsilon_j+z}\,. \tag{3·3}$$

Then (3·2) *is solved by* $G^{(0)}(\zeta,\eta)=\mathcal{G}^{(0)}\big(R^{-1}(\zeta),R^{-1}(\eta)\big)$, *where* $\mathcal{G}^{(0)}:\hat{\mathbb{C}}\times\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ *is the rational function*

$$\mathcal{G}^{(0)}(z,w) = \frac{1}{(R(w)-R(-z))(R(z)-R(-w))}\Bigg\{R(z)+R(w) + \frac{\lambda}{N}\sum_{k=1}^{d}\left(\frac{r_k}{R(\varepsilon_k)-R(z)}+\frac{r_k}{R(\varepsilon_k)-R(w)}\right) + \frac{\lambda^2}{N^2}\sum_{k,l=1}^{d}\frac{r_k r_l \mathcal{G}_{kl}}{(R(\varepsilon_k)-R(z))(R(\varepsilon_l)-R(w))}\Bigg\} \qquad (3\cdot4)$$

with

$$\mathcal{G}_{kl} = \frac{\left(\prod_{j,m=1}^{d}\frac{(-\widehat{\varepsilon_k}^j-\widehat{\varepsilon_l}^m)}{\varepsilon_j+\varepsilon_m}\right)\left(\prod_{\substack{j=1\\ j\neq k}}^{d}\frac{\varepsilon_k-\varepsilon_j}{R(\varepsilon_k)-R(\varepsilon_j)}\right)\left(\prod_{\substack{m=1\\ m\neq l}}^{d}\frac{\varepsilon_l-\varepsilon_m}{R(\varepsilon_l)-R(\varepsilon_m)}\right)}{R'(\varepsilon_k)R'(\varepsilon_l)(\varepsilon_k+\varepsilon_l)}\,. \qquad (3\cdot5)$$

Here $z\in\{u,\hat{u}^1,\dots,\hat{u}^d\}$ *is the list of the different roots of* $R(z)=R(u)$*, and the correct branch of* R^{-1} *is chosen by the implicitly defined solutions above (i.e.* $\varepsilon_k\in R^{-1}(\mathcal{U}_k)$ *for this branch). In particular,* $G^{(0)}_{|kl|}=\mathcal{G}^{(0)}(\varepsilon_k,\varepsilon_l)\equiv\mathcal{G}_{kl}$ *solves* (3·1).

Existence of $(\varepsilon_k(\lambda),\varrho_k(\lambda))$ in a neighbourhood of $\lambda=0$ is guaranteed by the implicit function theorem. We will prove several equivalent formulae for $\mathcal{G}^{(0)}(z,w)$: (4·5), (4·9), (4·14) and eventually (3·4). Some of them were already proved in [**12**]. There, inspired by the solution of a particular case [**26**], equation (3·2) was interpreted as an integral equation for a Dirac measure. Approximating the Dirac measure by a Hölder-continuous function allowed to employ boundary values techniques for sectionally holomorphic functions. Residue theorem and Lagrange–Bürmann resummation gave a solution formula whose limit back to Dirac measure was arranged into (4·14).

In this paper we provide a more elementary proof of these equations which solely needs properties of *Cauchy matrices* established by Schechter [**28**]:

PROPOSITION 3·2 ([**28**]). *For two d-tuples* $(a_1,\dots,a_d)$ *and* $(b_1,\dots,b_d)$*, with all* a_i,b_j *distinct, consider the* $d\times d$*-matrix* $H=\left(\frac{1}{a_k-b_l}\right)_{kl}$*. Let* $A(x):=\prod_{i=1}^{d}(x-a_i)$ *and* $B(y):=\prod_{j=1}^{d}(y-b_j)$*. Then the inverse of H is given by*

$$\left(H^{-1}\right)_{kl}=(a_l-b_k)A_l(b_k)\,B_k(a_l)\,, \qquad (3\cdot6)$$

where A_l,B_k *are the Lagrange interpolation polynomials*

$$A_l(x)=\frac{A(x)}{(x-a_l)\,A'(a_l)}\,,\qquad B_k(y)=\frac{B(y)}{(y-b_k)\,B'(b_k)}\,. \qquad (3\cdot7)$$

The inverse of H satisfies

$$\frac{B_k(x)A(b_k)}{A(x)}=\sum_{l=1}^{d}\frac{\left(H^{-1}\right)_{kl}}{a_l-x}\,,\qquad \frac{A_l(x)B(a_l)}{B(x)}=\sum_{k=1}^{d}\frac{\left(H^{-1}\right)_{kl}}{x-b_k}\,. \qquad (3\cdot8)$$

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Moreover, the row sums and column sums of H^{-1} are given by

$$\sum_{j=1}^{d}(H^{-1})_{kj} = -\frac{A(b_k)}{B'(b_k)}, \qquad \sum_{i=1}^{d}(H^{-1})_{il} = \frac{B(a_l)}{A'(a_l)}, \tag{3·9}$$

and one has, for all $j = 1, \dots, d$,

$$\sum_{k=1}^{d}\frac{A(b_k)}{(b_k - a_j)B'(b_k)} = 1 \quad \text{and} \quad \sum_{l=1}^{d}\frac{B(a_l)}{(a_l - b_j)A'(a_l)} = 1\,. \tag{3·10}$$

4. *Proof of Theorem 3·1*

We are going to construct a non-constant rational function $R \in \mathbb{C}(z)$ viewed as a branched cover $R : \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \to \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ of Riemann surfaces (with $z = \mathrm{id}_{\mathbb{C}}$ the standard coordinate on $\mathbb{C}$) via the following:

ANSATZ 4·1. *A branched cover $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is supposed to be determined by conventions* (i)–(vi) *and an algebraic relation* (vii):

(i) *R has degree $d + 1$;*

(ii) *all ramification points of R do not belong to $R^{-1}(\{E_1, \dots, E_d\})$;*

(iii) *without loss of generality, $R(\infty) = \infty$ with residue -1 in the sense that $Res_{z=\infty}R(z)dz = -1$;*

(iv) *for every $k = 1, \dots, d$, distinguish any of the $d+1$ distinct points of the fibre $R^{-1}(E_k)$ as ε_k. Take any connected neighbourhood $\mathcal{U}_k \subset \mathbb{C}$ of E_k for which $R^{-1}(\mathcal{U}_k)$ has $d+1$ connected components, and let $\mathcal{V}_k$ be the connected component of $R^{-1}(\mathcal{U}_k)$ which contains ε_k. Then the choice of $\{\varepsilon_k\}$ determines a holomorphic function $\mathcal{G}^{(0)} : \bigcup_{k,l=1}^{d}(\mathcal{V}_k \times \mathcal{V}_l) \to \mathbb{C}$ by $\mathcal{G}^{(0)}(z, w) = G^{(0)}(R(z), R(w))$, where $G^{(0)} : \bigcup_{k,l=1}^{d}(\mathcal{U}_k \times \mathcal{U}_l) \to \mathbb{C}$ satisfies* (3·2);

(v) *for any $w \in \bigcup_{j=1}^{d}\mathcal{V}_j$, let $\hat{w}^1, \dots, \hat{w}^d$ be the d other distinct preimages of $R(w) \in \bigcup_{j=1}^{d}\mathcal{U}_j$ under R, i.e. $R(\hat{w}^k) = R(w)$. Assume that*

$$\infty \neq R(-\hat{w}^l) \quad \text{and} \quad R(-\hat{w}^l) \neq R(-\hat{w}^{l'})$$

for all $l, l' = 1, \dots, d$ with $l \neq l'$ and w close to some ε_k;

(vi) *for any w close to some ε_k, $\mathcal{G}^{(0)}(-\hat{w}^l, w)$ is defined and finite for all $l = 1, \dots, d$. This is the case e.g. if $-\hat{w}^l \in \bigcup_{j=1}^{d}\mathcal{V}_j$ for all $l = 1, \dots, d$, or if $\mathcal{G}^{(0)}$ extends to a suitable rational function on $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$;*

(vii) *for any $z \in \mathcal{V}_l$ one has*

$$R(z) + \frac{\lambda}{N}\sum_{k=1}^{d} r_k \mathcal{G}^{(0)}(z, \varepsilon_k) + \frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k}{R(\varepsilon_k) - R(z)} = -R(-z). \tag{4·1}$$

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With the properties (iv) and (vii) in this Ansatz 4·1 we turn (3·2) into

$$\big(R(w)-R(-z)\big)\mathcal{G}^{(0)}(z,w)=1+\frac{\lambda}{N}\sum_{k=1}^{d} r_k\frac{\mathcal{G}^{(0)}(\varepsilon_k,w)}{R(\varepsilon_k)-R(z)}\,, \tag{4·2}$$

where $(z,w)\in\bigcup_{k,l=1}^{d}(\mathcal{V}_k\times\mathcal{V}_l)$. Next, setting $z=-\hat{w}^l$ in (4·2) for $l=1,\ldots,d$ and a given w close to some ε_k, requirements (v) and (vi) of Ansatz 4·1 give (by $\infty\neq R\big(-\hat{w}^l\big)$ and $\mathcal{G}^{(0)}\big(-\hat{w}^l,w\big)$ is defined and finite) the d equations

$$\frac{\lambda}{N}\sum_{k=1}^{d} r_k\frac{\mathcal{G}^{(0)}(\varepsilon_k,w)}{R\big(-\hat{w}^l\big)-R(\varepsilon_k)}=1\,. \tag{4·3}$$

This identifies $(\lambda/N)r_k\mathcal{G}^{(0)}(\varepsilon_k,w)$ as row sums of the inverse of a Cauchy matrix. Setting $a_j=R\big(-\hat{w}^j\big)$ and $b_i=R(\varepsilon_i)$ in the first identity (3·9) in Proposition 3·2 we conclude (since the a_j,b_i for $j,i=1,\ldots,d$ are pairwise distinct by requirement (v) of Ansatz 4·1):

COROLLARY 4·2. *With Ansatz 4·1 one has*

$$\frac{\lambda}{N}r_k\mathcal{G}^{(0)}(\varepsilon_k,w)=-\frac{\prod_{j=1}^{d}\big(R(\varepsilon_k)-R\big(-\hat{w}^j\big)\big)}{\prod_{j=1,j\neq k}^{d}\big(R(\varepsilon_k)-R\big(\varepsilon_j\big)\big)}\,. \tag{4·4}$$

Inserted back into (4·2) expresses $\mathcal{G}^{(0)}(z,w)$ in terms of R. The result simplifies:

LEMMA 4·3. *With Ansatz 4·1 one has*

$$\mathcal{G}^{(0)}(z,w)=\frac{1}{(R(w)-R(-z))}\prod_{j=1}^{d}\frac{R(z)-R\big(-\hat{w}^j\big)}{R(z)-R\big(\varepsilon_j\big)}\,. \tag{4·5}$$

Proof. This follows from (3·10) for $(d+1)$-tuples with index 0 prepended. Setting $b_0=R(z)$, $b_k=R(\varepsilon_k)$, $a_l=R\big(-\hat{w}^l\big)$ for $k,l=1,\ldots,d$, then the case $j=0$ of the first identity (3·10) reads (independent of a_0)

$$\frac{\prod_{j=1}^{d}\big(R(z)-R\big(-\hat{w}^j\big)\big)}{\prod_{j=1}^{d}\big(R(z)-R\big(\varepsilon_j\big)\big)}+\sum_{k=1}^{d}\frac{\prod_{j=1}^{d}\big(R(\varepsilon_k)-R\big(-\hat{w}^j\big)\big)}{(R(\varepsilon_k)-R(z))\prod_{j=1,j\neq k}^{d}\big(R(\varepsilon_k)-R\big(\varepsilon_j\big)\big)}=1\,. \tag{4·6}$$

Equation (4·5) results from this identity when inserting (4·4) into (4·2).

LEMMA 4·4. *With Ansatz 4·1 the rational function $R\in\mathbb{C}(z)$ is necessarily given by*

$$R(z)=z+c_0-\frac{\lambda}{N}\sum_{k=1}^{d}\frac{\varrho_k}{\varepsilon_k+z}\quad \textit{for some } c_0\in\mathbb{C}\,,\qquad \varrho_k=\frac{r_k}{R'(\varepsilon_k)}\,. \tag{4·7}$$

Proof. Comparing the limit $z\to\varepsilon_k$ of (4·5) with (4·4) shows that R has a simple pole at every $-\varepsilon_k$ with

$$R'(\varepsilon_k)\operatorname*{Res}_{z=-\varepsilon_k}R(z)dz=-\frac{\lambda r_k}{N}\neq 0\,.$$

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By construction, R has also a pole at ∞. Since R has degree $d+1$ by (i) in Ansatz 4·1, $\{-\varepsilon_1,\dots,-\varepsilon_d,\infty\}$ is already the complete list of poles (i.e. preimages of ∞) of R. Moreover, the pole at ∞ has to be simple with $\lim_{z\to\infty} R(z)/z=1$ by (iii) in Ansatz 4·1. Therefore, $R(z)-z+(\lambda/N)\sum_{k=1}^d \varrho_k/(\varepsilon_k+z)$ is a bounded holomorphic function on $\hat{\mathbb{C}}$, which by Liouville's theorem is a constant c_0.

COROLLARY 4·5. *For $u\in\bigcup_{j=1}^d \mathcal{V}_j$ one has an equality of rational functions in z:*

$$R(z)-R(u)=(z-u)\prod_{k=1}^d \frac{z-\hat{u}^k}{z+\varepsilon_k}\,, \tag{4·8}$$

where $\hat{u}^k$ are the other preimages of $R(u)$ under R.

Proof. Both sides are a rational function $r(z)$, with zeros only in $u,\hat{u}^1,\dots,\hat{u}^d$ and poles only in $-\varepsilon_1,\dots,-\varepsilon_d,\infty$, all of which are simple. So they differ by a constant factor, which has to be 1 because both sides satisfy $\lim_{z\to\infty} r(z)/z=1$.

PROPOSITION 4·6. *With Ansatz 4·1 the two-point function is symmetric, $\mathcal{G}^{(0)}(z,w)=\mathcal{G}^{(0)}(w,z)$. One has $\mathcal{G}^{(0)}(\varepsilon_k,\varepsilon_l)=\mathcal{G}_{kl}$ with $\mathcal{G}_{kl}$ given in (3·5).*

Proof. Inserting (4·8) into (4·5) gives for $z,w\in\bigcup_{j=1}^d \mathcal{V}_j$:

$$\begin{aligned}\mathcal{G}^{(0)}(z,w)&=\frac{\prod_{k=1}^d(\varepsilon_k-z)}{(z+w)\prod_{k=1}^d\big(-z-\hat{w}^k\big)}\prod_{k=1}^d\frac{\dfrac{\big(z+\hat{w}^k\big)\prod_{l=1}^d\big(-\hat{w}^k-\hat{z}^l\big)}{\prod_{l=1}^d\big(\varepsilon_l-\hat{w}^k\big)}}{\dfrac{(z-\varepsilon_k)\prod_{l=1}^d\big(\varepsilon_k-\hat{z}^l\big)}{\prod_{l=1}^d(\varepsilon_k+\varepsilon_l)}}\\&=\frac{1}{(z+w)}\prod_{k,l=1}^d\frac{(\varepsilon_k+\varepsilon_l)\big(-\hat{w}^k-\hat{z}^l\big)}{\big(\varepsilon_k-\hat{z}^l\big)\big(\varepsilon_l-\hat{w}^k\big)}\\&=\frac{1}{(z+w)}\prod_{k,l=1}^d\frac{\big(-\hat{w}^k-\hat{z}^l\big)}{(\varepsilon_k+\varepsilon_l)}\frac{(\varepsilon_k-z)}{(R(\varepsilon_k)-R(z))}\frac{(\varepsilon_l-w)}{(R(\varepsilon_l)-R(w))}\,.\end{aligned} \tag{4·9}$$

The limit $z\to\varepsilon_k$ and $w\to\varepsilon_l$ gives $\mathcal{G}^{(0)}(\varepsilon_k,\varepsilon_l)=\mathcal{G}_{kl}$.

We prove that (ii), (iv), (v), (vi) and (vii) of Ansatz 4·1 are automatic. We start with

PROPOSITION 4·7. *Relation* (vii) *of Ansatz 4·1 is consistent provided that $c_0=0$.*

Proof. With Lemma 4·4 and Lemma 4·3, both a consequence of Ansatz 4·1, each side of (4·1) is a rational function, and all poles are simple. For the term $r_k/(R(\varepsilon_k)-R(z))$ this follows from the assumption (ii) of Ansatz 4·1. We show that both sides of (4·1) have the same simple poles with the same residues. Then by Liouville's theorem their difference is a constant, which is easy to control.

First, it follows from (4·7) and (4·5) that both sides of (4·1) approach z for $z\to\infty$. Near ∞ the difference between both sides of (4·1) is $\pm 2c_0$, which shows that $c_0=0$ in (4·7) is necessary.

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Next, (4·7) shows that the only other poles of the right-hand side of (4·1) are simple and located at $z=\varepsilon_k$ with residue $-(\lambda/N)\varrho_k$. The same simple poles with the same residues are produced by $(\lambda/N)\sum_{k=1}^{d} r_k/(R(\varepsilon_k)-R(z))$ on the left-hand side, taking $r_k/R'(\varepsilon_k)=\varrho_k$ into account.

But the left-hand side of (4·1) could also have poles at $z=-\varepsilon_j$ and $z=\widehat{\varepsilon_m}^j$ (see (4·7) and (4·9)). We have $\mathrm{Res}_{z=-\varepsilon_j}\ R(z)dz=-(\lambda/N)\varrho_j$. Setting $w\mapsto\varepsilon_l$ in (4·5), then with $\lim_{z\to-\varepsilon_j}\big(R(z)-R\big(-\widehat{\varepsilon_l}^k\big)\big)/(R(z)-R(\varepsilon_k))=1$ for any $k,\,l$ (here (v) of Ansatz 4·1 is used) one easily finds that $\mathcal{G}^{(0)}(-\varepsilon_j,\varepsilon_l)$ is finite for $j\neq l$ and that $\mathrm{Res}_{z=-\varepsilon_j}(\lambda/N)r_j\mathcal{G}^{(0)}\big(z,\varepsilon_j\big)dz=(\lambda/N)\big(r_j/R'\big(\varepsilon_j\big)\big)$, which thus cancels $\mathrm{Res}_{z=-\varepsilon_j}\ R(z)dz=-(\lambda/N)\varrho_j$.

Finally, from (4·5) we conclude

$$
\begin{aligned}
\operatorname*{Res}_{z=\widehat{\varepsilon_m}^j}\mathcal{G}^{(0)}\big(z,\varepsilon_k\big)dz&=\frac{1}{\big(R(\varepsilon_k)-R\big(-\widehat{\varepsilon_m}^j\big)\big)R'\big(\widehat{\varepsilon_m}^j\big)}\frac{\prod_{i=1}^{d}\big(R(\varepsilon_m)-R\big(-\widehat{\varepsilon_k}^i\big)\big)}{\prod_{i=1,i\neq m}^{d}\big(R(\varepsilon_m)-R(\varepsilon_i)\big)}\\
&=\frac{1}{\big(R(\varepsilon_k)-R\big(-\widehat{\varepsilon_m}^j\big)\big)R'\big(\widehat{\varepsilon_m}^j\big)}\Big(-\frac{\lambda}{N}r_m\mathcal{G}^{(0)}\big(\varepsilon_m,\varepsilon_k\big)\Big)\\
&=\frac{r_m}{r_kR'\big(\widehat{\varepsilon_m}^j\big)}\frac{1}{\big(R(\varepsilon_k)-R\big(-\widehat{\varepsilon_m}^j\big)\big)}\Big(-\frac{\lambda}{N}r_k\mathcal{G}^{(0)}\big(\varepsilon_k,\varepsilon_m\big)\Big)\\
&=\frac{r_m}{r_kR'\big(\widehat{\varepsilon_m}^j\big)}\frac{1}{\big(R(\varepsilon_k)-R\big(-\widehat{\varepsilon_m}^j\big)\big)}\frac{\prod_{i=1}^{d}\big(R\big(\varepsilon_k\big)-R\big(-\widehat{\varepsilon_m}^i\big)\big)}{\prod_{i=1,i\neq k}^{d}\big(R\big(\varepsilon_k\big)-R(\varepsilon_i)\big)}\,,
\end{aligned}
$$

where (4·4), the symmetry $\mathcal{G}^{(0)}(\varepsilon_m,\varepsilon_k)=\mathcal{G}^{(0)}(\varepsilon_k,\varepsilon_m)$ and again (4·4) have been used. The first identity (3·10) for $b_k=R(\varepsilon_k)$ and $a_j=R\big(-\widehat{\varepsilon_m}^j\big)$ gives

$$
\operatorname*{Res}_{z=\widehat{\varepsilon_m}^j}\sum_{k=1}^{d}r_k\mathcal{G}^{(0)}(z,\varepsilon_k)dz=\frac{r_m}{R'\big(\widehat{\varepsilon_m}^j\big)}\,,
$$

which precisely cancels $\mathrm{Res}_{z=\widehat{\varepsilon_m}^j}\sum_{k=1}^{d}\frac{r_k}{R(\varepsilon_k)-R(z)}dz=-\frac{r_m}{R'\big(\widehat{\varepsilon_m}^j\big)}$.

Let us consider now the rational function

$$
R(z)=z-\frac{\lambda}{N}\sum_{k=1}^{d}\frac{\varrho_k}{\varepsilon_k+z} \tag{4·10}
$$

from equation (4·7) with $c_0=0$. We are interested in the real and complex solutions $\{\varepsilon_k,\varrho_k\}_{k=1,\dots,d}$ (depending on λ) of the $2d$ equations

$$
0=R(\varepsilon_l)-E_l=\varepsilon_l-E_l-\frac{\lambda}{N}\sum_{k=1}^{d}\frac{\varrho_k}{\varepsilon_k+\varepsilon_l}=:f_l(\varepsilon_1,\varrho_1,\dots,\varepsilon_d,\varrho_d,\lambda)\,, \tag{4.11}
$$

$$
0=R'(\varepsilon_l)-\frac{r_l}{\varrho_l}=1-\frac{r_l}{\varrho_l}+\frac{\lambda}{N}\sum_{k=1}^{d}\frac{\varrho_k}{(\varepsilon_k+\varepsilon_l)^2}=:g_l(\varepsilon_1,\varrho_1,\dots,\varepsilon_d,\varrho_d,\lambda) \tag{4.12}
$$

for $l=1,\dots,d$ from Theorem 3·1 for the given positive real numbers $E_l>0$ and $r_l>0$ from Section 2. In the following we only use that all these E_l,r_l are positive (but not that r_l

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is an integer counting the multiplicity of E_l). So we consider for $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$ the real or complex algebraic subset

$$Z(\mathbb{K}) := \{(\varepsilon_1, \varrho_1, \ldots, \varepsilon_d, \varrho_d, \lambda) \in U(\mathbb{K}) | f_l = 0, g_l = 0,\ l = 1, \ldots, d\}$$

in

$$U(\mathbb{K}) := \{(\varepsilon_1, \varrho_1, \ldots, \varepsilon_d, \varrho_d, \lambda) |\ \varepsilon_k + \varepsilon_l \neq 0,\ \varrho_l \neq 0,\ k, l = 1, \ldots, d\} \subset \mathbb{K}^{2d+1},$$

i.e. in the complement of the corresponding central hyperplane arrangement in $\mathbb{K}^{2d+1}$. Note that $(\varepsilon_1, \varrho_1, \ldots, \varepsilon_d, \varrho_d, 0) \in Z(\mathbb{K})$ iff $\varepsilon_l = E_l$ and $\varrho_l = r_l$ for all $l = 1, \ldots, d$, and this real point belongs to the chamber

$$U_+(\mathbb{R}) := \{(\varepsilon_1, \varrho_1, \ldots, \varepsilon_d, \varrho_d, \lambda) |\ \varepsilon_l > 0,\ \varrho_l > 0,\ l = 1, \ldots, d\} \subset U(\mathbb{R}) \subset \mathbb{R}^{2d+1}.$$

Note that the complex dimension of any irreducible component of $Z(\mathbb{C})$ is at least $1 = (2d+1) - 2d$, since we are considering $2d$ equations in a Zariski-open subset of $\mathbb{C}^{2d+1}$.

LEMMA 4·8. *$Z(\mathbb{K})$ is a one-dimensional (real or complex) algebraic submanifold of $U(\mathbb{K})$ near the reference point $(E_1, r_1, \ldots, E_d, r_d, 0)$, with the projection*

$$pr : \mathbb{K}^{2d+1} \supset Z(\mathbb{K}) \longrightarrow \mathbb{K}$$

onto the last λ-coordinate a submersion near $(E_1, r_1, \ldots, E_d, r_d, 0)$ in the Zariski topology. In particular, the reference point $(E_1, r_1, \ldots, E_d, r_d, 0)$ only belongs to one irreducible component of $Z(\mathbb{C})$, which is of dimension one. Moreover, the map (of pointed sets)

$$pr : Z(\mathbb{K}) \longrightarrow \mathbb{K} \quad \text{with} \quad pr(E_1, r_1, \ldots, E_d, r_d, 0) = 0$$

becomes locally near $(E_1, r_1, \ldots, E_d, r_d, 0)$ a (real or complex) analytic isomorphism onto an open interval or disc around $\lambda = 0 \in \mathbb{K}$, fitting with the description given in Theorem 3·1 in terms of the implicit function theorem.

Proof. The claim follows from

$$\left(\frac{\partial f_l}{\partial \varepsilon_k}(E_1, r_1, \ldots, E_d, r_d, 0), \frac{\partial f_l}{\partial \varrho_k}(E_1, r_1, \ldots, E_d, r_d, 0)\right) = (\delta_{lk}, 0)$$

$$\text{and} \quad \left(\frac{\partial g_l}{\partial \varepsilon_k}(E_1, r_1, \ldots, E_d, r_d, 0), \frac{\partial g_l}{\partial \varrho_k}(E_1, r_1, \ldots, E_d, r_d, 0)\right) = \left(0, \delta_{lk} \cdot \frac{1}{r_l}\right)$$

for $l, k = 1, \ldots, d$.

Remark 4·9. Let us rewrite for a fixed $\lambda \in \mathbb{C}$ the equations (4·11) and (4·12) in terms of the d polynomials

$$F_l\big(\varepsilon_1, \varrho_1, \ldots, \varepsilon_d, \varrho_d\big) := f_l\big(\varepsilon_1, \varrho_1, \ldots, \varepsilon_d, \varrho_d, \lambda\big) \cdot \prod_{k=1}^{d} (\varepsilon_k + \varepsilon_l)$$

of degree $d+1$ and the d polynomials

$$G_l\big(\varepsilon_1, \varrho_1, \ldots, \varepsilon_d, \varrho_d\big) := g_l(\varepsilon_1, \varrho_1, \ldots, \varepsilon_d, \varrho_d, \lambda) \cdot \varrho_l \prod_{k=1}^{d} (\varepsilon_k + \varepsilon_l)^2$$

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of degree $2d+1$ (for $l=1,\ldots,d$), so that:

$$Z(\mathbb{C})\cap\{pr=\lambda\}\subset\Big\{\big(\varepsilon_1,\varrho_1,\ldots,\varepsilon_d,\varrho_d\big)\in\mathbb{C}^{2d}|\, F_l=0, G_l=0,\ l=1,\ldots,d\Big\}\times\{\lambda\}\,.$$

If for a given $\lambda\in\mathbb{C}$ the set

$$\Big\{\big(\varepsilon_1,\varrho_1,\ldots,\varepsilon_d,\varrho_d\big)\in\mathbb{C}^{2d}|\, F_l=0, G_l=0,\ l=1,\ldots,d\Big\}$$

is finite, then one gets by the affine Bezout inequality [**29**, Thm 3·1] the upper estimate $(d+1)^d(2d+1)^d$ for the number of solutions of the equations (4·11) and (4·12) (for this λ).

Let us come back to the rational function $R(z)$ from (4·10) for the case of positive real $E_l>0$ and $r_l>0$ for $l=1,\ldots,d$ related to the solution of Theorem 3·1 as discussed before. Then

$$R'(z)=1+\frac{\lambda}{N}\sum_{k=1}^{d}\frac{\varrho_k}{(\varepsilon_k+z)^2}>0 \tag{4·13}$$

for all $\lambda\geqslant 0$ and $z\in\mathbb{R}\backslash\{-\varepsilon_1,\ldots,-\varepsilon_d\}$.

LEMMA 4·10. *$R^{-1}(E_k)$ consists for all $E_k>0$ of $d+1$ different real points so that assumption* (ii) *of Ansatz 4·1 holds. Moreover we can choose $\mathcal{U}=\mathcal{U}_1=\ldots=\mathcal{U}_d$ as a small simply connected open neighbourhood of* $(R(0),\infty)\subset\mathbb{R}$ *in $\mathbb{C}$, with $\mathcal{V}=\mathcal{V}_1=\ldots=\mathcal{V}_d$ also containing* $(0,\infty)$. *By shrinking of $\mathcal{U}$ we can even assume that $\mathcal{V}\subset\{z\in\mathbb{C}|\, Re(z)>0\}$. Then the assumptions* (v) *and* (vi) *of Ansatz 4·1 hold for all $w\in\mathcal{V}$ with $\mathcal{V}$ small enough, as well as the assumption* (iv) *with $\mathcal{G}^{(0)}$ as in* (4·5) *resp.* (4·9).

Proof. If we order the numbering of the ε_l as $\varepsilon_i<\varepsilon_{i+1}$ for $i=0,\ldots,d$ with $\varepsilon_0:=-\infty$ and $\varepsilon_{d+1}:=+\infty$, then

$$R:(-\varepsilon_{i+1},-\varepsilon_i)\longrightarrow\mathbb{R}$$

is for all $i=0,\ldots,d$ strictly monotone increasing and bijective by the estimate $R'(z)>0$ above and the intermediate value theorem. This proves the first claim. Similarly, $R^{-1}(R(w))$ consists for any $w\in\mathcal{V}\cap(0,\infty)$ of $d+1$ real points which we can order as $\hat{w}^l\in(-\varepsilon_{l+1},-\varepsilon_l)$. Therefore, $-\hat{w}^l\in(0,\infty)$ and all $R\big(-\hat{w}^l\big)$ are distinct for $w\in\mathcal{V}\cap(0,\infty)$, because R is injective on $(-\varepsilon_1,\infty)$. Moreover, R is an injective immersion in $(-\varepsilon_1,\infty)$, which is an open condition so that also the second claim follows. Finally the assumption (iv) with $\mathcal{G}^{(0)}$ as in (4·9) follows from $0\neq z+w$ for all $z,w\in\mathcal{V}$, since $\mathrm{Re}(z+w)=\mathrm{Re}(z)+\mathrm{Re}(w)>0$.

We finish this section with the:

Proof of Theorem 3·1. We have seen that equation (3·2) can be solved by Ansatz 4·1 to $G^{(0)}(R(z),R(w))=\mathcal{G}^{(0)}(z,w)$, where R is given in (4·10) and $\mathcal{G}^{(0)}$ in (4·5) resp. (4·9). This solution depends on the choice of preimages $\varepsilon_k\in R^{-1}(E_k)$ made in (iv) of Ansatz 4·1. *Any* solution $\{\varepsilon_1,\ldots,\varepsilon_d\}$ of the system of equations (3·3) provides a solution of (4·5), if also the assumptions (iv), (v) and (vi) of Ansatz 4·1 hold. Theorem 3·1 selects one particular solution of (3·2) which satisfies the assumptions (ii), (iv), (v) and (vi) of Ansatz 4·1 by Lemma 4·10. Hence also relation (vii) of Ansatz 4·1 holds by Proposition 4·7. The choice $\lim_{\lambda\to 0}\varepsilon_k=E_k$ and $\lim_{\lambda\to 0}\varrho_k=r_k$ is made to recover in the limit $\lambda\to 0$ the moments of the Gaußian measure (2·1).

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It remains to show (3·4). On the right-hand side of (4·2) we use the the symmetry $\mathcal{G}^{(0)}(\varepsilon_k, w) = \mathcal{G}^{(0)}(w, \varepsilon_k)$ from Proposition 4·6 and express $\mathcal{G}^{(0)}(w, \varepsilon_k)$ as (4·5) for $w \mapsto \varepsilon_k$ and $z \mapsto w$. Dividing by $(R(w) - R(z))$ gives

$$\mathcal{G}^{(0)}(z, w) = \frac{1 - \dfrac{\lambda}{N}\displaystyle\sum_{k=1}^{d} \frac{r_k}{(R(z) - R(\varepsilon_k))(R(\varepsilon_k) - R(-w))} \prod_{j=1}^{d} \frac{R(w) - R\big(-\widehat{\varepsilon_k}^j\big)}{R(w) - R\big(\varepsilon_j\big)}}{R(w) - R(-z)} . \tag{4·14}$$

This equation was obtained in [12] by another method. We rearrange it as

$$\begin{aligned}\mathcal{G}^{(0)}(z, w) = \frac{1}{(R(w) - R(-z))(R(z) - R(-w))} \Bigg\{ & R(z) - R(-w) \\ & - \frac{\lambda}{N}\sum_{l=1}^{d} \frac{r_l}{(R(\varepsilon_l) - R(-w))} \prod_{j=1}^{d} \frac{R(w) - R\big(-\widehat{\varepsilon_l}^j\big)}{R(w) - R\big(\varepsilon_j\big)} \\ & - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{r_k}{(R(z) - R(\varepsilon_k))} \prod_{j=1}^{d} \frac{R(w) - R\big(-\widehat{\varepsilon_k}^j\big)}{R(w) - R\big(\varepsilon_j\big)} \Bigg\} .\end{aligned}$$

The second line is $-(\lambda/N)\sum_{l=1}^{d} \mathcal{G}^{(0)}(w, \varepsilon_l)$ by (4·5). We combine it with the term $-R(-w)$ inside $\{\ \}$ according to our main algebraic relation (4·1). In the last line, the factor $\prod_{j=1}^{d}\big(R(w) - R\big(-\widehat{\varepsilon_k}^j\big)\big)/\big(R(w) - R\big(\varepsilon_j\big)\big)$ is rewitten via (4·6), with $w \mapsto \varepsilon_k$ and $z \mapsto w$. We arrive at

$$\begin{aligned}\mathcal{G}^{(0)}(z, w) = \frac{1}{(R(w) - R(-z))(R(z) - R(-w))} \Bigg\{ & R(z) + R(w) \\ & + \frac{\lambda}{N}\sum_{k=1}^{d} \frac{r_k}{(R(\varepsilon_k) - R(z))} + \frac{\lambda}{N}\sum_{l=1}^{d} \frac{r_l}{(R(\varepsilon_l) - R(w))} \\ & + \frac{\lambda}{N}\sum_{k,l=1}^{d} \frac{r_k}{(R(\varepsilon_k) - R(z))(R(\varepsilon_l) - R(w))} \frac{\prod_{j=1}^{d}\big(R(\varepsilon_l) - R\big(-\widehat{\varepsilon_k}^j\big)\big)}{\prod_{j\neq l}^{d}\big(R(\varepsilon_l) - R\big(\varepsilon_j\big)\big)} \Bigg\} .\end{aligned}$$

The result (3·4) follows from equation (4·4) for $\mathcal{G}^{(0)}(\varepsilon_k, \varepsilon_l)$.

5. *The diagonal* 2*-point function*

The diagonal planar cumulant $z \mapsto \mathcal{G}^{(0)}(z, z)$ of length 2 and cycle type (0, 1) admits a simpler formula due to properties of the rational function $\tilde{R}$ with $\tilde{R}(z) := R(z) - R(-z)$. Let $z \in \big\{0, \pm\alpha_1, \ldots, \pm\alpha_d\big\}$ be the list of roots of

$$0 = R(z) - R(-z) = 2z - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{\varrho_k}{\varepsilon_k + z} - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{\varrho_k}{-\varepsilon_k + z} ,$$

with the convention $\alpha_k > 0$ and $\alpha_k \neq \alpha_l$ for $k \neq l$. Since here all $\varepsilon_k > 0$ and $\varrho_k > 0$ are positive real numbers, we can argue as for the rational function R that also the rational function $\tilde{R}$ maps each of the $2d + 1$ connected components of $\mathbb{R}\backslash\{\pm\varepsilon_1, \ldots, \pm\varepsilon_d\}$ bijectively onto $\mathbb{R}$. So

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the odd function $\tilde{R}$ has indeed $2d+1$ different real roots $\{0, \pm\alpha_1, \ldots, \pm\alpha_d\}$ of the equation $\tilde{R}=0$. Taking its poles $\{\infty, \pm\varepsilon_k\}$ into account, we have

$$(R(z)-R(-z)) = 2z\prod_{k=1}^{d}\frac{(z^2-\alpha_k^2)}{(z^2-\varepsilon_k^2)}\,. \tag{5·1}$$

PROPOSITION 5·1. *For any $z \in \hat{\mathbb{C}}$, the diagonal planar cumulant of cycle type* (0, 1) *can be simplified to*

$$\begin{aligned}\mathcal{G}^{(0)}(z,z) &= \frac{2R(z)-2R(0)}{(R(z)-R(-z))^2}\prod_{k=1}^{d}\frac{(R(z)-R(\alpha_k))^2}{(R(z)-R(\varepsilon_k))^2} \\ &\equiv \frac{1}{2z}\prod_{k=1}^{d}\frac{(z-\hat{0}^k)(z+e_k)\prod_{j=2}^{d}(z-\widehat{\alpha_k}^j)^2}{\prod_{l=1}^{d}(z-\widehat{e_k}^l)^2}\,,\end{aligned} \tag{5·2}$$

in the convention $\widehat{\alpha_k}^1 \equiv -\alpha_k$.

Proof. The $d+1$ fold product of (5·1) for $\{z, \hat{z}^1, \ldots, \hat{z}^d\}$ is inserted into (4·5):

$$\begin{aligned}&(R(z)-R(-z))^2\mathcal{G}^{(0)}(z,z)\prod_{k=1}^{d}(R(z)-R(\varepsilon_k)) \\ &= (R(z)-R(-z))\prod_{l=1}^{d}\big(R(z)-R(-\hat{z}^l)\big) \\ &= 2z\Big(\prod_{l=1}^{d}2\hat{z}^l\Big)\prod_{k=1}^{d}\frac{(z^2-\alpha_k^2)\prod_{l=1}^{d}((\hat{z}^l)^2-\alpha_k^2)}{(z^2-\varepsilon_k^2)\prod_{l=1}^{d}((\hat{z}^l)^2-\varepsilon_k^2)}\,.\end{aligned} \tag{5·3}$$

We use cases of (4·8); the third one takes $R(\alpha_k)=R(-\alpha_k)$ into account:

$$\begin{aligned}2(R(z)-R(0)) &= \frac{2z\prod_{l=1}^{d}(2\hat{z}_l)}{\prod_{k=1}^{d}(-2\varepsilon_k)}\,, \\ (R(z)-R(\varepsilon_k)) &= (z-\varepsilon_k)\frac{\prod_{l=1}^{d}(\varepsilon_k-\hat{z}^l)}{\prod_{l=1}^{d}(\varepsilon_k+\varepsilon_l)}\,, \\ (R(z)-R(\alpha_k))^2 &= (z^2-\alpha_k^2)\frac{\prod_{l=1}^{d}((\hat{z}^l)^2-\alpha_k^2)}{\prod_{l=1}^{d}(\varepsilon_l^2-\alpha_k^2)}\,.\end{aligned}$$

We identify in (5·3) the first equation and the product over k of the second and third equations:

$$\begin{aligned}&(R(z)-R(-z))^2\mathcal{G}^{(0)}(z,z)\prod_{k=1}^{d}(R(z)-R(\varepsilon_k)) \\ &= \frac{2(R(z)-R(0))}{\prod_{k=1}^{d}((z+\varepsilon_k)\prod_{l=1}^{d}(\hat{z}^l+\varepsilon_k))}\prod_{k=1}^{d}\frac{(R(z)-R(\alpha_k))^2}{(R(z)-R(\varepsilon_k))}\cdot\prod_{k=1}^{d}\frac{(2\varepsilon_k)\prod_{l=1}^{d}(\varepsilon_l^2-\alpha_k^2)}{\prod_{l=1}^{d}(\varepsilon_l+\varepsilon_k)}\,.\end{aligned}$$

Now observe that the residue of (4·8) at $z=-\varepsilon_k$ is the identity

$$\frac{(u+\varepsilon_k)\prod_{l=1}^{d}(\hat{u}^l+\varepsilon_k)}{\prod_{j\neq k}^{d}(\varepsilon_j-\varepsilon_k)} = -\frac{\lambda}{N}\varrho_k\,, \tag{5·4}$$

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for any $u \notin R^{-1}(\{\infty\})$. Consequently,

$$\frac{(R(z)-R(-z))^2\mathcal{G}^{(0)}(z,z)}{2(R(z)-R(0))}\prod_{k=1}^{d}\frac{(R(z)-R(\varepsilon_k))^2}{(R(z)-R(\alpha_k))^2}=\prod_{k=1}^{d}\frac{N\prod_{l=1}^{d}\big(\varepsilon_l^2-\alpha_k^2\big)}{(-\lambda\varrho_k)\prod_{j\neq k}^{d}\big(\varepsilon_j^2-\varepsilon_k^2\big)}=C$$

is a constant independent of z, which for $z\to\infty$ is identified as $C=1$.

The following result will be needed in the next section:

LEMMA 5·2. *For any $w\in\hat{\mathbb{C}}$ one has*

$$\frac{1}{R(w)-R(-w)}+\sum_{k=1}^{d}\frac{1}{R(w)-R\big(-\hat{w}^k\big)}=\frac{1}{2(R(w)-R(0))}+\sum_{k=1}^{d}\frac{1}{R(w)-R(\alpha_k)}\,. \tag{5·5}$$

Proof. Taking $R(w)=R\big(\hat{w}^k\big)$ into account, all terms on the lhs of (5·5) are of the form (5·1) so that the lhs of (5·5) has simple poles at $w\in\{0,\pm\alpha_l\}$ and $\hat{w}^k\in\{0,\pm\alpha_l\}$. Applying R shows that these $\hat{w}^k$ correspond to additional poles at $w\in\big\{\hat{0}^l,\widehat{\alpha_l}^j,\mp\alpha_l\big\}$ for $l=1,\ldots,d$ and $j=2,\ldots,d$. We evaluate the residues at these poles and check that the rhs of (5·5) has the same poles (clear) with the same residues.

Note that $R(w)=R\big(\hat{w}^k\big)$ implies $R'(w)=R'\big(\hat{w}^k\big)\big(\hat{w}^k\big)'(w)$ or $\big(\hat{w}^k\big)'(w)=R'(w)/R'\big(\hat{w}^k\big)$. Consider the pole at $w=\pm\alpha_l$. Then there is precisely one $k_l\in\{1,\ldots,d\}$ with $\hat{w}^{k_l}=\mp\alpha_l$. Therefore,

$$\begin{aligned}&\operatorname*{Res}_{w=\pm\alpha_l}\left(\frac{dw}{R(w)-R(-w)}+\sum_{k=1}^{d}\frac{dw}{R(w)-R\big(-\hat{w}^k\big)}\right)\\&=\left(\frac{1}{R'(w)+R'(-w)}+\frac{1}{R'(w)+R'\big(-\hat{w}^{k_l}\big)\frac{R'(w)}{R'(\hat{w}^{k_l})}}\right)\Bigg|_{w=\pm\alpha_l,\hat{w}^{k_l}=\mp\alpha_l}=\frac{1}{R'(\pm\,\alpha_l)}\,.\end{aligned}$$

Consider in case of $d\geqslant 2$ the pole at $w=\widehat{\alpha_l}^j$. There are precisely two distinct $k_+,k_-\in\{1,\ldots,d\}$ with $\hat{w}^{k_+}=\alpha_l$ and $\hat{w}^{k_-}=-\alpha_l$. Therefore,

$$\begin{aligned}&\operatorname*{Res}_{w=\widehat{\alpha_l}^j}\left(\frac{dw}{R(w)-R(-w)}+\sum_{k=1}^{d}\frac{dw}{R(w)-R\big(-\hat{w}^k\big)}\right)\\&=\left(\frac{1}{R'(w)+R'\big(-\hat{w}^{k_+}\big)\frac{R'(w)}{R'\big(\hat{w}^{k_+}\big)}}+\frac{1}{R'(w)+R'\big(-\hat{w}^{k_-}\big)\frac{R'(w)}{R'\big(\hat{w}^{k_-}\big)}}\right)\Bigg|_{w=\widehat{\alpha_l}^j,\hat{w}^{k_\pm}=\pm\alpha_l}\\&=\frac{1}{R'\big(\widehat{\alpha_l}^j\big)}\,.\end{aligned}$$

The rhs of (5·5) has exactly the same residues.

Finally, it is also clear that both sides of (5·5) have the same residue $1/2R'(0)$ at $w=0$. For $w=\hat{0}^l$ there is a unique $k_l\in\{1,\ldots,d\}$ with $\hat{w}^{k_l}=0$. Then

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$$\operatorname*{Res}_{w=\widehat{0}^l}\left(\frac{dw}{R(w)-R(-w)}+\sum_{k=1}^{d}\frac{dw}{R(w)-R\big(-\hat{w}^k\big)}\right)$$
$$=\frac{1}{R'(w)+R'\big(-\hat{w}^{k_l}\big)\frac{R'(w)}{R'\big(\hat{w}^{k_l}\big)}}\Bigg|_{w=\hat{0}^l,\hat{w}^{k_l}=0}=\frac{1}{2R'\big(\hat{0}^l\big)}\,,$$

which agrees with the residue of the rhs of (5·5). Therefore, the difference between lhs and rhs of (5·5) is a bounded entire function, i.e. a constant by Liouville's theorem, which is zero when considering $w\to\infty$. This completes the proof.

6. *The planar* $1+1$-*point function*

The $1+1$-point function $G_{|a|b|}$ is the cumulant of length 2 and cycle type (2, 0) (i.e. two cycles a and b of length 1), see Appendix A. We derive in Appendix B its equation of motion (B8) whose restriction to the planar sector (of degree or genus $g=0$) reads

$$(E_a+E_a)G^{(0)}_{|a|b|}=-\frac{\lambda}{N}\sum_{k=1}^{N}G^{(0)}_{|ak|}G^{(0)}_{|a|b|}+\frac{\lambda}{N}\sum_{k=1}^{N}\frac{G^{(0)}_{|k|b|}-G^{(0)}_{|a|b|}}{E_k-E_a}+\lambda\frac{G^{(0)}_{|bb|}-G^{(0)}_{|ab|}}{E_b-E_a}\,. \tag{6·1}$$

We interpret this equation as evaluation $G^{(0)}_{|a|b|}=\mathcal{G}^{(0)}(\varepsilon_a|\varepsilon_b)$ of a function[2] $\mathcal{G}^{(0)}(z|w)$ which satisfies

$$\big(R(z)-R(-z)\big)\mathcal{G}^{(0)}(z|w)-\frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k\mathcal{G}^{(0)}(\varepsilon_k|w)}{R(\varepsilon_k)-R(z)}=\lambda\frac{\mathcal{G}^{(0)}(z,w)-\mathcal{G}^{(0)}(w,w)}{R(z)-R(w)}\,. \tag{6·2}$$

The identity (4·1) was decisive here, and multiplicities r_k of the $E_k=R(\varepsilon_k)$ were admitted.

Since $\mathcal{G}^{(0)}(\alpha_k|w)$ must be regular[3] for any $w>0$, evaluation at $z=\alpha_k$ produces d equations

$$\frac{\lambda}{N}\sum_{l=1}^{d}\frac{r_l\mathcal{G}^{(0)}(\varepsilon_l|w)}{R(\alpha_k)-R(\varepsilon_l)}=\lambda\frac{\mathcal{G}^{(0)}(\alpha_k,w)-\mathcal{G}^{(0)}(w,w)}{R(\alpha_k)-R(w)}\,. \tag{6·3}$$

Equation (6·3) is with Proposition 3·2 solved by

$$\frac{r_k}{N}\mathcal{G}^{(0)}(\varepsilon_k|w)=\sum_{l=1}^{d}(R(\alpha_l)-R(\varepsilon_k))\mathbf{A}_l(R(\varepsilon_k))\mathbf{E}_k(R(\alpha_l))\frac{\mathcal{G}^{(0)}(\alpha_l,w)-\mathcal{G}^{(0)}(w,w)}{R(\alpha_l)-R(w)}\,,$$

where

$$\mathbf{A}_i(x)=\frac{\mathbf{A}(x)}{(x-R(\alpha_i))\mathbf{A}'(R(\alpha_i))}\qquad\text{and}\qquad\mathbf{E}_j(y)=\frac{\mathbf{E}(y)}{\big(y-R(\varepsilon_j)\big)\mathbf{E}'\big(R(\varepsilon_j)\big)}\,,$$
$$\text{with}\quad\mathbf{A}(x):=\prod_{k=1}^{d}(x-R(\alpha_k))\qquad\text{and}\qquad\mathbf{E}(y)=\prod_{k=1}^{d}(y-R(\varepsilon_k))\,.$$

[2] Be careful to distinguish $\mathcal{G}^{(0)}(z|w)$ from $\mathcal{G}^{(0)}(z,w)$.

[3] Regularity of $\mathcal{G}^{(0)}(\alpha_k|w)$ is here a technical assumption which is justified by viewing (6·2) as limiting case of singular integral equations of Carleman type (see e.g. [32, section 4·4]). Their solutions are regular for any $z,w>0$.

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Here, the $R(\alpha_k)$ and $R(\varepsilon_l)$ are pairwise distinct, since R is injective on $(-\varepsilon_1, \infty)$. Inserting this back into (6·2) gives the 1 + 1-point function

$$\mathcal{G}^{(0)}(z|w) = \frac{\lambda}{R(z)-R(-z)}\Bigg\{\frac{\mathcal{G}^{(0)}(z,w)-\mathcal{G}^{(0)}(w,w)}{R(z)-R(w)} - \sum_{k,l=1}^{d}\frac{(R(\alpha_l)-R(\varepsilon_k))\mathbf{A}_l(R(\varepsilon_k))\mathbf{E}_k(R(\alpha_l))}{R(z)-R(\varepsilon_k)}\frac{\mathcal{G}^{(0)}(\alpha_l,w)-\mathcal{G}^{(0)}(w,w)}{R(\alpha_l)-R(w)}\Bigg\} \tag{6·4}$$

in terms of the 2-point function $\mathcal{G}^{(0)}(z,w)$ known from Theorem 3·1. We convert the solution (6·4) into a manifestly symmetric form:

PROPOSITION 6·1. *The planar cumulant of length* 2 *and cycle type* (2, 0) *has the solution*

$$\mathcal{G}^{(0)}(z|w) = \frac{\lambda}{(R(z)-R(w))^2}\Bigg(\mathcal{G}^{(0)}(z,w) - \frac{R(z)+R(w)-2R(0)}{(R(z)-R(-z))(R(w)-R(-w))}\prod_{k=1}^{d}\frac{(R(z)-R(\alpha_k))(R(w)-R(\alpha_k))}{(R(z)-R(\varepsilon_k))(R(w)-R(\varepsilon_k))}\Bigg). \tag{6·5}$$

Proof. Using (3·8) we evaluate the k-sum in (6·4) to

$$\mathcal{G}^{(0)}(z|w) = \frac{\lambda}{R(z)-R(-z)}\Bigg\{\frac{\mathcal{G}^{(0)}(z,w)-\mathcal{G}^{(0)}(w,w)}{R(z)-R(w)} - \sum_{l=1}^{d}\frac{\mathbf{A}_l(R(z))\mathbf{E}(R(\alpha_l))}{\mathbf{E}(R(z))}\frac{\mathcal{G}^{(0)}(\alpha_l,w)-\mathcal{G}^{(0)}(w,w)}{R(\alpha_l)-R(w)}\Bigg\}. \tag{6·6}$$

In the second line we have

$$\frac{\mathbf{A}_l(R(z))\mathbf{E}(R(\alpha_l))}{\mathbf{E}(R(z))} = -\frac{\mathbf{A}(R(z))}{\mathbf{E}(R(z))}\cdot\frac{\prod_{k=1}^{d}(R(\alpha_l)-R(\varepsilon_k))}{(R(\alpha_l)-R(z))\prod_{j=1,j\neq l}^{d}\big(R(\alpha_l)-R(\alpha_j)\big)}$$

and we recall

$$\mathcal{G}^{(0)}(\alpha_l,w) = -\frac{1}{(R(\alpha_l)-R(w))}\frac{\prod_{j=1}^{d}R(\alpha_l)-R\big(-\hat{w}^j\big)}{\prod_{j=1}^{d}\big(R(\alpha_l)-R(\varepsilon_j)\big)}$$

from (4·5). Inserting both identities into (6·6) gives after a first partial fraction decomposition

$$\mathcal{G}^{(0)}(z|w) = \frac{\lambda}{(R(z)-R(-z))(R(z)-R(w))}\Bigg\{\mathcal{G}^{(0)}(z,w)-\mathcal{G}^{(0)}(w,w) - \frac{\mathbf{A}(R(z))}{\mathbf{E}(R(z))}\mathcal{G}^{(0)}(w,w)\Bigg(\sum_{l=1}^{d}\frac{\prod_{k=1}^{d}(R(\alpha_l)-R(\varepsilon_k))}{(R(\alpha_l)-R(z))\prod_{j=1,j\neq l}^{d}\big(R(\alpha_l)-R(\alpha_j)\big)} - \sum_{l=1}^{d}\frac{\prod_{k=1}^{d}(R(\alpha_l)-R(\varepsilon_k))}{(R(\alpha_l)-R(w))\prod_{j=1,j\neq l}^{d}\big(R(\alpha_l)-R(\alpha_j)\big)}\Bigg)$$

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$$-\frac{\mathbf{A}(R(z))}{\mathbf{E}(R(z))}\Bigg(\sum_{l=1}^{d}\frac{\prod_{k=1}^{d}\big(R(\alpha_l)-R(-\hat{w}^k)\big)}{(R(\alpha_l)-R(z))(R(\alpha_l)-R(w))\prod_{j=1,j\neq l}^{d}\big(R(\alpha_l)-R(\alpha_j)\big)}$$
$$-\sum_{l=1}^{d}\frac{\prod_{k=1}^{d}\big(R(\alpha_l)-R(-\hat{w}^k)\big)}{(R(\alpha_l)-R(w))^2\prod_{j=1,j\neq l}^{d}\big(R(\alpha_l)-R(\alpha_j)\big)}\Bigg)\Bigg\}. \qquad (6\cdot7)$$

The second and third line are converted via an identity (4·6) with substitution $\varepsilon_i\mapsto\alpha_i$ and $-\hat{w}^j\mapsto\varepsilon_j$. One of the surviving terms cancels $\mathcal{G}^{(0)}(w,w)$ in the first line of (6·7). Another partial fraction decomposition in the fourth line of (6·7) and $1/((R(\alpha_l)-R(w))^2)=\lim_{u\to w}1/((R(u)-R(w)))\big(1/((R(\alpha_l)-R(u)))-1/((R(\alpha_l)-R(w)))\big)$ in the fifth line of (6·7) also give rise to expressions (4·6) with substitution $\varepsilon_i\mapsto\alpha_i$. We thus find

$$\mathcal{G}^{(0)}(z|w)=\frac{\lambda}{(R(z)-R(-z))(R(z)-R(w))}\Bigg\{\mathcal{G}^{(0)}(z,w)-\frac{\mathbf{A}(R(z))\mathbf{E}(R(w))}{\mathbf{E}(R(z))\mathbf{A}(R(w))}\mathcal{G}^{(0)}(w,w)$$
$$+\frac{\prod_{k=1}^{d}\frac{R(z)-R(-\hat{w}^k)}{R(z)-R(\varepsilon_k)}-\frac{\mathbf{A}(R(z))\mathbf{E}(R(w))}{\mathbf{E}(R(z))\mathbf{A}(R(w))}\prod_{k=1}^{d}\frac{R(w)-R(-\hat{w}^k)}{R(w)-R(\varepsilon_k)}}{R(z)-R(w)}$$
$$-\frac{\mathbf{A}(R(z))}{\mathbf{E}(R(z))}\lim_{u\to w}\frac{\prod_{k=1}^{d}\frac{R(u)-R(-\hat{w}^k)}{R(u)-R(\alpha_k)}-\prod_{k=1}^{d}\frac{R(w)-R(-\hat{w}^k)}{R(w)-R(\alpha_k)}}{R(u)-R(w)}\Bigg\}.$$

After evaluation of the limit we reconstruct in the last two lines $\mathcal{G}^{(0)}(z,w)$ and $\mathcal{G}^{(0)}(w,w)$ via (4·5):

$$\mathcal{G}^{(0)}(z|w)=\frac{\lambda}{(R(z)-R(w))^2}\Bigg(\mathcal{G}^{(0)}(z,w)$$
$$-\frac{\mathbf{A}(R(z))\mathbf{E}(R(w))}{\mathbf{E}(R(z))\mathbf{A}(R(w))}\frac{R(w)-R(-w)}{R(z)-R(-z)}\mathcal{G}^{(0)}(w,w)\Bigg\{1$$
$$+\frac{R(z)-R(w)}{R(w)-R(-w)}+\sum_{k=1}^{d}\frac{R(z)-R(w)}{R(w)-R(-\hat{w}^k)}-\sum_{k=1}^{d}\frac{R(z)-R(w)}{R(w)-R(\alpha_k)}\Bigg\}\Bigg).$$

With Lemma 5·2 the terms in $\{\ \}$ can be reduced to $\{\ \}=(R(z)+R(w)-2R(0))/(2(R(w)-R(0)))$. Inserting (5·2) for $\mathcal{G}^{(0)}(w,w)$ gives the final result (6·5).

7. *Outlook*

We have developed a new algebraic solution strategy for the two initial cumulants of a quartic analogue of the Kontsevich model. Our results have been extended in [**3**] to an algorithm which allows to recursively compute all other cumulants. The key discovery of [**3**] was to understand that one first has to focus on three families $\Omega^{(g)}_m(u_1,\dots,u_m)$, $\mathcal{T}^{(g)}(u_1,\dots,u_m\|z,w|)$ and $\mathcal{T}^{(g)}(u_1,\dots,u_m\|z|w|)$ of auxiliary functions. Their simplest cases are the functions $\mathcal{T}^{(0)}(\emptyset\|z,w|):=\mathcal{G}^{(0)}(z,w)$ and $\mathcal{T}^{(0)}(\emptyset\|z|w|):=\mathcal{G}^{(0)}(z|w)$ analysed in this paper. The auxiliary functions are special polynomials [**2**] in the original cumulants. One first solves a coupled system of equations for $(\Omega^{(g)}_m,\mathcal{T}^{(g)})$ and then uses the result to turn the

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Dyson–Schwinger equations for the cumulants into a problem which can easily be solved by inversion of Cauchy matrices.

Of particular interest are the functions $\Omega_n^{(g)}$ which give rise to a family of meromorphic differentials

$$\omega_{g,n}(z_1, ..., z_n) = \Omega_n^{(g)}(z_1, ..., z_n) dR(z_1) \ldots dR(z_n) \tag{7·1}$$

which starts with $\omega_{0,2}(z_1, z_2) = dz_1\, dz_2/(z_1 - z_2)^2 + dz_1\, dz_2/(z_1 + z_2)^2$. Also the next forms $\omega_{0,3}$, $\omega_{0,4}$ and $\omega_{1,1}$ have been found in [3], where $\omega_{1,1}$ needs Propositions 5·1 and 6·1 of this paper. Remarkably, all forms computed so far satisfy abstract loop equations [1] if one sets $\omega_{0,1}(z) = y(z)dx(z)$ with $x(z) = R(z)$ and $y(z) = -R(-z)$. It was shown in [7] that the solution of abstract loop equations is *blobbed topological recursion*, a systematic extension of topological recursion [10, 11] by additional terms which are holomorphic at ramification points of x. The natural conjecture is that all $\omega_{g,n}$ of the quartic analogue of the Kontsevich model obey blobbed topological recursion. The conjecture was proved for genus $g = 0$ in [18] by relating it to an equation which expresses $\omega_{g,n+1}(z_1, ..., z_n, -z)$ in terms of $\omega_{g,m+1}(z_1, ..., z_m, +z)$ with $m \leqslant n$.

In an early version of this paper we had speculated that the exact solution of the non-linear equation (3·2) might be caused by a hidden integrable structure. The discovery in [3, 18] that the quartic analogue of the Konsevich model obeys blobbed topological recursion questions this interpretation: integrability is not known in blobbed topological recursion. The relation to intersection theory on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable complex curves extends, however, to blobbed topological recursion [7]. The discovery in [18] that (at least the planar sector of) the quartic analogue of the Kontsevich model is completely governed by the behaviour of the $\omega_{g,n}$ under a global (and canonical) involution makes us confident that the intersection numbers generated by this model will have a geometric significance. It will be an exciting programme to make this precise.

Appendix A. *Decomposition of moments via cumulants*

The moments (2·3) decompose into cumulants (see e.g. [23, 30]),

$$\left\langle \prod_{i=1}^{n} e_{k_i l_i} \right\rangle = \sum_{\substack{\text{partitions}\\ \pi \text{ of}\{1,\ldots,n\}}} \prod_{\text{blocks } \beta \in \pi} \left\langle \prod_{i \in \beta} e_{k_i l_i} \right\rangle_c . \tag{A1}$$

There is a similar formula expressing the cumulants in terms of moments [30, equation (1·2)], related to (A1) via *Möbius inversion* on the partially ordered set of (partitions of) subsets of indices (the *partition lattice* of $[N] \times [N]$)

$$I := \{k_1 l_1, \ldots, k_n l_n\} \subset [N] \times [N] ,$$

with $|I| = n$ and $[N] := \{1, \ldots, N\}$. For a quartic potential (2·2), moments and cumulants are only non-zero if n is even and every block β is of even length. For example,

$$\begin{aligned}\langle e_{k_1 l_1} e_{k_2 l_2} e_{k_3 l_3} e_{k_4 l_4} \rangle &= \langle e_{k_1 l_1} e_{k_2 l_2} e_{k_3 l_3} e_{k_4 l_4} \rangle_c + \langle e_{k_1 l_1} e_{k_2 l_2} \rangle_c \langle e_{k_3 l_3} e_{k_4 l_4} \rangle_c \\ &\quad + \langle e_{k_1 l_1} e_{k_3 l_3} \rangle_c \langle e_{k_2 l_2} e_{k_4 l_4} \rangle_c + \langle e_{k_1 l_1} e_{k_4 l_4} \rangle_c \langle e_{k_2 l_2} e_{k_3 l_3} \rangle_c .\end{aligned}$$

Note that in our context the moments $\left\langle \prod_{i=1}^{n} e_{k_i l_i} \right\rangle$ are invariant under permutations of $I := \{k_1 l_1, \ldots, k_n l_n\}$ so that they only depend on the subset $I \subset [N] \times [N]$, but not on the

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choice of a labelling $I=\{k_1l_1,\dots,k_nl_n\}\simeq[n]$. By [30, equation (1·2)] the same is then true for the cumulants, i.e. $\big\langle\prod_{i\in\beta}e_{k_il_i}\big\rangle_c$ only depends on the subset $\{k_il_i|\ i\in\beta\}\subset[N]\times[N]$.

We restrict our attention to the case that all k_i are pairwise different. Then the structure of the Gaußian measure (2·1) (together with the invariance of a trace under cyclic permutations) implies that the cumulant $\big\langle\prod_{i=1}^n e_{k_il_i}\big\rangle_c$ corresponding to $I=\{k_1l_1,\dots,k_nl_n\}$ with $|I|=n>0$ is only non-zero if I has a permutation σ with $pr_2=pr_1\circ\sigma$. Here

$$pr_i:[N]\times[N]\supset I\longrightarrow[N]$$

is the projection onto the corresponding factor for $i=1,2$. By choosing a labelling

$$I:=\{k_1l_1,\dots,k_nl_n\}\simeq[n]$$

as before, this corresponds to a permutation σ in the symmetric group $\mathcal{S}_n$, with $(l_1,\dots,l_n)=\big(k_{\sigma(1)},\dots,k_{\sigma(n)}\big)$.

Therefore, the cumulant $\big\langle\prod_{i=1}^n e_{k_il_i}\big\rangle_c$ only depends on I and the *conjugacy class* of a permutation in $\mathcal{S}_n$ (corresponding to the permutation σ of I with $pr_2=pr_1\circ\sigma$), which is again independent of the choice of the labelling of I. In fact such conjugacy classes in $\mathcal{S}_n$ just correspond to the different *cycle types* of a permutation in the symmetric group $\mathcal{S}_n$. The cycle type of σ is the n-tuple $(\ell_1(\sigma),\ell_2(\sigma),\dots,\ell_n(\sigma))$ where $\ell_k(\sigma)$ is the number of cycles of length k in σ, with $\sum_{i=1}^n i\ell_i(\sigma)=n$. The number of cycles in a permutation σ is $b(\sigma)=\sum_{i=1}^n\ell_i(\sigma)$. The number of different cycle types is the partition number $p(n)$, and there are $n!/1^{\ell_1}\ell_1!2^{\ell_2}\ell_2!\dots n^{\ell_n}\ell_n!$ permutations with the same cycle type $(\ell_1,\dots,\ell_n)$.

Conversely, the l-indices of a non-vanishing cumulant $\big\langle e_{k_1l_1}\dots e_{k_nl_n}\big\rangle_c$ are completely determined by the cycle type and the information which k's belong in which cyclic order to the same cycle. If, after renaming the k's, $\big(k_1^1,\dots,k_{n_1}^1\big)$ belong to one cycle, $\big(k_1^2,\dots,k_{n_2}^2\big)$ belong to another cycle, and so on up to the b^{th} cycle, this information uniquely encodes a cumulant (with $n=n_1+\dots+n_b$)

$$N^n\Big\langle\Big(e_{k_1^1k_2^1}e_{k_2^1k_3^1}\dots e_{k_{n_1}^1k_1^1}\Big)\dots\Big(e_{k_1^bk_2^b}e_{k_2^bk_3^b}\dots e_{k_{n_b}^bk_1^b}\Big)\Big\rangle_c=:N^{2-b}G_{|k_1^1\dots k_{n_1}^1|\dots|k_1^b\dots k_{n_b}^b|}\,. \tag{A2}$$

The power series expansion of the Fourier transform $\mathcal{Z}(M)$ into moments (2·3) can be compared with the insertion of (A2) into (A1). The first terms are:

$$\begin{aligned}\mathcal{Z}(M)=1&-\frac{1}{N^2}\sum_{j,k=1}^N\left\{\frac{N}{2}G_{|jk|}M_{jk}M_{kj}+\frac{1}{2}G_{|j|k|}M_{jj}M_{kk}\right\}\\&+\frac{1}{N^4}\sum_{j,k,l,m=1}^N\left\{\frac{N}{4}G_{|jklm|}M_{jk}M_{kl}M_{lm}M_{mj}+\frac{1}{3}G_{|j|klm|}M_{jj}M_{kl}M_{lm}M_{mk}\right.\\&+\frac{1}{8}G_{|jk|lm|}M_{jk}M_{kj}M_{lm}M_{ml}+\frac{1}{4N}G_{|j|k|lm|}M_{jj}M_{kk}M_{lm}M_{ml}\\&+\frac{1}{24N^2}G_{|j|k|l|m|}M_{jj}M_{kk}M_{ll}M_{mm}+\frac{N^2}{8}G_{|jk|}G_{|lm|}M_{jk}M_{kj}M_{lm}M_{ml}\\&\left.+\frac{N}{4}G_{|jk|}G_{|l|m|}M_{jk}M_{kj}M_{ll}M_{mm}+\frac{1}{8}G_{|j|k|}G_{|l|m|}M_{jj}M_{kk}M_{ll}M_{mm}\right\}\\&+\mathcal{O}(M^6)\,.\end{aligned} \tag{A3}$$

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Appendix B. *Equations for the second cumulant*

We derive here equations for the two non-vanishing second-order cumulants $G_{|ab|} = \frac{1}{N}\langle e_{ab}e_{ba}\rangle_c$ of cycle type (0, 1) (i.e. one cycle ab of length 2) and $G_{|a|b|} = \langle e_{aa}e_{bb}\rangle_c$ of cycle type (2, 0) (i.e. two cycles a and b of length 1). To distinguish $G_{|ab|}$ and $G_{|a|b|}$ we require $a \neq b$.

We start from (2·7) with $\mathcal{Z}(M)$ given by (A3), apply $(N(E_a+E_b)/\mathrm{i})(\partial/\partial M_{ba})$ and put $M=0$. For $a \neq b$ this gives the following result (the underlining should be ignored for the moment; we explain it later):

$$
\begin{aligned}
(E_a+E_b)G_{|ab|} = 1 - \lambda \Bigg\{ & \frac{1}{N^2}\sum_{k,l=1}^{N} \underline{\underline{G_{|bakl|}}} + \frac{1}{N}\sum_{k=1}^{N}\Big(G_{|ab|}G_{|ak|} + \underline{\underline{G_{|ab|}G_{|bk|}}}\Big) \\
& + \frac{1}{N^2}\Big(G_{|abab|} + \underline{G_{|abbb|}} + G_{|baaa|} + G_{|ab|}\Big(G_{|a|a|} + \underline{G_{|a|b|}} + \underline{G_{|b|b|}}\Big)\Big) \\
& + \frac{1}{N^3}\sum_{k=1}^{N}\Big(\underline{\underline{G_{|k|bak|}}} + \underline{\underline{G_{|a|bak|}}} + \underline{G_{|b|bak|}} + \underline{\underline{G_{|ab|bk|}}} + G_{|ab|ak|}\Big) \\
& + \frac{1}{N^4}\Big(G_{|a|a|ab|} + \underline{G_{|a|b|ab|}} + \underline{G_{|b|b|ab|}}\Big)\Bigg\}. \qquad \text{(B1)}
\end{aligned}
$$

Next, we set $b \equiv a$ in (2·7) for $\mathcal{Z}(M)$ given by (A3), apply $(N^2(E_a+E_a)/\mathrm{i})(\partial/\partial M_{bb})$ for $a \neq b$ and obtain for $M=0$ (ignore again the underlining):

$$
\begin{aligned}
(E_a+E_a)G_{|a|b|} = -\lambda \Bigg\{ & \underline{\underline{G_{|bb|}G_{|ab|}}} + \frac{1}{N^2}\sum_{k,l=1}^{N} \underline{G_{|b|akl|}} \\
& + \frac{1}{N}\sum_{k=1}^{N}\Big(\underline{G_{|bbka|}} + \underline{\underline{G_{|bbak|}}} + G_{|ak|}G_{|a|b|} + \underline{G_{|ak|}G_{|a|b|}} + \underline{G_{|ak|}G_{|b|k|}}\Big) \\
& + \frac{1}{N^2}\Big(G_{|b|aaa|} + G_{|a|abb|} + \underline{\underline{G_{|a|abb|}}} + \underline{\underline{G_{|b|abb|}}} + \underline{\underline{G_{|bb|ab|}}} + 3G_{|a|b|}G_{|a|a|}\Big) \\
& + \frac{1}{N^3}\sum_{k=1}^{N}\Big(G_{|a|b|ak|} + \underline{G_{|a|b|ak|}} + \underline{G_{|b|k|ak|}}\Big) + \frac{1}{N^4}G_{|b|a|a|a|}\Bigg\}. \qquad \text{(B2)}
\end{aligned}
$$

Equations (B1) and (B2) are the analogues of Dyson–Schwinger equations in quantum field theory. In this form they provide little information because the right-hand sides are too complicated. We will now establish from the equations of motion (2·7) two other identities which collect the underlined terms in (B1) and (B2) into a function of the left-hand sides.

To establish the identities, set $b \mapsto k$ in (2·7) and apply $(N(E_a+E_k)/\mathrm{i})(\partial/\partial M_{kb})$. Next, set $a \mapsto k$ in (2·7) and apply $(N(E_b+E_k)/\mathrm{i})(\partial/\partial M_{ak})$. Take the difference of both equations and sum over k:

$$
-N\sum_{k=1}^{N}(E_a-E_b)\frac{\partial^2\mathcal{Z}(M)}{\partial M_{ak}\partial M_{kb}} = \sum_{k=1}^{N}\left(M_{ka}\frac{\partial\mathcal{Z}(M)}{\partial M_{kb}} - M_{bk}\frac{\partial\mathcal{Z}(M)}{\partial M_{ak}}\right). \qquad \text{(B3)}
$$

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This is a Ward–Takahashi identity first discovered in [9]. The strategy which we follow here was suggested in [17]. We insert (A3) into (B3) and evaluate the derivatives for $a \neq b$:

$$\begin{aligned}
&\frac{1}{N}\sum_{k=1}^{N}\left\{\big(G_{|kb|}-G_{|ak|}\big)M_{bk}M_{ka}+\frac{1}{N}\big(G_{|b|k|}-G_{|a|k|}\big)M_{ba}M_{kk}\right\}\\
&=\frac{1}{N}\sum_{k=1}^{N}(E_a-E_b)\left\{\frac{1}{N}\sum_{l=1}^{N}G_{|bkal|}+G_{|ak|}G_{|bk|}\right.\\
&\qquad\left.+\frac{1}{N^2}\Big(G_{|b|abk|}+G_{|a|abk|}+G_{|bk|ak|}\Big)\right\}M_{bk}M_{ka}\\
&\qquad+\frac{1}{N^2}\sum_{k=1}^{N}(E_a-E_b)\left\{G_{|bakk|}+\frac{1}{N}\sum_{l=1}^{N}G_{|k|bal|}+G_{|ab|}\big(G_{|b|k|}+G_{|a|k|}\big)\right.\\
&\qquad\left.+\frac{1}{N^2}\big(G_{|b|k|ab|}+G_{|a|k|ab|}\big)\right\}M_{ba}M_{kk}+\mathcal{O}\big(M^4\big)\,.
\end{aligned}\tag{B4}$$

For the next steps we assume that the functions $G_{..k_i..}$ under consideration are evaluations of holomorphic functions in several complex variables at points E_{k_i} in the holomorphicity domain. See the discussion after (3·1). Applying to (B4) the operators $N\partial^2/\big(\partial M_{bp}\partial M_{pa}\big)$ or $N^2\partial^2/\big(\partial M_{ba}\partial M_{pp}\big)$ for $a \neq p \neq b$ gives two independent equations. Under the holomorphicity assumption they extend continuously to $p = a$ and $p = b$. After exchanging $p \leftrightarrow b$, these equations read

$$-\frac{G_{|pb|}-G_{|ab|}}{E_p-E_a}=\frac{1}{N}\sum_{k=1}^{N}G_{|bakp|}+G_{|ab|}G_{|bp|}+\frac{1}{N^2}\big(G_{|p|bap|}+G_{|a|bap|}+G_{|bp|ab|}\big)\,,\tag{B5}$$

$$\begin{aligned}
-\frac{G_{|p|b|}-G_{|a|b|}}{E_p-E_a}&=G_{|bbpa|}+\frac{1}{N}\sum_{k=1}^{N}G_{|b|akp|}+G_{|ap|}\big(G_{|p|b|}+G_{|a|b|}\big)\\
&\qquad+\frac{1}{N^2}\big(G_{|b|p|ap|}+G_{|a|b|ap|}\big)\,;
\end{aligned}\tag{B6}$$

they hold for $p \neq a$. By the holomorphicity assumption the equations (B5) and (B6) extend continuously to $p = a$. Then, summing (B5) over p collects the double-underlined terms in (B1) into $-(1/N)\sum_{p=1}^{N}\big(G_{|pb|}-G_{|ab|}\big)\big/\big(E_p-E_a\big)$, and the case $p = b$ of (B6) collects the single-underlined terms in (B1). Similarly, summing (B6) over p collects the single-underlined terms in (B2) into $-\frac{1}{N}\sum_{p=1}^{N}\big(G_{|p|b|}-G_{|a|b|}\big)\big/\big(E_p-E_a\big)$, and the case $p = b$ of (B5) collects the double-underlined terms in (B2):

$$\begin{aligned}
(E_a+E_b)G_{|ab|}&=1-\frac{\lambda}{N}\sum_{p=1}^{N}G_{|ab|}G_{|ap|}+\frac{\lambda}{N}\sum_{p=1}^{N}\frac{G_{|pb|}-G_{|ab|}}{E_p-E_a}\\
&\qquad-\frac{\lambda}{N^2}\left(-\frac{G_{|b|b|}-G_{|a|b|}}{E_b-E_a}+G_{|abab|}+G_{|baaa|}+G_{|ab|}G_{|a|a|}\right.\\
&\qquad\left.+\frac{1}{N}\sum_{p=1}^{N}G_{|ab|ap|}\right)-\frac{\lambda}{N^4}G_{|a|a|ab|}\,,
\end{aligned}\tag{B7}$$

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$$
\begin{aligned}
(E_a+E_a)G_{|a|b|} = & -\frac{\lambda}{N}\sum_{p=1}^{N} G_{|ap|}G_{|a|b|} + \frac{\lambda}{N}\sum_{p=1}^{N}\frac{G_{|p|b|}-G_{|a|b|}}{E_p-E_a} + \lambda\frac{G_{|bb|}-G_{|ab|}}{E_b-E_a} \\
& -\frac{\lambda}{N^2}\Big(G_{|b|aaa|}+G_{|a|abb|}+3G_{|a|b|}G_{|a|a|}+\frac{1}{N}\sum_{p=1}^{N}G_{|a|b|ap|}\Big) \\
& -\frac{\lambda}{N^4}G_{|b|a|a|a|}\,.
\end{aligned} \tag{B8}
$$

These identities have been found in [17] (by a faster, but less elementary approach).

Identities of such type can be solved by a further expansion of all arising functions $G_{\ldots}$ as formal power series in N^{-2},

$$
G_{\ldots} = \sum_{g=0}^{\infty}\frac{1}{N^{2g}}G^{(g)}_{\ldots}\,. \tag{B9}
$$

With the convention that $(1/N)\sum_1^N$ is of order N^0, the coefficient of N^{-2g} in (B7) reads

$$
\begin{aligned}
(E_a+E_b)G^{(g)}_{|ab|} = & \,\delta_{g,0} - \frac{\lambda}{N}\sum_{p=1}^{N}\sum_{g_1+g_2=g}G^{(g_1)}_{|ab|}G^{(g_2)}_{|ap|} + \frac{\lambda}{N}\sum_{p=1}^{N}\frac{G^{(g)}_{|pb|}-G^{(g)}_{|ab|}}{E_p-E_a} \\
& -\lambda\Big(-\frac{G^{(g-1)}_{|b|b|}-G^{(g-1)}_{|a|b|}}{E_b-E_a} + G^{(g-1)}_{|abab|} + G^{(g-1)}_{|baaa|} + \sum_{g+g_2=g-1}G^{(g_1)}_{|ab|}G^{(g_2)}_{|a|a|} \\
& + \frac{1}{N}\sum_{p=1}^{N}G^{(g-1)}_{|ab|ap|}\Big) - \lambda G^{(g-2)}_{|a|a|ab|}\,.
\end{aligned} \tag{B10}
$$

For the degree or genus $g=0$ we thus obtain the closed equation (3·1) for $G^{(0)}_{|ab|}$. Similarly, the restriction of (B8) to the degree or genus $g=0$ is (6·1). Both equations have been solved in this paper.

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