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BINDING NUMBER AND MINIMUM DEGREE FOR FRACTIONAL (k, m)-DELETED GRAPHS

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Abstract

Let G be a graph of order n, and let $k \geq 1$ be an integer. Let $h : E(G) \to [0, 1]$ be a function. If $\sum_{e \ni x} h(e) = k$ holds for any $x \in V(G)$, then we call $G[F_h]$ a fractional k-factor of G with indicator function h where $F_h = \{e \in E(G) : h(e) > 0\}$. A graph G is called a fractional (k, m)-deleted graph if for every $e \in E(H)$, there exists a fractional k-factor $G[F_h]$ of G with indicator function h such that $h(e) = 0$, where H is any subgraph of G with m edges. The minimum degree of a vertex in G is denoted by $\delta(G)$. For $X \subseteq V(G)$, $N_G(X) = \bigcup_{x \in X} N_G(x)$. The binding number of G is defined by

$$\mathrm{bind}(G) = \min\left\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subset V(G), N_G(X) \neq V(G)\right\}.$$

In this paper, it is proved that if

$$n \geq 4k - 6 + \frac{2m}{k-1}, \quad \mathrm{bind}(G) > \frac{(2k-1)(n-1)}{kn-2m} \quad \text{and} \quad \delta(G) \geq 1 + \frac{(k-1)n + 2m}{2k-1},$$

then G is a fractional (k, m)-deleted graph. Furthermore, it is shown that this result is best possible in some sense.

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1. Introduction

Many physical structures can conveniently be modelled by networks. Examples include a communication network with the nodes and links modelling cities and communication channels, respectively, or a railroad network with nodes and links representing railroad stations and railways between two stations, respectively. Factors and factorizations in networks are very useful in combinatorial design, network design, circuit layout, and so on. It is well known that a network can be represented by a graph.

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Vertices and edges of the graph correspond to nodes and links between the nodes, respectively. Henceforth we use the term *graph* instead of *network*.

We investigate the fractional factor problem in graphs, which can be considered as a relaxation of the well-known cardinality matching problem. The fractional factor problem has wide-ranging applications in areas such as network design, scheduling and combinatorial polyhedra. For instance, in a communication network, if we allow several large data packets to be sent to various destinations through several channels, the efficiency of the network will be improved if we allow the large data packets to be partitioned into small parcels. The feasible assignment of data packets can be seen as a fractional flow problem, and it becomes a fractional matching problem when the destinations and sources of a network are disjoint (that is, the underlying graph is bipartite).

We consider only finite undirected graphs without loops or multiple edges. Let G be a graph. We use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. For $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G, and by $N_G(x)$ the set of vertices adjacent to x in G. For any subset X of $V(G)$, we write $N_G(X)$ for $\bigcup_{x\in X} N_G(x)$. We use $G[X]$ to denote the subgraph of G induced by X, and $G - X = G[V(G) \setminus X]$. A vertex set $X \subseteq V(G)$ is called independent if $G[X]$ has no edges. The minimum degree of a vertex in G is denoted by $\delta(G)$. The binding number $\operatorname{bind}(G)$ of G is defined by

$$\operatorname{bind}(G) = \min\left\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subset V(G), N_G(X) \neq V(G)\right\}.$$

We refer the reader to [1] for standard graph-theoretic terms not defined in this paper.

Let $k \geq 1$ be an integer. Then a spanning subgraph F of G is called a k-factor if $d_F(x) = k$ for each $x \in V(G)$. Let $h : E(G) \to [0, 1]$ be a function. If $\sum_{e\ni x} h(e) = k$ holds for any $x \in V(G)$, then we call $G[F_h]$ a fractional k-factor of G with indicator function h where $F_h = \{e \in E(G) : h(e) > 0\}$. A graph G is called a fractional (k, m)-deleted graph if for every $e \in E(H)$, there exists a fractional k-factor $G[F_h]$ of G with indicator function h such that $h(e) = 0$, where H is any subgraph of G with m edges. A fractional (k, m)-deleted graph is simply called a fractional k-deleted graph if $m = 1$.

Many authors have investigated factors of graphs [2–4, 6, 9, 13]. Liu and Zhang [5] gave a toughness condition for a graph to have a fractional k-factor. Yu *et al.* [7] obtained a degree condition for graphs to have fractional k-factors. Zhou [11] obtained a minimum degree and independence number condition for a graph to have a fractional k-factor. Zhou [10, 12] showed two sufficient conditions for graphs to be fractional (k, m)-deleted graphs. In this paper, we study fractional (k, m)-deleted graphs and obtain a binding number and minimum degree condition for a graph to be a fractional (k, m)-deleted graph. The main result is the following theorem.

Theorem 1.1. *Let $k \geq 1$ and $m \geq 0$ be integers, and let G be a graph of order n with $n \geq 4k - 6 + 2m/(k - 1)$. If G satisfies*

$$\operatorname{bind}(G) > \frac{(2k-1)(n-1)}{kn-2m} \quad \text{and} \quad \delta(G) \geq 1 + \frac{(k-1)n+2m}{2k-1},$$

then G is a fractional (k, m)-deleted graph.

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If $m = 0$ in Theorem 1.1, then we obtain the following corollary.

COROLLARY 1.2. *Let $k \geq 1$ be an integer, and let G be a graph of order n with $n \geq 4k - 6$. If G satisfies*

$$\text{bind}(G) > \frac{(2k-1)(n-1)}{kn} \quad \textit{and} \quad \delta(G) \geq 1 + \frac{(k-1)n}{2k-1},$$

then G has a fractional k-factor.

Obviously,

$$1 + \frac{k-1}{k} = \frac{(2k-1)(n-1)}{k(n-1)} > \frac{(2k-1)(n-1)}{kn}.$$

Thus, from Corollary 1.2 we get the following corollary.

COROLLARY 1.3 [8]. *Let $k \geq 1$ be an integer, and let G be a graph of order n with $n \geq 4k - 4$. If G satisfies*

$$\text{bind}(G) \geq 1 + \frac{k-1}{k} \quad \textit{and} \quad \delta(G) \geq 1 + \frac{(k-1)n}{2k-1},$$

then G has a fractional k-factor.

If $m = 1$ in Theorem 1.1, then we obtain the following corollary.

COROLLARY 1.4. *Let $k \geq 1$ be an integer, and let G be a graph of order n with $n \geq 4k - 6 + 2/(k-1)$. If G satisfies*

$$\text{bind}(G) > \frac{(2k-1)(n-1)}{kn-2} \quad \textit{and} \quad \delta(G) \geq 1 + \frac{(k-1)n+2}{2k-1},$$

then G is a fractional k-deleted graph.

If $k \geq 2$, then we have

$$1 + \frac{k-1}{k} = \frac{(2k-1)(n-1)}{k(n-1)} > \frac{(2k-1)(n-1)}{kn-2}.$$

Thus, by Corollary 1.4 we get the following corollary.

COROLLARY 1.5 [8]. *Let $k \geq 2$ be an integer, and let G be a graph of order n with $n \geq 4k - 3$. If G satisfies*

$$\text{bind}(G) > 1 + \frac{k-1}{k} \quad \textit{and} \quad \delta(G) \geq 1 + \frac{(k-1)n+2}{2k-1},$$

then G is a fractional k-deleted graph.

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2. The proof of Theorem 1.1

Let G be a graph. For all disjoint subsets S and T of $V(G)$, define

$$\delta_G(S,T) = k|S| + \sum_{x\in T}(d_{G-S}(x) - k).$$

The proof of Theorem 1.1 depends on the following lemma.

Lemma 2.1 Zhou [10]. *Let $k \geq 1$ and $m \geq 0$ be two integers, and let G be a graph and H a subgraph of G with m edges. Then G is a fractional (k, m)-deleted graph if and only if*

$$\delta_G(S,T) \geq \sum_{x\in T}(d_H(x) - e_H(x,S))$$

for all disjoint subsets S and T of $V(G)$.

Proof of Theorem 1.1. Suppose that G satisfies the assumption of Theorem 1.1, but is not a fractional (k, m)-deleted graph. Then by Lemma 2.1 there exist disjoint subsets S and T of $V(G)$ such that

$$\delta_G(S,T) \leq \sum_{x\in T}(d_H(x) - e_H(x,S)) - 1, \tag{2.1}$$

where H is some subgraph of G with m edges. Since $|E(H)| = m$, we obtain $\sum_{x\in T}(d_H(x) - e_H(x,S)) \leq 2m$. Thus, from (2.1) we have

$$\delta_G(S,T) \leq 2m - 1. \tag{2.2}$$

We choose subsets S and T such that $|T|$ is minimum. Clearly, $T \neq \emptyset$ by (2.1). □

We next show that $d_{G-S}(x) \leq k-1$ for any $x \in T$. This follows since, if $d_{G-S}(x) \geq k$ for some $x \in T$, then the subsets S and $T \setminus \{x\}$ satisfy (2.2), contradicting the choice of S and T. Now define $h = \min\{d_{G-S}(x) : x \in T\}$. From the above and the definition of h,

$$0 \leq h \leq k-1$$

and

$$\delta(G) \leq h + |S|. \tag{2.3}$$

In the following, the proof splits into two cases by the value of h.

Case 1. $h = 0$.

Let t be the number of vertices x in T such that $d_{G-S}(x) = 0$, and let $Y = V(G) \setminus S$. Since $h = 0$, we have $N_G(Y) \neq V(G)$. Obviously, $Y \neq \emptyset$ since $T \neq \emptyset$. Hence, by the

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definition of $\mathrm{bind}(G)$, we obtain

$$\frac{|N_G(Y)|}{|Y|} \geq \mathrm{bind}(G). \tag{2.4}$$

Note that $|N_G(Y)| \leq n - t$. Combining this with (2.4) and the condition of Theorem 1.1, we get

$$n - t \geq |N_G(Y)| \geq \mathrm{bind}(G)|Y| > \frac{(2k-1)(n-1)}{kn-2m}(n - |S|),$$

which implies that

$$(2k-1)(n-1)|S| > (2k-1)(n-1)n - (n-t)(kn-2m). \tag{2.5}$$

Note that $t \geq 1$. Using $|S| + |T| \leq n$, $n \geq 4k - 6 + 2m/(k-1)$ and (2.5), we obtain

$$\begin{aligned}
(n-1)\delta_G(S,T) &= (n-1)\Big(k|S| + \sum_{x\in T}(d_{G-S}(x) - k)\Big)\\
&\geq (n-1)(k|S| + (1-k)|T| - t)\\
&\geq (n-1)(k|S| - (k-1)(n-|S|) - t)\\
&= (n-1)((2k-1)|S| - (k-1)n - t)\\
&= (2k-1)(n-1)|S| - (k-1)(n-1)n - (n-1)t\\
&> (2k-1)(n-1)n - (n-t)(kn-2m) - (k-1)(n-1)n - (n-1)t\\
&= -kn + 2mn + ((k-1)n - 2m + 1)t\\
&\geq -kn + 2mn + (k-1)n - 2m + 1\\
&= (n-1)(2m-1),
\end{aligned}$$

that is,

$$\delta_G(S,T) > 2m - 1,$$

which contradicts (2.2).

Case 2. $1 \leq h \leq k - 1$.

According to (2.3) and the condition of Theorem 1.1,

$$|S| \geq \delta(G) - h \geq \frac{(k-1)n + 2m}{2k-1} + 1 - h,$$

that is,

$$(2k-1)|S| \geq (k-1)n + 2m - (2k-1)(h-1). \tag{2.6}$$

In view of (2.2), (2.6) and $|S| + |T| \leq n$, we obtain

$$\begin{aligned}
0 &\geq (2k-1)(\delta_G(S,T) - 2m + 1)\\
&= (2k-1)\Big(k|S| + \sum_{x\in T}(d_{G-S}(x) - k) - 2m + 1\Big)\\
&\geq (2k-1)(k|S| - (k-h)|T| - 2m + 1)
\end{aligned}$$

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$$\begin{aligned}
&\geq (2k-1)(k|S| - (k-h)(n-|S|) - 2m + 1) \\
&= (2k-1)((2k-h)|S| - (k-h)n - 2m + 1) \\
&\geq (2k-h)((k-1)n + 2m - (2k-1)(h-1)) - (2k-1)((k-h)n + 2m - 1) \\
&= (kn - (2k-1)(2k-h) - 2m)(h-1) + 2k - 1,
\end{aligned}$$

that is,

$$0 \geq (kn - (2k-1)(2k-h) - 2m)(h-1) + 2k - 1. \tag{2.7}$$

Subcase 2.1. $h = 1$.

Using (2.7),

$$0 \geq 2k - 1 > 0.$$

This is a contradiction.

Subcase 2.2. $2 \leq h \leq k - 1$.

From (2.7), $2 \leq h \leq k-1$ and $n \geq 4k - 6 + 2m/(k-1)$, we get

$$\begin{aligned}
0 &\geq (kn - (2k-1)(2k-h) - 2m)(h-1) + 2k - 1 \\
&\geq (kn - (2k-1)(2k-2) - 2m)(h-1) + 2k - 1 \\
&\geq \left(\frac{2m}{k-1} - 2\right)(h-1) + 2k - 1,
\end{aligned}$$

that is,

$$0 \geq \left(\frac{2m}{k-1} - 2\right)(h-1) + 2k - 1. \tag{2.8}$$

Subcase 2.2.1. $m \geq k - 1$.

In this case, by $2 \leq h \leq k-1$ and (2.8), we obtain

$$0 \geq \left(\frac{2m}{k-1} - 2\right)(h-1) + 2k - 1 \geq 2k - 1 > 0,$$

which is a contradiction.

Subcase 2.2.2. $0 \leq m \leq k - 2$.

In this case, using $2 \leq h \leq k-1$ and (2.8), we have

$$\begin{aligned}
0 &\geq \left(\frac{2m}{k-1} - 2\right)(h-1) + 2k - 1 \\
&\geq -2(h-1) + 2k - 1 \geq -2(k-2) + 2k - 1 = 3 > 0,
\end{aligned}$$

also a contradiction.

From all the cases above, we deduce contradictions. Hence, G is a fractional (k, m)-deleted graph. This completes the proof of Theorem 1.1.

Remark. Let us show that the conditions

$$\mathrm{bind}(G) > \frac{(2k-1)(n-1)}{kn-2m} \quad \text{and} \quad \delta(G) \geq 1 + \frac{(k-1)n+2m}{2k-1}$$

in Theorem 1.1 cannot be replaced by

$$\mathrm{bind}(G) \geq \frac{(2k-1)(n-1)}{kn-2m} \quad \text{and} \quad \delta(G) \geq \frac{(k-1)n+2m}{2k-1}.$$

Let k and m be nonnegative integers such that $k \geq 2$ is odd. Now we construct a graph $G = K_t \vee (K_1 \cup (lK_2))$, where $t = (k-1)(2k-1) + 2m$ and $l = (k(2k-1) + 2m - 1)/2$. We write $n = t + 2l + 1 = (2k-1)^2 + 4m$. Let $Y = V(K_1 \cup (lK_2))$; then $|N_G(Y)| = n - 1$. In view of the definition of bind(G),

$$\mathrm{bind}(G) = \frac{|N_G(Y)|}{|Y|} = \frac{n-1}{2l+1} = \frac{(2k-1)(n-1)}{kn-2m}.$$

Furthermore,

$$\delta(G) = t = \frac{(k-1)n+2m}{2k-1}.$$

Let $S = V(K_t)$, $T = V(K_1 \cup (lK_2))$ and H be any subgraph of $G[T]$ with m edges. Then $|S| = t$, $|T| = 2l + 1$ and $\sum_{x\in T}(d_H(x) - e_H(x, S)) = 2m$. Thus,

$$\begin{aligned}
\delta_G(S,T) &= k|S| + \sum_{x\in T}(d_{G-S}(x) - k) \\
&= k|S| + |T| - 1 - k|T| = k|S| - (k-1)|T| - 1 \\
&= kt - (k-1)(2l+1) - 1 \\
&= k((k-1)(2k-1) + 2m) - (k-1)(k(2k-1) + 2m) - 1 \\
&= 2m - 1 < 2m = \sum_{x\in T}(d_H(x) - e_H(x,S)).
\end{aligned}$$

By Lemma 2.1, G is not a fractional (k, m)-deleted graph. In the above sense, the result of Theorem 1.1 is best possible.

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