PROBLEMS FOR SOLUTION

P. 156. Let G be a group with right invariant metric d_R. Suppose right multiplication is continuous. Then

(i) inversion if continuous at the identity e ;

(ii) if left multiplication is also continuous, then inversion is continuous everywhere (i.e. G is a topological group;

(iii) if G possesses a left invariant metric d_L equivalent to d_R, then left multiplication is continuous and G is a topological group (equivalent means gives the same topology).

J. Marsden, University of California, Berkeley

P. 157. Find a topological space X which is locally compact Hausdorff and second countable and an equivalence relation $\sim$ on X such that the quotient space $X/\sim$ is not locally compact.

J. Marsden, University of California, Berkeley

P. 158. Does there exist an infinitely differentiable function of a real variable, which is nowhere analytic?

R. Giles, Queen's University, Kingston

SOLUTIONS

P. 144. Prove that a normed linear space X is an inner product space if and only if for each set $S \subset X$ and $z \in S$, S_z is convex where

$$S_z = \{x : ||x - z|| = \inf_{y \in s} ||x - y||\}$$

K. L. Singh, Memorial University

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Solution by the Proposer

A normed linear space is an inner product space if and only if each two dimensional subspace has an inner product [1]. Let X be two dimensional; then X is an inner product space if and only if for each set S and $z \in S$, S_z is convex [2].

Combining these two results gives the answer.

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P. 146. (i) Let $n_1 < n_2 < \cdots$ be an infinite sequence of integers such that $\sigma(n_i) - n_i$ is a constant, where $\sigma(n)$ is the sum of the divisors of n. Prove that each n_i is prime.

(ii) For each $k \geq 1$, show that there exist integers $n_1 < n_2 < \cdots < n_k$, none of which is a prime, such that $\sigma(n_i) - n_i$ is constant.

P. Erdös, McGill University

Solution by R. Breusch, Amherst College, Mass.

(i) If n is prime $\sigma(n) - n = 1$. If n is composite, n has at least one proper divisor $\geq \sqrt{n}$; thus $\sigma(n) - n \geq n + \sqrt{n} + 1 - n > \sqrt{n} > 1$. Either $\sigma(n_j) - n_j = 1$ for $j = 1, 2, \ldots$. Then all the n_j are prime. Or $\sigma(n_j) - n_j = m > 1$ for $j = 1, 2, \ldots$. Then all the n_j are composite, and thus $m > \sqrt{n_j}$ for $j = 1, 2, \ldots$. But this is impossible for a strictly increasing sequence $\{n_j\}$ of natural numbers.

(ii) Consider numbers n_j that are products of two distinct primes:

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$n_j = p_j q_j$. Then $\sigma(n_j) - n_j = 1 + p_j + q_j$. Let N be such that $N/(32 \log^2 N) > k$. Let S be the set of primes that are not greater than N. S has $M = \pi(N)$ members. For N sufficiently large, $M > N/(2 \log N)$. There are $\frac{1}{2}M(M-1) > M^2/4$ distinct pairs of distinct members of S. The sum of each pair is less than $2N$. Therefore at least one natural number less than $2N$ (call it Q) must be such that for more than $\frac{1}{4}M^2/(2N) > N/(32 \log^2 N) > k$ distinct pairs (p_j, q_j) of distinct members of S, $p_j + q_j = Q$. Now let $n_j = p_j q_j$ $(j = 1,2,\ldots,k)$. Then, for $j = 1,2,\ldots,k$, $\sigma(n_j) - n_j = 1 + p_j + q_j = 1 + Q$.

P. 148. Let X be a locally separable connected metric space. Prove that X is separable. Is this true if X is not metric?

J. Marsden, University of California, Berkeley

Solution by David J. Lutzer, University of Washington, Seattle

Our approach is based on Lemma 1 of R.W. Heath's paper [1]. A space X is metalindelöf if every open cover of X has an open point-countable refinement [2]. Clearly, any metric space is metalindelöf.

PROPOSITION: A connected, locally separable metalindelöf space is separable.

Proof. Suppose X is locally separable and metalindelöf. Then there is a point-countable open cover of X, say $\mathcal{U}$, by separable subspaces of X. Let $U_0 \in \mathcal{U}$ and let D_0 be a countable dense subset of U_0. Then $\mathrm{st}(U_0, \mathcal{U}) = \{U \in \mathcal{U} \mid U \cap U_0 \neq \emptyset\} = \{U \in \mathcal{U} \mid U \cap D_0 \neq \emptyset\}$ and so $\mathrm{st}(U_0, \mathcal{U})$ is a countable subcollection of $\mathcal{U}$. Let $\mathcal{U}(1) = \{U_0\}$. For each $n \geq 2$, let $\mathcal{U}(n) = \mathrm{st}(U(n-1), \mathcal{U})$ where $U(n-1) = \cup\, \mathcal{U}(n-1)$. Each collection $\mathcal{U}(n)$ is countable. Let $\mathcal{V} = \cup \{\mathcal{U}(n) \mid n \geq 1\}$. If $\mathcal{V} \neq \mathcal{U}$, let $V = \cup\, \mathcal{V}$ and $W = \cup(\mathcal{U} \setminus \mathcal{V})$. Then V and W would be disjoint, non-empty open subsets of X which cover X. Thus, if we assume that X is connected, $\mathcal{V}$ covers X. Therefore, X is a countable union of separable subspaces and so X is separable.

Also solved by J.P. Penot, J.B. Wilker and the proposer.

Case X not metric (by tbe proposer)

The result is false. For example, consider the "long line" (i.e. the first uncountable ordinal space with each point joined to its successor by a copy of the real line).

Examples are also given by J.P. Penot and J.B. Wilker.

Note. A short and elementary proof of the property as stated appears in [3, Theorem 1.7, page 75]. Mention of this property can also be found in a paper by P.S. Alexandroff [4] and a proof in a paper by W. Sierpiński [5].

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