A REMARK ON THE UNIQUENESS OF WEIGHTED MARKOV BRANCHING PROCESSES

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Abstract

We present an elegant uniqueness criterion for the weighted Markov branching process in the potentially explosive case.

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1. Introduction

The basic property that governs the evolution of Markov branching processes (MBPs) is the *branching property*, i.e. that particles behave independently producing descendants according to the same rule. This greatly simplifies their analysis; their study has proved to be a very fruitful area of research in stochastic processes (standard references, among others, are Harris (1963), Athreya and Ney (1972), and Asmussen and Hering (1983)). However, since particles may interact, through collision or some other mechanism, the branching property may be lost. For this reason, more general branching models have been proposed (see, for example, Athreya and Jagers (1997)). A particularly interesting class, which we call here weighted Markov branching processes (WMBPs), was considered by R. R. Chen (1997). Using the methods and techniques developed by M. F. Chen (1992), R. R. Chen addressed questions concerning regularity and eventual extinction (see also A. Y. Chen (2002a), (2002b)), but left open the question of uniqueness in the potentially explosive case (see Definition 1, below). Our aim here is to establish a uniqueness criterion which is easy to check. We begin with some definitions.

Definition 1. A q-matrix $\boldsymbol{Q} = (q_{ij}, i, j \in \mathbb{Z}_+)$ is called a *weighted branching q-matrix* (*WB-q-matrix*) if

$$q_{ij} = \begin{cases} w_i b_{j-i+1}, & i \geq 1, \ j \geq i-1, \\ 0, & \text{otherwise,} \end{cases}$$

where $w_j > 0$ $(j \geq 1)$, $b_0 > 0$, $b_j \geq 0$ $(j \geq 2)$, and $-b_1 = \sum_{j \neq 1}^{\infty} b_j$. It is called *potentially explosive* if $\sum_{k=1}^{\infty} 1/w_k < \infty$ and *subexplosive* if $\sum_{k=1}^{\infty} 1/w_k = \infty$.

In order that the branching property holds for the ordinary MBP it is necessary that its transition function obeys the Kolmogorov forward equation. Guided by this fact, we use the following definition.

Definition 2. A Q-process is called a *weighted Markov branching process* (*WMBP*) if its transition function $\boldsymbol{P}(t) = (p_{ij}(t), i, j \in \mathbb{Z}_+, t > 0)$ satisfies the Kolmogorov forward equation $\boldsymbol{P}'(t) = \boldsymbol{P}(t)\boldsymbol{Q}$.

Chen *et al.* (2005) derived regularity conditions for WMBPs in terms of $d := b_0$ (*death rate*) and $m_b := \sum_{j=1}^{\infty} j b_{j+1}$ (*net birth rate*). They proved that the subexplosive WB-q-matrix is almost regular (meaning that it is regular if m_b is finite), and they obtained a simple criterion for regularity in the potentially explosive case.

Proposition 1. *The potentially explosive WB-q-matrix $\boldsymbol{Q}$ is regular if and only if $d \geq m_b$.*

From Proposition 1 we can see that a potentially explosive $\boldsymbol{Q}$ is explosive whenever $d < m_b \leq \infty$, and general theory then dictates that there are infinitely many $\boldsymbol{Q}$-processes, including infinitely many honest ones (see Anderson (1991) or Yang (1990)). However, for uniqueness of the WMBP, we require the forward equation to hold. The following uniqueness criterion was obtained by Chen *et al.* (2005).

Proposition 2. *If $\boldsymbol{Q}$ is potentially explosive and $d < m_b \leq \infty$, then there is only one WMBP if and only if $\sum_{n=1}^{\infty} R_n = \infty$, where*

$$R_0 = 1,$$

$$R_n = \frac{1}{b_0 w_{n+1}} \left(1 + \sum_{k=1}^{n} w_k \tau_{n-k+2} R_{k-1} \right) \quad for \ n \geq 1, \tag{1}$$

with $\tau_n = \sum_{j=n}^{\infty} b_j$, $n \geq 2$.

However, this criterion is not easy to check in all cases. We shall provide an alternative criterion, which is much easier to verify.

2. The main result

Let U be the generating function defined by $U(s) = \sum_{j=0}^{\infty} b_j s^j$. It is well known that if $d < m_b \leq \infty$ then $U(s) = 0$ has two distinct roots, q and 1, with $0 < q < 1$; we shall always denote the smaller of these by q.

Theorem 1. *If the WB-q-matrix $\boldsymbol{Q}$ is potentially explosive and $d < m_b \leq \infty$, then there is more than one WMBP if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{w_n q^n} < \infty. \tag{2}$$

Moreover, if (2) holds then there are infinitely many WMBPs, exactly one of which is honest (it is not the minimal $\boldsymbol{Q}$-process). If (2) fails to hold then there is only one WMBP, being the (dishonest) minimal $\boldsymbol{Q}$-process.

The following immediate corollary improves on the conclusions of Chen *et al.* (2005).

Corollary 1. *Suppose that Q is potentially explosive and $d < m_b \leq \infty$. If*

$$\liminf_{n \to \infty} \sqrt[n]{w_n} < \frac{1}{q}$$

then there is only one WMBP, being the (dishonest) minimal Q-process; while if

$$\liminf_{n \to \infty} \sqrt[n]{w_n} > \frac{1}{q}$$

then there is only one honest WMBP (which is not the minimal Q-process), together with infinitely many dishonest WMBPs.

In order to prove Theorem 1, we will need two technical lemmas. Define G by

$$G(s) = \frac{q - s}{U(s)}. \tag{3}$$

Lemma 1. *Viewed as a complex function, $G(z)$ is analytic, at least on the unit disk $\mathcal{D} = \{z, \ |z| < 1\}$, and has a Taylor expansion*

$$G(z) = \sum_{n=0}^{\infty} g_n z^n \tag{4}$$

whose coefficients satisfy

$$0 < g_n \leq g_0 = \frac{q}{d}, \qquad n \geq 0. \tag{5}$$

Proof. Since $d < m_b \leq \infty$, we may let $\rho_0 = d/q \ (= b_0/q)$ and $\rho_k = \sum_{m=1}^{\infty} b_{k+m} q^{m-1}$, $k \geq 1$. It is easy to see that $(\rho_k, \ k \geq 0)$ is a nonnegative sequence with $\rho_0 > 0$ and $\sum_{k=1}^{\infty} \rho_k = \rho_0$. A little algebra together with (3) then immediately yields

$$G(z) = \left(\rho_0 - \sum_{k=1}^{\infty} \rho_k z^k \right)^{-1}. \tag{6}$$

Since $\rho_0 - \sum_{k=1}^{\infty} \rho_k z^k$ is analytic on $\mathcal{D}$, and $|\rho_0 - \sum_{k=1}^{\infty} \rho_k z^k| \geq \rho_0(1 - |z|) > 0$ for all $z \in \mathcal{D}$, we deduce that G is analytic on $\mathcal{D}$ and, so, has the Taylor expansion (4).

Since (4) and (6) hold for all $z \in \mathcal{D}$, we obtain $\rho_0 g_0 = 1$ and $\rho_0 g_n = \sum_{k=1}^{n} \rho_k g_{n-k}$, $n \geq 1$, and then (5) follows immediately.

Lemma 2. *Define $(T_n, \ n \geq 0)$ by $T_0 = w_1$ and $T_n = w_{n+1} R_n$, $n \geq 1$, where R_n is given by (1). Then $T(s) := \sum_{n=0}^{\infty} T_n s^n$ has radius of convergence q and*

$$T(s)U(s) = s + dw_1(1 - s), \qquad 0 \leq s < q. \tag{7}$$

Proof. Theorem 5 of Chen *et al.* (2005) established that the power series $T(s)$ has radius of convergence q. Now we rewrite (1) as $T_n = (1 + \sum_{k=1}^{n} T_{k-1} \tau_{n-k+2})/b_0$. A little algebra then establishes (7).

We are now ready to prove Theorem 1.

Proof of Theorem 1. By (3) and (4) we have, for $|s| < q$,

$$\frac{1}{U(s)} = \frac{G(s)}{q-s} = \frac{1}{q} \sum_{m=0}^{\infty} \left(\frac{s}{q}\right)^m \sum_{n=0}^{\infty} g_n s^n = \frac{1}{q} \sum_{n=0}^{\infty} C_n s^n, \tag{8}$$

where

$$C_n = \sum_{k=0}^{n} g_k \left(\frac{1}{q}\right)^{n-k} \equiv \sum_{k=0}^{n} \left(\frac{1}{q}\right)^k g_{n-k}.$$

Now, (8) together with (7) yields, for $|s| < q$,

$$T(s) = \frac{d w_1}{q} \sum_{n=0}^{\infty} C_n (1-s) s^n + \frac{1}{q} \sum_{n=0}^{\infty} C_n s^{n+1}. \tag{9}$$

But, $T(s) = \sum_{n=0}^{\infty} T_n s^n = \sum_{n=0}^{\infty} w_{n+1} R_n s^n$ and, since (9) holds for all $|s| < q$, we find that

$$R_n = \frac{C_n - C_{n-1}}{w_{n+1}} \frac{d w_1}{q} + \frac{C_{n-1}}{w_{n+1}} \frac{1}{q}, \qquad n \geq 1. \tag{10}$$

The sequence $(C_n, n \geq 0)$ is nonnegative and increasing because

$$C_n - C_{n-1} = g_n + \frac{1-q}{q} \sum_{k=1}^{n} \left(\frac{1}{q}\right)^{k-1} g_{n-k} > 0.$$

Therefore, both terms on the right-hand side of (10) are nonnegative. Thus, by Proposition 2, the WMBP is not unique if and only if both $\sum_{n=1}^{\infty} (C_n - C_{n-1})/w_{n+1} < \infty$ and $\sum_{n=1}^{\infty} C_{n-1}/w_{n+1} < \infty$, or, equivalently,

$$\sum_{n=1}^{\infty} \frac{C_n}{w_{n+1}} < \infty. \tag{11}$$

The proof will be complete if we can show that this is equivalent to (2). First assume that (2) holds. Since $(g_n, n \geq 0)$ is bounded above by q/d (see (5)), we have

$$C_n \leq \frac{q}{d} \sum_{k=0}^{n} \left(\frac{1}{q}\right)^k = \frac{q^2}{d(1-q)} \left(\left(\frac{1}{q}\right)^{n+1} - 1\right) \leq \frac{q^2}{d(1-q)} \left(\frac{1}{q}\right)^{n+1},$$

for all $n \geq 0$. Therefore, by (2),

$$\sum_{n=1}^{\infty} \frac{C_n}{w_{n+1}} \leq \frac{q^2}{d(1-q)} \sum_{n=2}^{\infty} \frac{1}{w_n q^n} < \infty,$$

thus establishing (11). Conversely, if (11) holds then, since $C_n \geq g_0 (1/q)^n$ for all $n \geq 0$, we have

$$\sum_{n=1}^{\infty} \frac{g_0}{w_{n+1}} \left(\frac{1}{q}\right)^n < \infty,$$

and so (2) follows because $0 < g_0 < \infty$ and $q > 0$. This completes the proof.

Acknowledgements

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