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THE MINIMAL NUMBER OF PERIODIC ORBITS OF PERIODS GUARANTEED IN SHARKOVSKII'S THEOREM

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Let $f(x)$ be a continuous function from a compact real interval into itself with a periodic orbit of minimal period m, where m is not an integral power of 2. Then, by Sharkovskii's theorem, for every positive integer n with $m \to n$ in the Sharkovskii ordering defined below, a lower bound on the number of periodic orbits of $f(x)$ with minimal period n is 1. Could we improve this lower bound from 1 to some larger number? In this paper, we give a complete answer to this question.

1. Introduction

Let I be a compact real interval and let $f \in C^0(I, I)$. For any x_0 in I and any positive integer k, we let $f^k(x_0)$ denote the kth iterate of x_0 under f and call $\left\{f^k(x_0) \mid k \geq 0\right\}$ the orbit of x_0 (under f). If $f^m(x_0) = x_0$ for some positive integer m, we call x_0 a periodic point of f and call the cardinality of the orbit of x_0 (under f) the minimal period of x_0 and of the orbit (under f). If f has a periodic orbit of a period m, must f also have periodic orbits of periods $n \neq m$? In 1964, Sharkovskii [11] (see [1], [3], [7], [9],

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[12], [13], also) had given a complete answer to this question. Arrange the positive integers according as the following new order (called Sharkovskii ordering):

$$\begin{array}{cccccccccccc} & 3 & \Delta & 5 & \Delta & 7 & \Delta & \ldots \\ \Delta & 2\cdot 3 & \Delta & 2\cdot 5 & \Delta & 2\cdot 7 & \Delta & \ldots \\ & \vdots & & \vdots & & \vdots \\ \Delta & 2^n\cdot 3 & \Delta & 2^n\cdot 5 & \Delta & 2^n\cdot 7 & \Delta & \ldots \\ & \vdots & & \vdots & & \vdots \\ \Delta & \ldots & \Delta & 2^3 & \Delta & 2^2 & \Delta & 2 & \Delta & 1 . \end{array}$$

Sharkovskii's theorem says that any function $f \in C^0(I, I)$ with a periodic orbit of minimal period m, must also have at least one periodic orbit of minimal period n precisely when $m \Delta n$ in the above Sharkovskii ordering. Therefore, for every positive integer n with $m \Delta n$, the number 1 is a lower bound on the number of distinct periodic orbits of f with minimal period n. One question arises naturally: could we improve this lower bound from 1 to some larger number?

In 1976 Bowen and Franks [2] showed, among other things, that if $f \in C^0(I, I)$ has a periodic orbit of minimal period $n = 2^d m$, where $m > 1$ is odd, then there is a number M_n (independent of f) such that, for all integers $k \geq M_n$, f has at least $(2^{k/m})/(2^d k)$ distinct periodic orbits of minimal period $2^d k$.

In 1979, Jonker [8] also obtained a similar result on a class of unimodal maps. If c is an interior point of I, let S_c denote the collection of all $f \in C^0(I, I)$ which has either one maximum or one minimum point at c, and is strictly monotone on each component of $I - \{c\}$ with $f(\partial I) \subset \partial I$. Jonker showed, among other things, that if m, n are any two odd integers with $1 < m < n$, and if $f \in S_c$ has a periodic orbit of minimal period $2^k m$, where $k \geq 0$ is any integer, then f must also have at least $2^{(n-m)/2}$ distinct periodic orbits of minimal period $2^k n$.

In [6], a result along this line is also obtained. However, that result is only a partial one. In this paper we give a complete answer to that question.

In Section 2 we state our main results (Theorems 1, 2, and 3). In Section 3 we describe the method used to prove them. This method is the same as that used in [5] and [6]. The proofs of Theorems 1 and 2 will appear in Sections 4 and 5. Theorem 3 then follows easily from Theorems 1 and 2.

2. Statement of main results

Let $\phi(m)$ be an integer-valued function defined on the set of all positive integers. If $m = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$, where the p_i's are distinct prime numbers, r and k_i's are positive integers, we define

$$\Phi(1, \phi) = \phi(1)$$

and

$$\Phi(m, \phi) = \phi(m) - \sum_{i=1}^{r} \phi\left(m/p_i\right) + \sum_{i_1<i_2} \phi\left(m/\left(p_{i_1} p_{i_2}\right)\right)$$
$$- \sum_{i_1<i_2<i_3} \phi\left(m/\left(p_{i_1} p_{i_2} p_{i_3}\right)\right) + \dots + (-1)^r \phi\left(m/\left(p_1 p_2 \dots p_r\right)\right) ,$$

where the summation $\sum_{i_1<i_2<\dots<i_j}$ is taken over all $i_1, i_2, \dots, i_j$ with $1 \le i_1 < i_2 < \dots < i_j \le r$. If, when considered as a sequence, $\langle \phi(m) \rangle$ is the Lucas sequence, that is if $\phi(1) = 1$, $\phi(2) = 3$, and $\phi(m+2) = \phi(m+1) + \phi(m)$ for all positive integers m, then, for simplicity, we denote $\Phi(m, \phi)$ as $\Phi_1(m)$. Note that if $f \in C^0(I, I)$ and if, for every positive integer m, $\phi(m)$ is the number of distinct solutions of the equation $f^m(x) = x$, then $\Phi(m, \phi)$ is, by the standard inclusion-exclusion argument, the number of periodic points of f with minimal period m. Now we can state the following theorem.

THEOREM 1. *Let* $f : [1, 3] \to [1, 3]$ *be defined by* $f(x) = -2x + 5$

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if $1 \le x \le 2$ *and* $f(x) = x - 1$ *if* $2 \le x \le 3$. *Then the following hold:*

(a) *for every positive integer* m , *if* a_m *is the number of distinct solutions of the equation* $f^m(x) = x$, *then the sequence* $\langle a_m \rangle$ *is the Lucas sequence;*

(b) *for every positive integer* m , f *has exactly* $\Phi_1(m)/m$ *distinct periodic orbits of minimal period* m ;

(c) *the sequence* $\langle \Phi_1(m)/m \rangle$ *is strictly increasing for* $m \ge 6$ *and* $\lim_{n\to\infty} [\Phi_1(m+1)/(m+1)]/[\Phi_1(m)/m] = (1+\sqrt{5})/2$ ..

Fix any integer $n > 1$ and let

$$Q_n = \{(1, n+1)\} \cup \{(m, 2n+2-m) \mid 2 \le m \le n\} \cup \{(m, 2n+1-m) \mid n+1 \le m \le 2n\} .$$

For all integers i, j , and k , with $1 \le i, j \le 2n$ and $k \ge 1$, we define $b_{k,i,j,n}$ recursively as follows:

$$b_{1,i,j,n} = \begin{cases} 1 , \text{ if } (i, j) \in Q_n , \\ 0 , \text{ otherwise,} \end{cases}$$

and

$$b_{k+1,i,j,n} = \begin{cases} b_{k,i,2n+1-j,n} + b_{k,i,n+1,n} , & \text{if } 1 \le j \le n-1 , \\ b_{k,i,n,n} + b_{k,i,n+1,n} , & \text{if } j = n , \\ b_{k,i,1,n} , & \text{if } j = n+1 , \\ b_{k,i,2n+2-j,n} , & \text{if } n+2 \le j \le 2n . \end{cases}$$

We also define $c_{k,n}$ by letting

$$c_{k,n} = \sum_{i=1}^{2n} b_{k,i,i,n} + b_{k,n+1,n,n} + \sum_{i=n+2}^{2n} b_{k,i,n+1,n} .$$

Note that these sequences $\langle b_{k,i,j,n} \rangle$ and $\langle c_{k,n} \rangle$ have the following six properties. Some of these will be used later in the proofs of our main results. (Recall that $n > 1$ is fixed.)

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(i) The sequence $\langle b_{k,1,n,n}\rangle$ is increasing, and for all integers $k \geq 2$, we have $b_{k,1,n,n} \geq b_{k,n+1,n,n}$ and $b_{k,1,i+1,n} \geq b_{k,1,i,n}$ for all $1 \leq i \leq n-1$.

(ii) The sequences $\langle b_{k,1,j,n}\rangle$, $1 \leq j \leq n$, and $\langle b_{k,n+1,n,n}\rangle$ can also be obtained by the following recursive formulas:

$$b_{1,1,j,n} = 0\ , \quad 1 \leq j \leq n\ ,$$

$$b_{2,1,j,n} = 1\ , \quad 1 \leq j \leq n\ ,$$

$$b_{1,n+1,n,n} = b_{2,n+1,n,n} = 1\ ,$$

$$b_{1,n+1,j,n} = b_{2,n+1,j,n} = 0\ , \quad 1 \leq j \leq n-1\ .$$

For $i = 1$ or $n+1$, and $k \geq 1$,

$$b_{k+2,i,n,n} = b_{k,i,1,n} + b_{k+1,i,n,n}\ ,$$

$$b_{k+2,i,j,n} = b_{k,i,1,n} + b_{k,i,j+1,n}\ , \quad 1 \leq j \leq n-1\ .$$

(iii) For every positive integer k, $c_{k+2n-2,n}$ can also be obtained by the following formulas:

$$c_{k+2n-2,n} = b_{k+2n-2,n+1,n,n} + 2\sum_{j=1}^{n} b_{k+2n-2j,1,j,n}$$

$$= b_{k+2n-2,n+1,n,n} + 2nb_{k,1,n,n} + \sum_{i=2}^{n} \left(2^{i}-2\right) b_{k,1,n+1-i,n}\ .$$

The first identity also holds for all integers k with $-2n+3 \leq k \leq 0$ provided we define $b_{k,1,j,n} = 0$ for all $-2n+3 \leq k \leq 0$ and $1 \leq j \leq n$.

(iv) For all integers k with $1 \leq k \leq 2n$, $c_{2k,n} = 2^{k+1} - 1$.

(v) For all integers k with $n+1 \leq k \leq 3n$, $c_{2k+1,n} = 2c_{2k+1,n+1} - 1$.

(vi) Since, for every positive integer $k \geq 2n+1$,

$$b_{k,1,n,n} = b_{k-1,1,n,n} + \sum_{i=2}^{2n} (-1)^{i} b_{k-i,1,n,n}\ ,$$

there exist $2n+1$ nonzero constants α_j's such that

$b_{k,1,n,n} = \sum_{j=1}^{2n+1} \alpha_j x_j^k$ for all positive integers k, where $\{x_j \mid 1 \le j \le 2n+1\}$ is the set of all zeros (including complex zeros) of the polynomial $x^{2n+1} - 2x^{2n-1} - 1$.

For all positive integers k, m, n, with $n > 1$, we let $\phi_n(k) = c_{k,n}$ and let $\Phi_n(m) = \Phi(m, \phi_n)$, where Φ is defined as above. Now we can state the following theorem.

THEOREM 2. *For every integer* $n > 1$, *let*

$$f_n : [1, 2n+1] \to [1, 2n+1]$$

be the continuous function with the following six properties:

(1) $f_n(1) = n + 1$,

(2) $f_n(2) = 2n + 1$,

(3) $f_n(n+1) = n + 2$,

(4) $f_n(n+2) = n$,

(5) $f_n(2n+1) = 1$, *and*

(6) f_n *is linear on each component of the complement of the set* $\{2, n+1, n+2\}$ *in* $[1, 2n+1]$.

Then the following hold:

(a) for every positive integer k, *the equation* $f_n^k(x) = x$ *has exactly* $c_{k,n}$ *distinct solutions;*

(b) for every positive integer m, $f_n(x)$ *has exactly* $\Phi_n(m)/m$ *distinct periodic orbits of minimal period* m;

(c) $\lim_{m\to\infty} \left(\log[\Phi_n(m)/m]\right)/m = \lambda_n$, *where* λ_n *is the (unique) positive (and the largest in absolute value) zero of the polynomial* $x^{2n+1} - 2x^{2n-1} - 1$.

From Theorems 1 and 2 above and Theorem 2 of [*12*, p. 243], we easily obtain the following result.

THEOREM 3. *Assume that* $f \in C^0(I, I)$ *has a periodic orbit of minimal period* $s = 2^k(2n+1)$ *, where* $n \geq 1$ *and* $k \geq 0$ *, and no periodic orbits of minimal period* r *with* $r \Delta s$ *in the Sharkovskii ordering. Then for every positive integer* t *with* $s \Delta t$ *in the Sharkovskii ordering,* f *has at least* $\Phi_n(t/2^k)/(t/2^k)$ *(sharp) distinct periodic orbits of minimal period* t *.*

REMARK 1. We call attention to the fact that there exist continuous functions from I into I with exactly one periodic orbit of minimal period 2^i for every positive integer i (and two fixed points), but no other periods (see [*10*]).

REMARK 2. With the help of Theorem 2 of [*12*, p. 243] on the distribution along the real line of points in a periodic orbit of odd period $n > 1$, when there are no periodic orbits of odd period m with $1 < m < n$, our results give a new proof of Sharkovskii's theorem.

REMARK 3. Table 1 (see p. 96) lists the first 31 values of $\Phi_n(m)/m$ for $1 \leq n \leq 5$. It seems that, for all positive integers n and m , we have

$$\Phi_n(2m+1)/(2m+1) = 2^{m-n} \quad \text{for} \quad n \leq m \leq 3n+1 ,$$

and

$$\Phi_n(2m+1)/(2m+1) > 2^{m-n} \quad \text{for} \quad m > 3n + 1 .$$

REMARK 4. For all positive integers k and m , let $\psi(k) = 2^k$ and $\Psi(m) = \Phi(m, \psi)$, where Φ is defined as in Section 2. It is obvious that $\Psi(m)/m$ is the number of distinct periodic orbits of minimal period m for, say, the mapping $g(x) = 4x(1-x)$ from $[0, 1]$ onto itself. Since, for all positive integers k and n with $1 \leq k \leq 2n$, $c_{2k,n} = 2^{k+1} - 1$ $c_{1,n} = 1$, we obtain that $\Phi_n(2k+2)/(2k+2) = \Psi(k+1)/(k+1)$ for all

TABLE 1

m	$\Phi_1(m)/m$	$\Phi_2(m)/m$	$\Phi_3(m)/m$	$\Phi_4(m)/m$	$\Phi_5(m)/m$	$\Psi(m)/m$
1	1	1	1	1	1	2
2	1	1	1	1	1	1
3	1	0	0	0	0	2
4	1	1	1	1	1	3
5	2	1	0	0	0	6
6	2	2	2	2	2	9
7	4	2	1	0	0	18
8	5	3	3	3	3	30
9	8	4	2	1	0	56
10	11	6	6	6	6	99
11	18	8	4	2	1	186
12	25	11	9	9	9	335
13	40	16	8	4	2	630
14	58	23	18	18	18	1161
15	90	32	16	8	4	2182
16	135	46	32	30	30	4080
17	210	66	32	16	8	7710
18	316	94	61	56	56	14560
19	492	136	64	32	16	27594
20	750	195	115	101	99	52377
21	1164	282	128	64	32	99858
22	1791	408	224	191	186	190557
23	2786	592	258	128	64	364722
24	4305	856	431	351	337	698870
25	6710	1248	520	256	128	1342176
26	10420	1814	850	668	635	2580795
27	16264	2646	1050	512	256	4971008
28	25350	3858	1673	1257	1177	9586395
29	39650	5644	2128	1026	512	18512790
30	61967	8246	3328	2402	2220	35790267
31	97108	12088	4320	2056	1024	69273666

$1 \le k \le 2n$. It seems that $\Phi_n(2k+2)/(2k+2) > \Psi(k+1)/(k+1)$ for all $k > 2n$. But note that

$$\lim_{k\to\infty} \left(\log[\Phi_n(2k+2)/(2k+2)]\right)/(2k+2)$$

$$= \log \lambda_n > \tfrac{1}{2} \log 2 = \tfrac{1}{2} \lim_{k\to\infty} \left(\log[\Psi(k+1)]/(k+1)\right)/(k+1) ,$$

where λ_n is the unique positive zero of the polynomial $x^{2n+1} - 2x^{2n-1} - 1$.

3. Symbolic representation for continuous piecewise linear functions

In this section we describe a method. This method was first introduced in [4], and then generalized in [5] to construct, for every positive integer n , a continuous piecewise linear function from $[0, 1]$ into itself which has a periodic orbit of minimal period 3 , but with the property that almost all (in the sense of Lebesgue) points of $[0, 1]$ are eventually periodic of minimal period n with the periodic orbit the same as the orbit of a fixed known period n point. The same method was also used in [6] to give a new proof of a result of Block *et al* [1] on the topological entropy of interval maps. In this paper we will use this method to prove our main results.

Throughout this section, let g be a continuous piecewise linear function from the interval $[c, d]$ into itself. We call the set $\{(x_i, y_i) \mid i = 1, 2, \ldots, k\}$ a set of nodes for (the graph of) $y = g(x)$ if the following three conditions hold:

(1) $k \ge 2$,

(2) $x_1 = c$, $x_k = d$, $x_1 < x_2 < \ldots < x_k$, and

(3) g is linear on $[x_i, x_{i+1}]$ for all $1 \le i \le k-1$ and $y_i = g(x_i)$ for all $1 \le i \le k$.

For any such set, we will use its y-coordinates $y_1, y_2, \ldots, y_k$ to represent the graph of $y = g(x)$ and call $y_1y_2 \ldots y_k$ (in that order) a (symbolic) representation for (the graph of) $y = g(x)$. For

$1 \le i < j \le k$, we will call $y_i y_{i+1} \cdots y_j$ the representation for $y = g(x)$ on $[x_i, x_j]$ obtained by restricting $y_1 y_2 \cdots y_k$ to $[x_i, x_j]$. For convenience, we will also call every y_i in $y_1 y_2 \cdots y_k$ a node. If $y_i = y_{i+1}$ for some i (that is, g is constant on $[x_i, x_{i+1}]$), we will simply write $y_1 \cdots y_i y_{i+2} \cdots y_k$ instead of $y_1 \cdots y_i y_{i+1} y_{i+2} \cdots y_k$. Therefore, every two consecutive nodes in a (symbolic) representation are distinct.

Now assume that $\{(x_i, y_i) \mid i = 1, 2, \ldots, k\}$ is a set of nodes for $y = g(x)$ and $a_1 a_2 \cdots a_r$ is a representation for $y = g(x)$ with $\{a_1, a_2, \ldots, a_r\} \subset \{y_1, y_2, \ldots, y_k\}$ and $a_i \ne a_{i+1}$ for all $1 \le i \le r-1$. If $\{y_1, y_2, \ldots, y_k\} \subset \{x_1, x_2, \ldots, x_k\}$, then there is an easy way to obtain a representation for $y = g^2(x)$ from the one $a_1 a_2 \cdots a_r$ for $y = g(x)$. The procedure is as follows. First, for any two distinct real numbers u and v , let $[u : v]$ denote the closed interval with endpoints u and v . Then let $b_{i,1} b_{i,2} \cdots b_{i,t_i}$ be the representation for $y = g(x)$ on $[a_i : a_{i+1}]$ which is obtained by restricting $a_1 a_2 \cdots a_r$ to $[a_i : a_{i+1}]$. We use the following notation to indicate this fact: $a_i a_{i+1} \to b_{i,1} b_{i,2} \cdots b_{i,t_i}$ (under g) if $a_i < a_{i+1}$, or $a_i a_{i+1} \to b_{i,t_i} \cdots b_{i,2} b_{i,1}$ (under g) if $a_i > a_{i+1}$. The above representation on $[a_i : a_{i+1}]$ exists since $\{a_1, a_2, \ldots, a_r\} \subset \{x_1, x_2, \ldots, x_k\}$. Finally, if $a_i < a_{i+1}$, let $z_{i,j} = b_{i,j}$ for all $1 \le j \le t_i$. If $a_i > a_{i+1}$, let $z_{i,j} = b_{i,t_i+1-j}$ for all $1 \le j \le t_i$. It is easy to see that $z_{i,t_i} = z_{i+1,1}$ for all $1 \le i \le r-1$. So, if we define

$$Z = z_{1,1} \cdots z_{1,t_1} z_{2,2} \cdots z_{2,t_2} \cdots z_{r,2} \cdots z_{r,t_r} ,$$

then it is obvious that Z is a representation for $y = g^2(x)$. It is

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also obvious that the above procedure can be applied to the representation Z for $y = g^2(x)$ to obtain one for $y = g^3(x)$, and so on.

4. Proof of Theorem 1

In this section we let $f(x)$ denote the map as defined in Theorem 1, that is $f(x) = -2x + 5$ if $1 \le x \le 2$, and $f(x) = x - 1$ if $2 \le x \le 3$. The proof of part *(a)* of Theorem 1 will follow from two easy lemmas.

LEMMA 4. *Under* f, *we have*

$$13 \to 312, \quad 31 \to 213,$$
$$12 \to 31, \quad 21 \to 13.$$

In the following when we say the representation for $y = f^k(x)$, we mean the representation obtained, following the procedure as described in Section 3, by applying Lemma 4 to the representation 312 for $y = f(x)$ successively until we get to the one for $y = f^k(x)$.

For every positive integer k, let $u_{1,k}$ ($u_{2,k}$ respectively) denote the number of 13's and 31's in the representation for $y = f^k(x)$ whose corresponding x-coordinates are $\le$ ($\ge$ respectively) 2. We also let $v_{1,k}$ ($v_{2,k}$ respectively) denote the number of 12's and 21's in the representation for $y = f^k(x)$ whose corresponding x-coordinates are $\le$ ($\ge$ respectively) 2. It is clear that $u_{1,1} = v_{2,1} = 1$ and $u_{2,1} = v_{1,1} = 0$. Now from Lemma 4, we have

LEMMA 5. *For every positive integer* k *and integers* $i = 1, 2$, $u_{i,k+1} = u_{i,k} + v_{i,k}$ *and* $v_{i,k+1} = u_{i,k}$. *Furthermore, if* $w_k = u_{1,k} + v_{1,k} + u_{2,k}$, *then* $w_1 = 1$, $w_2 = 3$, *and* $w_{k+2} = w_{k+1} + w_k$. *That is,* $\langle w_k \rangle$ *is the Lucas sequence.*

Since, for every positive integer k, the number of distinct solutions of the equation $f^k(x) = x$ equals w_k, part *(a)* of Theorem 1 follows from Lemma 5. Part *(b)* follows from the standard inclusion-exclusion argument. As for part *(c)*, we note that, for every positive

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integer k ,

$$a_{k+2} = \sum_{i=1}^{k} a_i + 3 .$$

So, for $k \geq 6$,

$$\begin{aligned}(k+2)\Phi_1(k+3) &> (k+2)\left(a_{k+3}-a_{[(k+3)/2]+1}\right) \\ &> (k+3)\left(a_{k+2}+a_{[(k+3)/2]+1}\right) \\ &> (k+3)\Phi_1(k+2) ,\end{aligned}$$

where $[(k+3)/2]$ is the largest integer less than or equal to $(k+3)/2$. The proof of the other statement of part *(c)* is easy and omitted. This completes the proof of Theorem 1.

5. Proof of Theorem 2

In this section we fix any integer $n > 1$ and let $f_n(x)$ denote the map as defined in Theorem 2. For convenience, we also let S_n denote the set of all these $4n$ symbolic pairs: $i(i+1)$, $(i+1)i$, $1 \leq i \leq n-1$; $n(n+2)$, $(n+2)n$, $(n+1)(2n+1)$, $(2n+1)(n+1)$, $j(j+1)$, $(j+1)j$, $n+2 \leq j \leq 2n$.

The following lemma is easy.

LEMMA 6. *Under* f_n *, we have*

$$n(n+2) \to (n+3)(n+2)n , \quad (n+2)n \to n(n+2)(n+3) ,$$

$$(n+1)(2n+1) \to (n+2)n(n-1)(n-2) \ldots 321 ,$$

$$(2n+1)(n+1) \to 123 \ldots (n-2)(n-1)n(n+2) ,$$

and $uv \to f_n(u)f_n(v)$ *for every* uv *in*

$$S_n - \{n(n+2), (n+2)n, (n+1)(2n+1), (2n+1)(n+1)\} .$$

In the following when we say the representation for $y = f_n^k(x)$, we mean the representation obtained, following the procedure as described in Section 3, by applying Lemma 6 to the representation

$$(n+1)(2n+1)(2n)(2n-1) \ldots (n+2)n(n-1)(n-2) \ldots 321$$

for $y = f_n(x)$ successively until we get to the one for $y = f_n^k(x)$.

For every positive integer k and all integers i, j with $1 \le i, j \le 2n$, let $b_{k,i,j,n}$ denote the number of uv's and vu's in the representation for $y = f_n^k(x)$ whose corresponding x-coordinates are in $[i, i+1]$, where $uv = j(j+1)$ if $1 \le j \le n-1$ or $n+2 \le j \le 2n$, $uv = n(n+2)$ if $j = n$, and $uv = (n+1)(2n+1)$ if $j = n + 1$. It is obvious that $b_{1,1,n+1,n} = 1$, $b_{1,i,2n+2-i,n} = 1$ if $2 \le i \le n$, $b_{1,i,2n+1-i,n} = 1$ if $n+1 \le i \le 2n$, and $b_{1,i,j,n} = 0$ elsewhere. From Lemma 6, we see that the sequences $\langle b_{k,i,j,n} \rangle$ are exactly the same as those defined in Section 2.

Since

$$c_{k,n} = \sum_{i=1}^{2n} b_{k,i,i,n} + b_{k,n+1,n,n} + \sum_{i=n+2}^{2n} b_{k,i,n+1,n} ,$$

it is clear that $c_{k,n}$ is the number of intersection points of the graph of $y = f_n^k(x)$ with the diagonal $y = x$. This proves part *(a)* of Theorem 2. Part *(b)* follows from the standard inclusion-exclusion argument. As for part *(c)*, we note that there exist $2n + 1$ nonzero constants α_j's such that

$$b_{k,1,n,n} = \sum_{j=1}^{2n+1} \alpha_j x_j^k$$

for all positive integers k , where $\{x_j \mid 1 \le j \le 2n+1\}$ is the set of all zeros (including complex zeros) of the polynomial $x^{2n+1} - 2x^{2n-1} - 1$. Since $c_{k+2n-2,n}$ can also be expressed as

$$b_{k+2n-2,n+1,n,n} + 2nb_{k,1,n,n} + \sum_{i=2}^{n} \left(2^i-2\right) b_{k,1,n+1-i,n} ,$$

part *(c)* follows from property (i) of the sequences $\langle b_{k,i,j,n} \rangle$ stated in Section 2. This completes the proof of Theorem 2.

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