UNIVERSAL FIELDS OF FRACTIONS FOR POLYCYCLIC GROUP ALGEBRAS

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Let G be a polycyclic-by-finite group and let $K[G]$ denote its group algebra over the field K. In this paper we discuss localization in $K[G]$ and in particular we prove that every faithful completely prime ideal is localizable. Furthermore, using a sequence of localizations, we show that, for G polyinfinite cyclic, the classical right quotient ring $\mathfrak{Q}(K[G])$ is in fact a universal field of fractions for $K[G]$. Finally we offer an example of a domain $K[G]$ which does not have a universal field of fractions.

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1. Location. In this section we develop some localization techniques in group rings analogous to Roseblade's trick [**3**] for lifting the AR-property. In particular we show that if G is a polycyclic-by-finite group and if P is a faithful completely prime ideal of the group algebra $K[G]$, then P is localizable.

In the following, R will always denote a right Noetherian ring and we will use basic Noetherian ring notation (see for example [**6**]). Thus let I be an ideal of R and let $\mathscr{C}(I)$ be the set of elements of R which are regular modulo I. Then I is *localizable* if $\mathscr{C}(I)$ is a right divisor set, that is if it satisfies the right Ore condition in R. Next the ideal I is said to be *completely prime* if R/I is a (not necessarily commutative) domain or equivalently if $\mathscr{C}(I)=R\backslash I$. Finally, I has the *AR-property* (weak Artin–Rees property) if for all right ideals $E\subseteq R$ we have $E\cap I^n\subseteq EI$ for some integer $n\geq 1$ depending upon E.

If T is a right divisor set of regular elements of R, then we can form the right ring of fractions RT^{-1} which is also Noetherian. Furthermore, we have the following basic properties.

LEMMA 1.1. *Let T be a right divisor set of regular elements of R and let I be an ideal of R with $T\subseteq\mathscr{C}(I)$. Then*

(i) *IT^{-1} is an ideal of RT^{-1} and $IT^{-1}\cap R=I$,*

(ii) *if I is prime, semiprime or completely prime respectively, then so is IT^{-1},*

(iii) *if I has the AR-property, then so does IT^{-1}.*

Now assume in addition that either IT^{-1} is semiprime or that $(RT^{-1})/(IT^{-1})$ is right Artinian. Then we have

(iv) *$\mathscr{C}(IT^{-1})=\mathscr{C}(I)T^{-1}$ and $\mathscr{C}(IT^{-1})\cap R=\mathscr{C}(I)$,*

(v) *if IT^{-1} is localizable, then I is localizable.*

Proof. Parts (i) and (ii) are simple computations. For part (iii), observe that any right ideal of RT^{-1} is of the form ET^{-1} for some right ideal $E\subseteq R$. Since I has the AR-property, we then have $E\cap I^n\subseteq EI$ for some $n\geq 1$, and since IT^{-1} is an ideal it

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follows easily that $(IT^{-1})^n = I^nT^{-1}$. Thus

$$ET^{-1} \cap (IT^{-1})^n = ET^{-1} \cap I^nT^{-1} = (E \cap I^n)T^{-1} \subseteq EIT^{-1} = (ET^{-1})(IT^{-1})$$

and hence IT^{-1} has the AR-property. Finally assume that either IT^{-1} is semiprime or that $(RT^{-1})/(IT^{-1})$ is right Artinian. Then it follows that any right regular element of $(RT^{-1})/(IT^{-1})$ is in fact regular. With this observation, parts (iv) and (v), in turn, are easily proved.

Most of the next result can be read off from [**6**, Proposition 2.1] but we will include a full proof.

LEMMA 1.2. *Let I be an ideal of R with the AR-property. Then any of the following implies that I is localizable.*

(i) *I/I^n is localizable in R/I^n for all $n \geq 1$.*

(ii) *$R \backslash I \subseteq \mathscr{C}(I^n)$ for all $n \geq 1$.*

(iii) *R/I is a division ring.*

Proof. (i) Let $r \in R$ and $t \in \mathscr{C}(I)$. Then for each $n \geq 1$, since I/I^n is localizable and $\mathscr{C}(I/I^n) = \mathscr{C}(I) + I^n$, there exists $r_n \in R$ and $t_n \in \mathscr{C}(I)$ with $e_n = rt_n - tr_n \in I^n$. Let $E = \sum_1^\infty e_nR$ and apply the AR-property to this right ideal. Then there exists $m \geq 1$ with $E \cap I^m \subseteq EI$. In particular, since $e_m \in E \cap I^m$ we have $e_m \in EI$ and hence $e_m = \sum_1^k e_js_j$ for suitable $s_j \in I$. Replacing e_j by $rt_j - tr_j$ in this formula, we deduce that $rt' = tr'$ where $t' = t_m - \sum_1^k t_js_j$ and $r' = r_m - \sum_1^k r_js_j$. Since $t' \equiv t_m \bmod I$ we have $t' \in \mathscr{C}(I)$ and hence I is localizable.

(ii) Fix $x \in R \backslash I$. We show first that if $y \in I^{n-1}$ then $xa \equiv yb \bmod I^n$ for some $a \in R$, $b \in R \backslash I$. To this end, we work in $\bar{R} = R/I^n$. If $\bar{y} = 0$ take $a = 0$, $b = 1$. Now let $\bar{y} \neq 0$ so that $\bar{y}\bar{R} \neq 0$. By assumption, $\bar{x}$ is regular and hence $\bar{x}\bar{R}$ is essential in $\bar{R}$. This yields $\bar{x}\bar{R} \cap \bar{y}\bar{R} \neq 0$; so there exist $a, b \in R$ with $\bar{x}\bar{a} = \bar{y}\bar{b} \neq 0$. Since $\bar{y} \in \bar{I}^{n-1}$ and $\bar{y}\bar{b} \neq 0$, we have $b \in R \backslash I$.

Now we show that I/I^n is localizable by induction on $n \geq 1$. The case $n = 1$ follows immediately from the above. Again fix $x \in R \backslash I = \mathscr{C}(I)$ and let $r \in R$ and observe by induction that there exist $a \in R$, $b \in R \backslash I$ with $y = rb - xa \in I^{n-1}$. Now the above implies that there exist $c \in R$, $d \in R \backslash I$ with $xc \equiv yd \bmod I^n$. Thus $x(c + ad) \equiv r(bd) \bmod I^n$ and since $b, d \in R \backslash I = \mathscr{C}(I)$ we have $bd \in R \backslash I$. Therefore I/I^n is localizable and hence, by (i), so is I.

(iii) Here we show that $R \backslash I \subseteq \mathscr{C}(I^n)$ by induction on $n \geq 1$. The case $n = 1$ is clear since R/I is a division ring. Now suppose $xr \in I^n$ with $x \in R \backslash I$. Then, by induction, $r \in I^{n-1}$ and, by assumption, there exists $y \in R$ with $yx \equiv 1 \bmod I$. Then clearly $r \equiv yxr \equiv 0 \bmod I^n$. Similarly $rx \in I^n$ yields $r \in I^n$; so part (ii) yields the result.

In the following, $R*G$ will denote a crossed product of the multiplicative group G over R. Observe that, if G is polycyclic-by-finite, then $R*G$ is also right Noetherian. Furthermore, if I is a G-invariant ideal of R, then $I*G$ is an ideal of $R*G$. The next lemma is well known.

LEMMA 1.3. *Let $R*G$ be a crossed product and let T be a G-invariant right divisor set of regular elements of R. Then T is a right divisor set of regular elements of $R*G$ and $(R*G)T^{-1}=(RT^{-1})*G$, where the latter is a suitable crossed product of G over the ring RT^{-1}.*

Recall that an ideal I of R is said to be *polycentral* if I is generated by elements $x_0=0, x_1, \ldots, x_n$ such that for all $i \geq 1$, x_i is central modulo the ideal generated by $x_0, x_1, \ldots, x_{i-1}$. Part (i) of the following lemma is due to P. F. Smith [**6**, Theorem 2.2, Corollary 1] while part (ii) is the result of Roseblade [**3**, §5] previously alluded to. One of our goals in this and the next section is essentially to show that ideals of the form $I*G$ as given below are also localizable.

LEMMA 1.4. *Let I be a polycentral ideal of R.*

(i) *If I is prime, then I is localizable.*

(ii) *If $R*G$ is a crossed product with G polycyclic-by-finite and if I is G-invariant, then $I*G$ has the AR-property.*

We now begin our work.

LEMMA 1.5. *Let $R*G$ be a crossed product with G finite and with R a domain. Let P be a completely prime ideal of $R*G$ having the AR-property. If $Q=P\cap R$ is localizable in R, then P is localizable in $R*G$.*

Proof. Since P is completely prime, Q is a completely prime ideal of R and thus $T=\mathscr{C}(Q)=R\setminus Q$. By assumption, Q is localizable and R is a domain; so T is certainly a right divisor set of regular elements of R. Furthermore, P is G-invariant, so is Q and hence also T. In view of Lemma 1.4 we can now form the ring of fractions $(R*G)T^{-1}=(RT^{-1})*G$. Observe that QT^{-1} is now the unique maximal ideal of RT^{-1} and that $(RT^{-1})/(QT^{-1})$ is a division ring.

Since P is completely prime we have $T\subseteq(R*G)\setminus P=\mathscr{C}(P)$ and hence Lemma 1.1 applies. In particular we know that PT^{-1} is completely prime and satisfies the AR-property. Furthermore it is easy to verify that $PT^{-1}\cap RT^{-1}=(P\cap R)T^{-1}=QT^{-1}$. Thus since G is finite, we see that the domain $(R*G)T^{-1}/(PT^{-1})$ is a finite module over the division ring $(RT^{-1})/(QT^{-1})$. Hence we conclude that $(R*G)T^{-1}/(PT^{-1})$ is also a division ring; so PT^{-1} is localizable by Lemma 1.2 (iii). Lemma 1.1 (v) now yields the result.

LEMMA 1.6. *Let $R*G$ be a crossed product with G polyinfinite cyclic and with R a domain. Let I be a G-invariant ideal of R which is polycentral and completely prime. Then $P=I*G$ is a completely prime localizable ideal of $R*G$.*

Proof. Since I is completely prime and G is polyinfinite cyclic, it is clear that $P=I*G$ is also completely prime.

Set $T=\mathscr{C}(I)=R\backslash I$ so that T is a G-invariant right divisor set in R by Lemma 1.4 (i). Thus, by Lemma 1.5, we can form the ring of fractions $(R*G)T^{-1}=(RT^{-1})*G$. Clearly, every element of $(RT^{-1})\backslash(IT^{-1})$ is a unit and IT^{-1} is a polycentral ideal of the domain RT^{-1}. Furthermore, since $T\subseteq(R*G)\backslash P=\mathscr{C}(P)$, Lemma 1.1 applies. In particular $PT^{-1}=(IT^{-1})*G$ is an ideal and, by Lemma 1.1 (v), it suffices to prove that PT^{-1} is localizable.

In view of the above, we may now assume in addition that every element of $R\backslash I$ is a unit in R. Now, by Lemma 1.4 (ii), $P=I*G$ has the AR-property. Thus, by Lemma 1.2 (ii), it suffices to show that every element of $(R*G)\backslash P$ is regular modulo $P^n=I^n*G$. To this end, let $\alpha\in(R*G)\backslash P$. We show by induction on $n\geq 0$ that $\alpha\beta\in P^n$ (or $\beta\alpha\in P^n$) implies that $\beta\in P^n$. The case $n=0$ is trivial; so let $n\geq 1$. If $\alpha\beta\in P^n$, then $\alpha\beta\in P^{n-1}$; so $\beta\in P^{n-1}$ by induction.

Write $\alpha=\alpha_0+\alpha_1$, where $\alpha_1\in P=I*G$ and all coefficients of α_0 are in $R\backslash I$, and write $\beta=\beta_0+\beta_1$, where $\beta_1\in P^n=I^n*G$ and all coefficients of β_0 are in $I^{n-1}\backslash I^n$. Then $\alpha_0\neq 0$, $\alpha_1\beta\in P^n$ and $\alpha_0\beta_1\in P^n$. Thus $\alpha\beta\in P^n$ yields $\alpha_0\beta_0\in P^n=I^n*G$. We now use the fact that G is right ordered and hence a unique product group. Thus if $\beta_0\neq 0$ we would clearly obtain $a_0b_0\in I^n$, where a_0 is some coefficient of α_0 (with group elements written on the left) and b_0 is some coefficient of β_0 (with group elements written on the right). Since $a_0\in R\backslash I$, a_0 is a unit in R. Thus $a_0b_0\in I^n$ yields $b_0\in I^n$, contradicting the definition of β_0. We conclude therefore that $\beta_0=0$; so $\beta\in P^n$. Lemma 1.2 (ii) now yields the result.

In the next section we will extend the localization aspect of the above result to all polycyclic-by-finite groups.

Let $K[G]$ denote the group algebra of G over the field K. If I is an ideal of this ring, then $I^\dagger=\{x\in G\mid x-1\in I\}$ is easily seen to be a normal subgroup of G. When $I^\dagger=\langle 1\rangle$, then I is said to be a *faithful* ideal. If $I^\dagger$ is finite, then I is *almost faithful*. Let $\Delta(G)$ denote the f.c. center of G.

The following lemma is an immediate consequence of the methods of Roseblade in [**4**]; so we will just briefly indicate a proof.

LEMMA 1.7. *Let $K[G]$ be the group algebra of a polycyclic-by-finite group G and let P be an almost faithful completely prime ideal of this ring. Then P is controlled by $\Delta(G)$, that is $P=(P\cap K[\Delta(G)])K[G]$.*

Proof. If $A=\mathfrak{Z}(G)$ is the Zalesskii subgroup of G then, since P is almost faithful and completely prime, we know that $P\cap K[A]$ is an almost faithful completely prime ideal of $K[A]$. With this observation, the orbitally sound assumption of [**4**, Theorem C1] is no longer needed and we deduce the controlling formula $P=(P\cap K[\Delta(G)])K[G]$.

Let $p>0$ be a prime. If G is a finite group, we say that G is *p-nilpotent* if G is the extension of a normal p'-group by a p-group. More generally, when G is polycyclic-by-finite, we say that G is *p-nilpotent* if all the finite homomorphic images of G are p-nilpotent. Since polycyclic-by-finite groups are residually finite, this is a reasonably restrictive condition. It is easy to see that this p-nilpotent property is inherited by quotient groups and normal subgroups.

Again let G be polycyclic-by-finite. Then $K[G]$ is said to be a *polycentral ring* if all its ideals are polycentral. According to [**2**, Corollary 11.3.12] this occurs if either char $K=0$ and G is finite-by-nilpotent or char $K=p>0$ and G is finite p'-by-(nilpotent-by-finite p). We now come to the main result of this section.

THEOREM 1.8. *Let $K[G]$ be the group algebra of the polycyclic-by-finite group G and let P be a completely prime ideal of this ring. Then any of the following implies that P is localizable.*

(i) *P is faithful.*

(ii) *P is almost faithful and* char $K=0$.

(iii) *P is almost faithful,* char $K=p>0$ *and $\Delta(G)$ is p-nilpotent.*

Proof. We first consider the structure of $\Delta=\Delta(G)$ and its group algebra $K[\Delta]$. Since G is polycyclic-by-finite we always have Δ center-by-finite and also finite-by-abelian. If char $K=0$, then the latter is of course sufficient to guarantee that $K[\Delta]$ is a polycentral ring. Now suppose that char $K=p>0$. If P is faithful, then U, the finite torsion subgroup of Δ, is isomorphic to a multiplicative subgroup of the finite dimensional characteristic p domain $K[U]/(P\cap K[U])$. Thus U is a cyclic p'-group and Δ is finite p'-by-abelian. Finally assume that Δ is p-nilpotent and let V be a torsion-free central subgroup of Δ of finite index. Then the finite quotient group Δ/V is p'-by-p and from this it follows easily that Δ is finite p'-by-(abelian-by-finite p). Thus we conclude that in all cases (i), (ii) and (iii), $K[\Delta]$ is a polycentral ring.

Next, by Lemma 1.7, we have $P=(P\cap K[\Delta])K[G]$ and by the above, $P\cap K[\Delta]$ is a polycentral ideal of $K[\Delta]$. Thus since $K[G]=K[\Delta]*(G/\Delta)$, a suitable crossed product of G/Δ over $K[\Delta]$, we conclude from Lemma 1.4 (ii) that P has the AR-property.

Finally let Z be a characteristic torsion-free central subgroup of Δ of finite index and choose $H\lhd G$ of finite index with H/Z polyinfinite cyclic. If $Q=P\cap K[H]$, then Q is a completely prime almost faithful ideal of $K[H]$. Thus, by Lemma 1.7, since clearly $Z=\Delta(H)$, we have $Q=(Q\cap K[Z])K[H]$. Note that $R=K[Z]$ is a commutative domain and that $I=Q\cap K[Z]$ is an H-invariant polycentral ideal of R. Thus since $K[H]=R*(H/Z)$ and $Q=I*(H/Z)$, we conclude from Lemma 1.6 that Q is localizable. Moreover, $K[G]=K[H]*(G/H)$ and P is a completely prime ideal with the AR-property. Thus since $P\cap K[H]=Q$ is localizable, Lemma 1.5 implies that P is localizable and the theorem is proved.

2. Specialization. In this section we prove that the group algebra $K[G]$ of a polyinfinite cyclic group G has a universal field of fractions. The local subrings of $\mathfrak{Q}(K[G])$ which are needed for this are obtained via a sequence of localizations and the transitivity lemma given below.

Let R and S be rings, no longer assumed to be Noetherian. A *specialization* $\alpha: R\to S$ is a homomorphism α from a subring R_0 of R onto S such that $\ker\alpha\subseteq JR_0$, the Jacobson radical of R_0. We call R_0 the domain of α and write $R_0=\mathfrak{D}(\alpha)$.

LEMMA 2.1. *Let R and S be rings and let T be a subring of R.*

(i) *Suppose S is Artinian and $\alpha: R \to S$ is a specialization with domain R_0. Then every element of $\mathscr{C}(\ker \alpha)$ is invertible in R_0 and, in particular, $\ker \alpha$ is localizable.*

(ii) *Let $\sigma: T \to S$ be an epimorphism which extends to a specialization $\alpha: R \to S$. Then there is a unique minimum specialization $\sigma': R \to S$ extending σ.*

(iii) *Suppose S is a division ring and $\sigma: T \to S$ is a homomorphism with S generated, as a division ring, by $\sigma(T)$. If σ extends to a specialization $\alpha: R \to S$, then there is a unique minimum specialization $\sigma': R \to S$ extending σ.*

Proof. (i) If $r \in \mathscr{C}(\ker \alpha)$, then $\alpha(r)$ is regular in the Artinian ring S and hence invertible. Since $\ker \alpha \subseteq JR_0$, every element in $1 + \ker \alpha$ is invertible in R_0 and hence clearly so is r.

For (ii) and (iii) let A denote the set of specializations extending σ and define

$$R_0 = \left\{ r \in \bigcap_{\alpha \in A} \mathscr{D}(\alpha) \mid \alpha(r) = \beta(r) \quad \text{for all} \quad \alpha, \beta \in A \right\}.$$

Then certainly R_0 is a subring containing T and the common value of all these specializations defines a homomorphism $\sigma': R_0 \to S$ extending σ. It suffices to show that σ' is a specialization.

(ii) If $r \in \ker \sigma'$, then $r \in \ker \alpha$ for all α so $(1+r)^{-1} \in \mathscr{D}(\alpha)$. Furthermore, since $\alpha((1+r)^{-1}) = 1$ is the same for all α, we have $(1+r)^{-1} \in R_0$. Thus $\ker \sigma' \subseteq JR_0$ and since $\sigma(T) = S$ we see that σ' is a specialization onto S.

(iii) If $r \in R_0 \backslash \ker \sigma'$, then $r \in \mathscr{D}(\alpha) \backslash \ker \alpha$ for all α so $r^{-1} \in \mathscr{D}(\alpha)$ by (i). Furthermore, since $\alpha(r^{-1}) = \alpha(r)^{-1} = \sigma'(r)^{-1}$ is the same for all α, we have $r^{-1} \in R_0$. Thus $\ker \sigma' \subseteq JR_0$ and $\sigma'(R_0) \simeq R_0/\ker \sigma'$ is a division ring. Finally, since $S \supseteq \sigma'(R_0) \supseteq \sigma(T)$, we conclude that $\sigma'(R_0) = S$.

The following transitivity lemma is crucial.

LEMMA 2.2. *Let $\alpha: R \to S$ and $\beta: S \to T$ be specializations. Then the composite map $\beta\alpha$ defined on $\alpha^{-1}(\mathscr{D}(\beta))$ determines a specialization $\beta\alpha: R \to T$.*

Proof. If $R_1 = \alpha^{-1}(\mathscr{D}(\beta))$, then since both α and β are onto we see that $\beta\alpha$ maps R_1 onto T. Now let $I_0 = \ker \alpha$ and $I_1 = \ker \beta\alpha$ so that $R_0 \supseteq R_1 \supseteq I_1 \supseteq I_0$. Since α is a specialization, $I_0 \subseteq JR_0$; so every element of $1 + I_0$ is invertible in R_0 and in fact $(1+I_0)^{-1} = 1 + I_0$. Thus since I_0 is also an ideal of R_1 and $(1+I_0)^{-1} = 1 + I_0 \subseteq R_1$, we see that $I_0 \subseteq JR_1$. Furthermore, $R_1/I_0 \simeq \mathscr{D}(\beta) = S_0$ and, since β is a specialization,

$$I_1/I_0 \simeq \ker \beta \subseteq JS_0 \simeq J(R_1/I_0).$$

Thus since $I_0 \subseteq JR_1$ we have $I_1 \subseteq JR_1$ and $\beta\alpha$ is indeed a specialization onto T.

If R has a classical right ring of quotients, then we denote this extension ring by $\mathfrak{Q}(R)$. In particular, by Goldie's theorem, $\mathfrak{Q}(R)$ exists and is Artinian if R is a semiprime right Noetherian ring. We require the following well known observation.

LEMMA 2.3. *Let $R * G$ be a crossed product with R a right Noetherian domain and with G polycyclic-by-finite. Then $\mathfrak{Q}(R * G)$ exists and is right Artinian. Furthermore, if H is a*

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normal subgroup of G of finite index and if $\{g_1, g_2, \ldots, g_n\}$ is a transversal for H in G, then

$$\mathfrak{Q}(R*G)=\oplus\sum_1^n \bar{g}_i\mathfrak{Q}(R*H).$$

Proof. Suppose that H is a normal subgroup of G of finite index such that $\mathfrak{Q}(R*H)$ exists and is right Artinian. Then $T=\mathscr{C}(0)$, the set of regular elements of $R*H$, is a right divisor set in $R*H$ which is clearly G-invariant. Thus, by Lemma 1.3, since $R*G=(R*H)*(G/H)$, we can form the ring of fractions

$$(R*G)T^{-1}=\oplus\sum_1^n \bar{g}_i(R*H)T^{-1}=\oplus\sum_1^n \bar{g}_i\mathfrak{Q}(R*H).$$

Note that $(R*G)T^{-1}$ is a finitely generated right module over the right Artinian ring $\mathfrak{Q}(R*H)$ so $(R*G)T^{-1}$ is also right Artinian. In particular, by Lemma 1.1 (iv), the regular elements of $R*G$ remain regular in the Artinian ring $(R*G)T^{-1}$ and hence are invertible in $(R*G)T^{-1}$. This shows that $(R*G)T^{-1}$ is indeed the classical right quotient ring of $R*G$.

Finally if G is an arbitrary polycyclic-by-finite group, choose H to be a normal polyinfinite cyclic subgroup of finite index. Then $R*H$ is a Noetherian domain; so $\mathfrak{Q}(R*H)$ exists and is a division ring by Goldie's theorem. Thus the above argument implies that $\mathfrak{Q}(R*G)$ exists and is right Artinian.

We now come to the promised extension of Lemma 1.6.

PROPOSITION 2.4. *Let $R*G$ be a crossed product with G polycyclic-by-finite and with R a right Noetherian domain. Let I be a G-invariant ideal of R which is polycentral and completely prime. Then $I*G$ is a localizable ideal of $R*G$. Furthermore, the epimorphism $R*G\to R*G/I*G$ extends to a specialization*

$$\mathfrak{Q}(R*G)\to\mathfrak{Q}(R*G/I*G).$$

Proof. Let H be a normal polyinfinite cyclic subgroup of G of finite index. Since $R*G/I*G=(R/I)*G$, the previous lemma implies that both $\mathfrak{Q}(R*G)$ and $\mathfrak{Q}(R*G/I*G)$ exist and that $\mathfrak{Q}(R*G/I*G)=\mathfrak{Q}(R*H/I*H)\bar{G}$. By Lemma 1.6, $I*H$ is a completely prime localizable ideal of $R*H$ and we set $T=\mathscr{C}(I*H)=(R*H)\backslash(I*H)$. Then T is a G-invariant right divisor set of regular elements of the domain $R*H$; so Lemma 1.3 implies that the ring of fractions $L_0=(R*G)T^{-1}$ exists and is a subring of $\mathfrak{Q}(R*G)$. Now the natural homomorphism $\alpha':R*G\to R*G/I*G$ maps the set T onto the nonzero elements of the domain $R*H/I*H$. Thus since $\mathfrak{Q}(R*G/I*G)=\mathfrak{Q}(R*H/I*H)\bar{G}$, the map α' clearly extends to an epimorphism $\alpha:(R*G)T^{-1}\to\mathfrak{Q}(R*G/I*G)$ with $\ker\alpha=(I*G)T^{-1}$. Moreover, observe that $L_0=L_1*(G/H)$, where $L_1=(R*H)T^{-1}$, and hence, by [**2**, Theorem 7.2.5], we have $JL_1\subseteq JL_0$. But $JL_1=(I*H)T^{-1}$; so we conclude that

$$\ker\alpha=(I*G)T^{-1}=(JL_1)*(G/H)\subseteq JL_0$$

and thus $\alpha:\mathfrak{Q}(R*G)\to\mathfrak{Q}(R*G/I*G)$ is a specialization.

Finally it is clear that $T \subseteq \mathscr{C}(I*G)$ and we note that $(R*G)T^{-1}/(I*G)T^{-1} \simeq \mathfrak{Q}(R*G/I*G)$ is right Artinian. Thus, by Lemma 2.1 (i), $(I*G)T^{-1}$ is localizable in $(R*G)T^{-1}$ and then, by Lemma 1.1 (v), we conclude that $I*G$ is localizable in $R*G$.

LEMMA 2.5. *Let G be a polyinfinite cyclic group. Then G has a characteristic series $\langle 1\rangle = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$ with G_i/G_{i-1} torsion free abelian.*

Proof. If $G \neq \langle 1\rangle$, then G/G' is infinite. Hence if $G \supseteq H \supseteq G'$, where H/G' is the torsion subgroup of G/G', then H is a characteristic subgroup of G with G/H infinite and torsion free abelian. Since H is also polyinfinite cyclic, induction on the Hirsch number yields the result.

We now come to the key result of this section. It is proved via a sequence of localizations and an application of the transitivity lemma. In the course of the proof we show that the classical ring of quotients indicated below does indeed exist.

LEMMA 2.6. *Let G be a polycyclic-by-finite group and let $H \lhd G$ with H polyinfinite cyclic. Furthermore let Q be a G-invariant completely prime ideal of the group algebra $K[H]$ and set $I = Q \cdot K[G]$. Suppose that $H/Q^\dagger$ is finite-by-nilpotent if* char $K = 0$ *or finite p'-by-nilpotent if* char $K = p > 0$. *Then the natural homomorphism $K[G] \to K[G]/I$ extends to a specialization $\alpha : \mathfrak{Q}(K[G]) \to \mathfrak{Q}(K[G]/I)$.*

Proof. Since H is polyinfinite cyclic, it follows from the preceding lemma that H has a characteristic series $\langle 1\rangle = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = H$ with each H_i/H_{i-1} torsion free abelian. Thus $H_i \lhd G$ and we set $Q_i = Q \cap K[H_i]$ and

$$I_i = Q_i \cdot K[G] = (Q \cap K[H_i]) \cdot K[G].$$

Hence we have $0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = I$. Our goal is to show that for each i with $1 \leq i \leq n$ the natural epimorphism $K[G]/I_{i-1} \to K[G]/I_i$ extends to a specialization

$$\alpha_i : \mathfrak{Q}(K[G]/I_{i-1}) \to \mathfrak{Q}(K[G]/I_i).$$

To this end, we fix subscript i with $1 \leq i \leq n$ and we start by studying the ring $R = K[H_i]/Q_{i-1}K[H_i]$. First, R is clearly Noetherian, being a homomorphic image of $K[H_i]$. Second, observe that

$$\begin{aligned} R &= K[H_i]/Q_{i-1}K[H_i] \\ &= (K[H_{i-1}]/Q_{i-1}) * (H_i/H_{i-1}) = R' * A \end{aligned}$$

is a suitable crossed product of the torsion free abelian group $A = H_i/H_{i-1}$ over the ring $R' = K[H_{i-1}]/Q_{i-1}$. Furthermore, R' is a domain since $R' \subsetneq K[H]/Q$ and Q is a completely prime ideal. Thus we deduce that R is also a domain. Finally, if $N = Q^\dagger$ then the augmentation ideal $\omega K[N]$ satisfies $\omega K[N] \subseteq Q$; so $\omega K[N \cap H_{i-1}] \subseteq Q_{i-1}K[H_i]$. This shows that R is a homomorphic image of the group ring $K[H_i/(N \cap H_{i-1})]$. Observe that

$$H_i/(N \cap H_{i-1}) \subsetneq H_i/(N \cap H_i) \times H_i/H_{i-1} \subsetneq (H/N) \times A$$

and, by assumption, $H/N = H/Q^\dagger$ is finite-by-nilpotent if char $K = 0$ or finite p'-by-nilpotent if char $K = p > 0$. Thus the same is true of $(H/N) \times A$ and then of $H_i/(N \cap H_{i-1})$. We conclude from this and the remarks preceding Theorem 1.8 that $K[H_i/(N \cap H_i)]$ is a polycentral ring and hence so is its homomorphic image R. Thus R is a Noetherian domain and a polycentral ring.

Now observe that $I_{i-1} = Q_{i-1}K[H_i] \cdot K[G]$; so

$$K[G]/I_{i-1} = (K[H_i]/Q_{i-1}K[H_i]) * (G/H_i) = R * (G/H_i)$$

is a suitable crossed product of G/H_i over the ring R. Furthermore, since $I_i = Q_iK[G]$, we have $I_i/I_{i-1} \subseteq K[G]/I_{i-1}$ with

$$I_i/I_{i-1} = (Q_i/Q_{i-1}K[H_i]) * (G/H_i) = \bar{Q}_i * (G/H_i).$$

Note that $\bar{Q}_i = Q_i/Q_{i-1}K[H_i]$ is a (G/H_i)-invariant ideal of R which is polycentral since R is a polycentral ring. In addition, $R/\bar{Q}_i \simeq K[H_i]/Q_i$; so $\bar{Q}_i$ is completely prime. We can now conclude from Proposition 2.4, applied to the crossed product $R * (G/H_i) = K[G]/I_{i-1}$, the ideal $\bar{Q}_i * (G/H_i) = I_i/I_{i-1}$ and the quotient ring

$$(R * (G/H_i))/(\bar{Q}_i * (G/H_i)) \simeq K[G]/I_i,$$

that $\mathfrak{Q}(K[G]/I_{i-1})$ and $\mathfrak{Q}(K[G]/I_i)$ exist and that the natural homomorphism $K[G]/I_{i-1} \to K[G]/I_i$ extends to a specialization

$$\alpha_i : \mathfrak{Q}(K[G]/I_{i-1}) \to \mathfrak{Q}(K[G]/I_i).$$

Finally, since $I_0 = 0$ and $I_n = I$, we conclude from transitivity, Lemma 2.2, that the homomorphism $K[G] \to K[G]/I$ extends to a specialization $\alpha : \mathfrak{Q}(K[G]) \to \mathfrak{Q}(K[G]/I)$. This completes the proof.

Let R be a domain contained in and generating a division ring D. Then D is said to be a *universal field of fractions* for R if every homomorphism $\alpha' : R \to D'$, where D' is a division ring generated by the image of R, can be extended to a specialization $\alpha : D \to D'$. In view of Lemma 2.1 (i), (iii), this agrees with the usual definition as given in [**1**, §7.2]. Furthermore, it is clear that if the domain R has a universal field of fractions D, then D is unique up to R-isomorphism. It is now a simple matter to prove our main result.

THEOREM 2.7. *Let G be a polyinfinite cyclic group. Then $\mathfrak{Q}(K[G])$ is the universal field of fractions of $K[G]$.*

Proof. We know that $K[G]$ is an Ore domain and hence that $\mathfrak{Q}(K[G])$ is a division ring containing $K[G]$ and in fact generated by it.

Let $\alpha' : K[G] \to D'$ be a homomorphism of $K[G]$ into a division ring D' such that D' is generated by $\alpha'(K[G])$. If $P = \ker \alpha$, then P is certainly a completely prime ideal of $K[G]$ and $\alpha'(K[G]) \simeq K[G]/P$ is an Ore domain. In particular, we must have $D' = \mathfrak{Q}(K[G]/P)$. Thus it suffices to show that for every completely prime ideal P of $K[G]$, the homomorphism $K[G] \to K[G]/P$ extends to a specialization $\mathfrak{Q}(K[G]) \to \mathfrak{Q}(K[G]/P)$.

To this end, let P be as above and let $N = P^\dagger$. Then P is the pullback in $K[G]$ of a faithful completely prime ideal of $K[G/N]$. Thus it follows easily from Lemma 1.7 that if $H/N = \Delta(G/N)$, then $P = (P \cap K[H])K[G]$. Certainly $Q = P \cap K[H]$ is a completely prime ideal of $K[H]$.

We now consider the structure of H and we note that, since G is polyinfinite cyclic, so is H. Now H/N is a finitely generated f.c. group and therefore it is finite-by-abelian. Furthermore, if char $K = p > 0$ and if U/N is the torsion subgroup of H/N, then U/N is embedded isomorphically as a multiplicative subgroup of the finite dimensional characteristic p domain $K[U]/(P \cap K[U])$. Thus U/N is a cyclic p'-group and H/N is finite p'-by-abelian here. We can now apply Lemma 2.6 to conclude that the specialization $\alpha : \mathfrak{Q}(K[G] \to \mathfrak{Q}(K[G]/P)$ does indeed exist. This completes the proof.

It is natural to ask whether the local subrings of $\mathfrak{Q}(K[G])$ needed above can be achieved via a single localization. Since the augmentation ideal $\omega K[G]$ is completely prime, the first parts of the next two theorems yield a negative answer. All of the first result and most of the second are due to P. F. Smith.

Proposition 2.8. *Let G be a polycyclic-by-finite group and let K be a field of characteristic* 0.

(i) *If $\omega K[G]$ is localizable, then G is finite-by-nilpotent.*

(ii) *If G is finite-by-nilpotent, then all prime ideals of $K[G]$ are localizable.*

Proof. Part (i) is a special case of [**7**, Theorem A] and part (ii) is immediate from Lemma 1.4 (i).

Proposition 2.9. *Let G be a polycyclic-by-finite group and let K be a field of characteristic $p > 0$.*

(i) *If $\omega K[G]$ is localizable, then G is p-nilpotent.*

(ii) *If G is p-nilpotent, then all prime ideals of $K[G]$ are localizable.*

Proof. For part (i) see [**2**, Theorem 11.2.15]. Part (ii) follows from [**5**] and [**6**, Corollary 3.3].

The next corollary is an immediate consequence of part (ii) above.

Corollary 2.10. *Let K be a field of characteristic $p > 0$ and let G be a p-nilpotent polycyclic-by-finite group. If $K[G]$ is a domain, then $\mathfrak{Q}(K[G])$ is its universal field of fractions.*

There is of course a similar corollary to Proposition 2.8 but it offers nothing new since finitely generated torsion free nilpotent groups are polyinfinite cyclic. We close this paper with an example of a domain $K[G]$, with G polycyclic, having no universal field of fractions.

Let
$$G = \langle x, y \mid x^{-1}y^2x = y^{-2},\ y^{-1}x^2y = x^{-2} \rangle$$
be the group studied in [**2**, Lemma 13.3.3] and set $a = x^2, b = y^2$ and $c = (xy)^2$. Then $A = \langle a, b, c \rangle$ is a normal free abelian subgroup of G of rank 3 and G/A is a 'fours' group.

Furthermore, G is torsion free and supersolvable; so a result of Formanek (see [**2**, Theorem 13.3.9]) implies that $K[G]$ is a domain. Observe that G is 2-nilpotent so, by the preceding corollary, if char $K=2$ then $K[G]$ has a universal field of fractions.

We assume now that char $K\neq 2$. Suppose by way of contradiction that $D\supseteq K[G]$ is a universal field of fractions for $K[G]$ and let $\alpha: D\to K$ be a specialization extending the augmentation map $K[G]\to K$. If $R=\mathfrak{D}(\alpha)$ is the domain of α, then $R\supseteq K[G]$ and every element of $K[G]\backslash\omega K[G]$ is a unit of R.

Let $z\in G$ with $z^2=a$. Then in D we have

$$(a-1)/(z-1)=(z^2-1)/(z-1)=z+1\in K[G]\backslash\omega K[G];$$

so

$$(z-1)(a-1)^{-1}=1/(z+1)\in R$$

and

$$\alpha((z-1)(a-1)^{-1})=\alpha((z+1)^{-1})=1/2$$

since $\alpha(z)=1$. In particular this applies to both $z=x$ and $z=xb$ since

$$(xb)^2=x^2b^xb=x^2=a.$$

Thus

$$\alpha((xb-1)(a-1)^{-1})=1/2=\alpha((x-1)(a-1)^{-1});$$

so

$$x(b-1)(a-1)^{-1}=(xb-1)(a-1)^{-1}-(x-1)(a-1)^{-1}\in\ker\alpha$$

and we have $(b-1)(a-1)^{-1}\in\ker\alpha$. By symmetry, $(a-1)(b-1)^{-1}\in\ker\alpha$. Multiplying these two expressions using $ab=ba$ yields $1\in\ker\alpha$, a contradiction. Therefore for char $K\neq 2$ the domain $K[G]$ does not have a universal field of fractions.

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